DIRECT SINGULARITIES AND COMPLETELY INVARIANT DOMAINS OF ENTIRE FUNCTIONS

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Abstract. Let f be a transcendental entire function which omits a point $a \in \mathbb{C}$. We show that if D is a simply connected domain which does not contain a, then the full preimage $f^{-1}(D)$ is disconnected. Thus, in dynamical context, if an entire function has a completely invariant domain and omits some value, then the omitted value belongs to the completely invariant domain. We conjecture that the same property holds if a is a *locally omitted value* (i.e., the projection of a direct singularity of f^{-1}). We were able to prove this conjecture for entire functions of finite order. We include some auxilliary results on singularities of f^{-1} for entire functions f, which can be of independent interest.

1. Introduction and results

The question considered in this paper is motivated by dynamics of entire functions [3, 6]. A component D of the Fatou set of an entire function f is called a *completely invariant domain* if $f^{-1}(D) = D$. This is a stronger property than simple invariance $f(D) \subset D$.

In what follows, all entire functions are assumed to be transcendental. It follows from a result of Baker [2, Theorem 1] that all invariant components of the Fatou set of such a function are simply connected. Baker also proved that at most one completely invariant domain can exist [1], and if f has a completely invariant domain, then all critical values (and thus all critical points) of f are contained in it [2, Theorem 2].

In [6, Lemma 11], the latter result of Baker was extended to the logarithmic singularities of f^{-1}: a completely invariant domain must contain all projections of logarithmic singularities of f^{-1}. In this paper, we consider possibilities of extension of this result to other types of singularities of f^{-1}.

A point $a \in \mathbb{C}$ is *omitted* by an entire function f if $f(z) \neq a$ for $z \in \mathbb{C}$. A point $a \in \mathbb{C}$ is *locally omitted* by f if there exists $r > 0$ and a component G

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of the set $f^{-1}(B(a,r))$ such that $f(z) \neq a$ in G. Here and in what follows, we use the notation $B(a,r)$ for a disc of radius r centered at $a \in \mathbb{C}$. According to Iversen's classification of singularities, which will be recalled in Section 2, a value is locally omitted if and only if it is the projection of a direct singularity of f^{-1}. In the special case that $f : G \to B(a,r) \setminus \{a\}$ is a universal covering we say that a is the projection of a logarithmic singularity of f^{-1}.

It is known that an omitted value does not have to be the projection of a logarithmic singularity. An example of this is

$$f(z) = \exp\left(\sum_{k=1}^{\infty}\left(\frac{z}{2^k}\right)^{2^k}\right). \tag{1}$$

We will analyse this example in the end of the paper.

Theorem 1. *Let f be an entire transcendental function omitting a point $a \in \mathbb{C}$, and let D be a simply connected region that does not contain a. Then $f^{-1}(D)$ is disconnected.*

Corollary. *Let f be a transcendental entire function having a completely invariant domain D. If f omits a point then this point belongs to D.*

We conjecture that Theorem 1 and the Corollary can be extended to locally omitted values. Paper [6] contains a statement that the Corollary can be proved for locally omitted values in the same way as for projections of logarithmic singularities. However, the argument given in [6] does not apply to locally omitted values of arbitrary entire functions. So the conjecture remains open.

In this paper we prove the conjecture for functions of finite order. Namely we establish the following.

Theorem 2. *Let f be an entire function of finite order, and let $a \in \mathbb{C}$ be either a critical value or a locally omitted value. If D is a simply connected region that does not contain a, then $f^{-1}(D)$ is disconnected.*

Iversen's Theorem (stated in Section 2) implies that a locally omitted value has to be an asymptotic value. There is an example [4] of an entire function of finite order with a completely invariant domain D and an asymptotic value that does not belong to D.

It is interesting that Theorem 2 has a converse:

Theorem 3. *Let f be an entire function of finite order, and let $a \in \mathbb{C}$ be neither a critical value nor a locally omitted value. Then there exists a simply connected region D which does not contain a, and such that $f^{-1}(D)$ is connected.*

The case of a locally omitted value in Theorem 2 is based on the following result which is of independent interest.

Theorem 4. *Let f be an entire function of finite order, and $a \in \mathbb{C}$ a locally omitted value. Then a is the projection of a logarithmic singularity of f^{-1}.*

The structure of the paper is the following. In Section 2 we recall auxilliary facts on the singularities of the inverses of entire functions. In Section 3 we prove Theorems 1, 2 and 4. In Section 4 we discuss some results needed for the proof of Theorem 3 and then we prove Theorem 3 in Section 5. In Section 6 we analyse the example (1).

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2. Preliminaries

We shall repeatedly use the following result of Iversen [11], which follows easily from the Gross Star Theorem [12, p. 292], or from the variant of the Gross Star Theorem stated as Proposition 1 in Section 4 below.

Iversen's Theorem. *Let ϕ be a holomorphic branch of the inverse f^{-1} defined in a neighborhood of some point w_0 and let $\gamma : [0, 1] \to \mathbb{C}$ be a curve with $\gamma(0) = w_0$. Then for every $\varepsilon > 0$ there exists a curve $\tilde{\gamma} : [0, 1] \to \mathbb{C}$ satisfying $\tilde{\gamma}(0) = w_0$ and $|\gamma(t) - \tilde{\gamma}(t)| < \varepsilon$ such that ϕ has an analytic continuation along $\tilde{\gamma}$.*

Now we recall Iversen's classification of singularities; see [5], [11] or [12, p. 289]. Let f be a transcendental meromorphic function and $a \in \mathbb{C}$. Consider the open discs $B(a, r)$ of radius r centered at a. For every $r > 0$, it is possible to choose a component U_r of the preimage $f^{-1}(B(a, r))$ in such a way that $r_1 < r_2$ implies $U_{r_1} \subset U_{r_2}$. The possibility of such a choice of (non-empty!) components U_r follows from Iversen's Theorem.

Now we have two possibilities:

a) $\bigcap_{r>0} U_r$ consists of one point, or

b) $\bigcap_{r>0} U_r = \emptyset$.

In the latter case we say that our choice $r \mapsto U_r$ defines a *transcendental singularity* of f^{-1} over a. We also say that a is the *projection* of the transcendental singularity, or that the transcendental singularity *lies over* a, and any of the sets U_r is called a *neighborhood* of the transcendental singularity. Projections of transcendental singularities coincide with asymptotic values of f. A transcendental singularity over a is called *direct* if for some $r > 0$ we have $f(z) \neq a$ for $z \in U_r$. Otherwise it is called *indirect*. A direct singularity is called *logarithmic* if the restriction $f : U_r \to B(a, r) \backslash \{a\}$ is a universal covering for some $r > 0$. All these definitions can be also given for $a = \infty$ using $B(\infty, r) = \{z \in \overline{\mathbf{C}} : |z| > 1/r\}$.

It is clear from these definitions that locally omitted values are exactly the projections of direct singularities.

For example, $\exp z$ has a logarithmic singularity over 0, and $(\sin z)/z$ has two indirect singularities over 0.

The importance of direct singularities comes to a great extent from the following result [12, §XI.4].

Denjoy-Carleman-Ahlfors Theorem. *A meromorphic function of finite order has only finitely many direct singularities.*

A corollary of this result is that an entire function of finite order has only finitely many asymptotic values [12, p. 313].

In Section 6, we will prove that for the function (1) the set of direct singularities over 0 has the power of continuum, but none of these singularities is logarithmic. According to Heins [7], the set of projections of direct singularities is always at most countable, but the set of direct singularities over one point can have the power of the continuum. Example (1) is a new example of this kind; unlike the previous examples, it is given by a simple explicit formula.

3. Proof of Theorems 1, 2 and 4

Proof of Theorem 1. Suppose that $f^{-1}(D)$ is connected. Using Iversen's Theorem, we can find a Jordan curve $\Gamma : [0,1] \to \mathbb{C}$ with $\Gamma(0) = \Gamma(1) = b$ for some point $b \in D$, such that Γ does not pass through a,

$$\frac{1}{2\pi}\int_\Gamma \frac{dw}{w-a} = 1,$$

and there exists a holomorphic branch ϕ of f^{-1} at b, such that ϕ has an analytic continuation along Γ. The preimage of Γ under this branch ϕ and its analytic continuation along Γ is a simple compact arc γ, which may be closed or not. Both endpoints of γ belong to $f^{-1}(D)$, and as $f^{-1}(D)$ is supposed to be connected, we can find an arc γ_1 in $f^{-1}(D)$, connecting the endpoints of γ. We have $f(\gamma') \subset D$, D is simply connected, and $a \notin D$. So

$$\frac{1}{2\pi}\int_{f(\gamma')} \frac{dw}{w-a} = 0.$$

So

$$\frac{1}{2\pi}\int_{\gamma\cup\gamma'} \frac{df(z)}{f(z)-a} = \frac{1}{2\pi}\int_{\Gamma\cup f(\gamma')} \frac{dw}{w-a} = 1,$$

which is a contradiction because $\gamma \cup \gamma'$ is a closed curve and $f(z) \neq a$ in the plane. This proves Theorem 1.

Now we prove the following result from which Theorem 4 follows:

Theorem 5. *If an entire function has a direct singularity over some point a which is not a logarithmic singularity, then every neighborhood of this singularity is also a neighborhood of other direct singularities over a.*

It follows that whenever we have a direct singularity over some point and no logarithmic singularities over the same point, then the set of direct singularities over this point has the power of the continuum.

As functions of finite order have only finitely many direct singularities by the Denjoy-Carleman-Ahlfors Theorem, we obtain Theorem 4.

The proof of Theorem 5 requires the following lemma.

Lemma 1. *Let μ be a singular measure on the unit circle, and $A = \{e^{i\theta} : \theta \in (a,b)\}$ an arc of the unit circle such that $\mu(A) > 0$. Then there exists a point $\theta \in (a,b)$ such*

$$\lim_{r\to 1} u(re^{i\theta}) = +\infty,$$

where

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(t-\theta)}d\mu(t)$$

is the Poisson integral of μ.

This is well-known and we include a standard proof for completeness (see, for example, [8, (6.3)]).

Proof. For a subinterval (x,y) of (a,b), we denote by $\mu(x,y)$ the measure of the arc $\{e^{i\theta} : \theta \in (x,y)\}$. We first prove that there exists $\theta \in (a,b)$ such that

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\mu(\theta-\varepsilon,\theta+\varepsilon) = +\infty. \tag{2}$$

Proving this by contradiction, suppose that such θ does not exist. Then the sets

$$E_n = \{x \in (a+1/n, b-1/n) : \liminf_{\varepsilon\to 0}\frac{1}{2\varepsilon}\mu(x-\varepsilon, x+\varepsilon) \le n\}, \quad n = n_0, n_0+1, \ldots$$

cover (a,b). Fix $n \ge n_0$. For every $x \in E_n$, there exists an interval of the form $(x-\varepsilon, x+\varepsilon)$ with $\varepsilon \in (0, n^{-2})$ whose μ-measure is at most $4\varepsilon n$. By the well-known covering lemma [10, Thm. 1.1], E_n can be covered by some of these intervals such that the multiplicity of this covering is an absolute constant K. Thus we obtain that $\mu(E_n) \le 4K/n$. As the sets E_n form an increasing sequence, we conclude that $\mu(E_n) = 0$ for all $n \ge n_0$. So $\mu(a,b) = 0$ and we obtain a contradiction, which proves the existence of the point θ satisfying (2).

Now it is easy to pass from (2) to the Poisson integral. For $0 < \varepsilon < \pi$ we have

$$\begin{aligned} u(re^{i\theta}) &\geq \frac{1}{2\pi}\int_{\theta-\varepsilon}^{\theta+\varepsilon} \frac{1-r^2}{1+r^2-2r\cos(t-\theta)} d\mu(t) \\ &\geq \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos\varepsilon}\mu(\theta-\varepsilon,\theta+\varepsilon). \end{aligned}$$

Putting $r = 1-\varepsilon$ and noting that then

$$\frac{1-r^2}{1+r^2-2r\cos\varepsilon} \geq \frac{1}{\varepsilon}$$

for sufficiently small ε we obtain

$$u(re^{i\theta}) \geq \frac{1}{2\pi\varepsilon}\mu(\theta-\varepsilon,\theta+\varepsilon),$$

which completes the proof.

Proof of Theorem 5. Suppose that $U = U_r$ is a neighborhood of exactly one direct singularity over a finite point a, where $r > 0$ is so small that $f(z) \neq a$ in U. We are going to prove that this singularity is logarithmic.

By the Maximum Principle, U is simply connected. It is easy to see that the closure of U in the Riemann sphere is locally connected. So a conformal map $\phi : B(0,1) \to U$ extends to a continuous map from the unit disc to the Riemann sphere. The preimage of infinity under ϕ is a closed subset E, of the unit circle, which by a theorem of Beurling [15, p. 344] has zero logarithmic capacity.

We consider the positive harmonic function

$$u(z) := \log\frac{r}{|f(\phi(z))-a|}, \quad z \in B(0,1).$$

It has a Poisson representation

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(t-\theta)} d\mu(t)$$

for some finite Borel measure μ, and we have $\lim_{r\to 1} u(re^{i\theta}) = 0$ if $e^{i\theta} \notin E$. As E has zero capacity, and thus zero length, the measure μ is singular. It is easy to see that if μ is a single atom so that u is proportional to the Poisson kernel, then $f : U \to B(r,a) \setminus \{a\}$ is a universal covering, and thus the singularity we consider is logarithmic.

Otherwise, there is a simple cross-cut σ in $B(0,1)$ beginning and ending in the complement of E, such that the two arcs on the unit circle bounded by the endpoints of σ both intersect the support of μ. Lemma 1 implies that u is unbounded in each of the two components G_1 and G_2 of $B(0,1)\backslash\sigma$. The image $\phi(\sigma)$ of this cross-cut separates U into two regions $D_j = \phi(G_j)$. The harmonic function

$$v(z) := u(\phi^{-1}(z)) = \log \frac{r}{|f(z) - a|}$$

is unbounded in each D_j and bounded on ∂D_j for $j = 1, 2$. Thus there exists $\varepsilon > 0$ such that

$$\left\{z \in U : v(z) > \log \frac{r}{\varepsilon}\right\} = \{z \in U : |f(z) - a| < \varepsilon\}$$

is disconnected. As $f(z) \neq a$ for $z \in U$ we conclude that U is a neighborhood of at least two singularities over 0. This completes the proof of Theorem 5.

As we already mentioned, Theorem 4 follows from Theorem 5 and the Denjoy-Carleman-Ahlfors Theorem.

Now Theorem 2 is an easy corollary: In the case of a critical value, we repeat Baker's argument [1] and in the case of a locally omitted value, we first use Theorem 4, to conclude that this singularity is in fact logarithmic, and then repeat the argument from [6]. Both [1] and [6] deal only with the case that D is completely invariant, but the arguments extend to the situation of Theorem 2 without difficulty.

4. Results needed for the proof of Theorem 3

The following definition will be used in the proof of Theorem 3. Let f be an entire function. A simple curve γ will be called *good for* f, if γ contains no critical values of f and all components of the full preimage $f^{-1}(\gamma)$ are compact.

It is easy to see that a simple curve which contains neither critical values nor asymptotic values is good. For entire functions of finite order, there can be only finitely many asymptotic values by the Denjoy-Carleman-Ahlfors Theorem. Thus we obtain the existence of good curves, and in fact we see that the conclusion of Proposition 1 and 2 below holds for entire functions of finite order.

In general, there are entire functions for which every point in the complex plane is an asymptotic value [9], so the existence of good curves for such functions is not evident. An instructive example is given by $f(z) = (\sin z)/z$. Here 0 is the projection of an indirect singularity. However, one can show that the segment $[-i, i]$ is good. On the other hand, $[-\varepsilon, \varepsilon]$ is not good for any positive ε.

In the remaining part of this section we will prove the existence of good curves in general. This material is not used anywhere else in the paper, but may be of independent interest.

Our Proposition 1 and 2 below are similar to the results of Shimizu [13, p. 186] and Terasaka [14, Lemma on p. 310]. We need the following version of the classical Gross Star Theorem [12, p. 292]:

Proposition 1. *Let ϕ be a holomorphic branch of the inverse f^{-1} defined in some disc B, and let ℓ be some direction in the plane. Then ϕ has an analytic continuation along almost all straight lines intersecting B and having the direction ℓ.*

Such lines can be parametrized by the points of their intersection with the diameter of B perpendicular to ℓ. "Almost all" refers to the Lebesgue measure on this diameter.

Proof of Proposition 1. We assume for simplicity that the direction ℓ is parallel to the real axis, and that the diameter of B perpendicular to this direction is (ia, ib), where $a < b$. Let $M > b - a$. Consider the rectangle

$$Q_M := \{z = x + iy : |x| < M, y \in (a, b)\}.$$

For each horizontal interval $\{x+iy_0 : |x| < M\}$, where $y_0 \in (a, b)$, we consider the maximal open subinterval containing the point iy_0 such that an analytic continuation of ϕ is possible along this subinterval. The union of these maximal subintervals over all $y_0 \in (a, b)$ forms a region $G_M \subset Q_M$. If a maximal horizontal interval in G_M has an endpoint inside Q_M, then we will call this endpoint a *singular point of* ϕ. It is enough to show that the Lebesgue measure of the projection of the set of singular points on the imaginary axis is zero, for every fixed $M > b-a$. The analytic continuation of ϕ along maximal horizontal intervals in G_M maps G_M univalently onto some region $G'_M \subset \mathbb{C}$. The singular points in G_M correspond to the critical values of f and to the accessible points at infinity of G'_M. Since the set of critical values is countable, the Lebesgue measure of its projection on the imaginary axis is zero.

Let σ'_r be the intersection of G'_M with the circle $\{z : |z| = r\}$. We may assume that ϕ is bounded in B. Then for $r > r_0$ the set $\sigma_r := f(\sigma'_r)$ is a union of cross-cuts in G_M which separate the diameter $[ia, ib]$ from the set of singular points of ϕ on the boundary of G_M. It is enough to show that the length of σ_r tends to zero as $r \to \infty$ on some sequence.

We have

$$\text{length}(\sigma_r) = \int_{\sigma'_r} |f'(z)|\,|dz|$$

and by Schwarz's inequality

$$\text{length}^2(\sigma_r) \le 2\pi r \int_{\sigma'_r} |f'(z)|^2\,|dz|.$$

Dividing by r and integrating with respect to r from r_0 to ∞, we obtain

$$\int_{r_0}^{\infty} \mathrm{length}^2(\sigma_r)\frac{dr}{r} \leq 2\pi \iint_{G'_M} |f'(z)|^2\, dxdy = 2\pi\, \mathrm{area}(G_M) \leq 2\pi\, \mathrm{area}(Q_M).$$

Thus the integral on the left hand side converges.

We conclude that $\mathrm{length}(\sigma_r) \to 0$ on some sequence $r = r_k \to \infty$. This proves the Proposition.

Proposition 2. *Let f be an entire function, and $Q = (a, b, c, d)$ a rectangle in the plane. Then almost every closed interval connecting the opposite sides $[a, b]$ and $[c, d]$ and parallel to the other sides is good for f.*

Proof. Consider the set of pairs $\{B_j, \phi_j\}$, where B_j is a disc contained in Q, having rational center and rational radius, and ϕ is a holomorphic branch of f^{-1} in this disc. According to the Poincaré–Volterra theorem, this set is countable. Applying Proposition 1 to the pair $\{B_j, \phi_j\}$ and the direction $[a, d]$ we obtain an exceptional set of lines E_j of measure zero. Then $E = \bigcup E_j$ is a set of measure zero, and all intervals which are intersections of Q with lines parallel to $[a, d]$ and not in E are good for f. This proves the Proposition.

As mentioned, we will use Propositions 1 and 2 only for entire functions of finite order, and for such functions the conclusion follows from the Denjoy-Carleman-Ahlfors Theorem.

5. Proof of Theorem 3

Lemma 2. *Let f be an entire function of finite order and let $a \in \mathbb{C}$. Then there exists $r_0 > 0$ such that if $0 < r \leq r_0$, then all components of $f^{-1}(B(a, r))$ have connected boundary.*

Proof. Let U be a component of $f^{-1}(B(a, r))$. By the Maximum Principle, U is simply connected. Each complementary component of U contains a neighborhood of a singularity of f^{-1} over ∞.

By the Denjoy-Carleman-Ahlfors Theorem, there are only finitely many singularities of f^{-1} over ∞. It follows that there exists $r_0 > 0$ such that $f^{-1}(\mathbb{C} \setminus \overline{B(a, r)})$ is connected for $0 < r \leq r_0$. As this set is a neighborhood of each singularity of f^{-1} over ∞ we deduce that the complement of U is connected. Hence the boundary of U is connected.

Proof of Theorem 3. We will construct a simple curve Γ connecting a with ∞ such that the preimage of Γ consists of infinitely many simple, pairwise disjoint curves which connect the preimages of a with ∞. With $D := \mathbb{C}\backslash\Gamma$ we then see that both D and $f^{-1}(D)$ are simply connected.

We choose r_0 according to Lemma 2. We begin with a simple curve Γ^0 (for example, a straight line segment) which is good for f and connects a point $w_0 \in B(a, r_0)$ to some point in $\mathbb{C}\backslash\overline{B(a, r_0)}$. In addition we may assume that

$\Gamma^0 \cap \partial B(a,r)$ consists of at most one point for all r. Curves with the latter property will be called *a-monotonic*. The existence of a curve Γ^0 with the properties mentioned follows from Proposition 2.

Take a point a point $b \in \Gamma^0 \cap \mathbb{C} \backslash \overline{B(a,r_0)}$ which is not the endpoint of Γ^0 and let (c_j) be the sequence of b-points of f. Let γ_1^0 be the component of $f^{-1}(\Gamma^0)$ which contains c_1. Then γ_1^0 is a simple curve connecting a w_0-point x_1 with c_1. Let U_1 be the component of $f^{-1}(B(a,r_0))$ that contains x_1. Since f has no direct singularity or critical point over a, there exists $z_1 \in U_1$ with $f(z_1)=a$ and $f'(z_1)\neq 0$. Thus there exists r_1 with $0<r_1<r_0$ such that there is a branch ϕ_1 of f^{-1} which is defined in $B(a,r_1)$ and maps a to z_1. We may also assume that ϕ_1 is bounded in $B(a,r_1)$.

We can connect z_1 by a curve σ_1 to ∂U_1 such that $f(\sigma_1)$ is a straight line connecting a to $\partial B(a,r_0)$. (Here we say that a curve γ *connects* a point z to a set S if γ is a simple curve such that one endpoint of γ is z while the other one is in S, and $S \cap \gamma$ consists only of that endpoint.) Since r_0 has been chosen according to Lemma 2, the boundary of U_1 is connected. Thus we can connect the endpoint of σ_1 in ∂U_1 by a curve $\sigma_1' \subset \partial U_1$ to the point v_1 which lies in the intersection of γ_1^0 and ∂U_1. (Note that the intersection of γ_1^0 and ∂U_1 consists of only one point since $\Gamma^0 = f(\gamma_1^0)$ is a-monotonic.) The curve $\sigma_1 + \sigma_1'$ thus connects z_1 to v_1. By deforming σ_1' slightly we can replace the curve $\sigma_1 + \sigma_1'$ by a curve τ_1 which connects z_1 to v_1 such that $f(\tau_1)$ is a-monotonic. Using Proposition 2, we can replace τ_1 by a curve τ_1' which connects a point $y_1^1 \in \phi_1(B(a,r_1))$ to $\gamma_1^0 \cap U_1$ and which has the property that $f(\tau_1')$ is good and a-monotonic. Combining $f(\tau_1')$ and Γ^0 we thus obtain a curve Γ^1 which is good and a-monotonic and which connects $w_1 := f(y_1^1) \in B(a,r_1)$ to b. More precisely, if u_1 is the endpoint of $f(\tau_1')$ in Γ^0 and if Σ^0 is the arc that connects u_1 and b in Γ^0, then we take $\Gamma^1 := f(\tau_1') \cup \Sigma^0$. Note that $u_1 \in B(a,r_0)$ since the endpoint of τ_1' is in U_1.

The component γ_1^1 of $f^{-1}(\Gamma^1)$ that contains c_1 consists of τ_1' and a subarc of γ_1^0, and it is a simple curve connecting $y_1^1 \in \phi(B(a,r_1))$ to c_1. In fact, since Γ^1 is good, we see that for all for $j\in\mathbb{N}$ the component γ_j^1 of $f^{-1}(\Gamma^1)$ which contains c_j is a simple curve connecting a w_1-point y_j^1 to c_j.

The following fact is important: *no matter how we extend Γ^1 by attaching a piece in $B(a,r_1)$, the component of the preimage of the extended curve that contains c_1 and hence γ_1^1 will be compact.* This follows since the part added to γ_1^1 will be contained in $\phi_1(B(a,r_1))$.

Now we repeat this process of extension. Suppose that $r_{n-1} < r_{n-2} < \cdots < r_1 < r_0$ and that Γ^{n-1} is a good and a-monotonic curve which connects a point $w_{n-1} \in B(a,r_{n-1})$ to b such that for $1 \leq j \leq n-1$ the component γ_j^{n-1} of $f^{-1}(\Gamma^{n-1})$ which contains c_j has the following property: *no matter how we extend Γ^{n-1} by attaching a piece in $B(a,r_{n-1})$, the component of the* *preimage of the extended curve that contains* c_j *and hence* γ_j^1 *will be compact for* $1 \leq j \leq n-1$.

The way we obtain Γ^n from Γ^{n-1} is essentially the same that we used to obtain Γ^1 from Γ^0: since Γ^{n-1} is good, the component γ_n^{n-1} of $f^{-1}(\Gamma^{n-1})$ that contains c_n is a simple curve connecting a w_n-point x_n to c_n. Let U_n be the component of $f^{-1}(B(a, r_{n-1}))$ that contains x_n. Then there exists $z_n \in U_n$ with $f(z_n) = a$ and $f'(z_n) \neq 0$, and hence there exists r_n with $0 < r_n < r_{n-1}$ such that there is a branch ϕ_n of f^{-1} in $B(a, r_n)$ with $\phi_n(a) = z_n$, where we may again assume that ϕ_n is bounded in $B(a, r_n)$. We connect z_n by a curve σ_n to ∂U_n such that $f(\sigma_n)$ is a straight line and we connect the endpoint of σ_n by a curve $\sigma_n' \subset \partial U_n$ to the point v_n which lies in the intersection of γ_n^{n-1} and ∂U_n. Again we can replace the curve $\sigma_n + \sigma_n'$ by a curve τ_n which connects z_n to v_n such that $f(\tau_n)$ is a-monotonic and, using Proposition 2, we can replace τ_n by a curve τ_n' which connects a point $y_n^n \in \phi_n(B(a, r_n))$ to $\gamma_n^{n-1} \cap U_n$ and which has the property that $f(\tau_n')$ is good and a-monotonic. From $f(\tau_n')$ and Γ^{n-1} we now obtain a curve Γ^n which is good and a-monotonic and which connects $w_n := f(y_n^n) \in B(a, r_n)$ to b. Moreover, the component γ_n^n of $f^{-1}(\Gamma^n)$ that contains c_n is a simple curve connecting $y_n^n \in \phi_n(B(a, r_n))$ to c_n, and no matter how we extend Γ^n by attaching a piece in $B(a, r_n)$, the component γ_n^n of the preimage of the extended curve which contains c_n will be compact. And it follows from our induction hypothesis that the same is true for the preimages γ_j^n of the extended curve which contain c_j, for $1 \leq j \leq n-1$.

Note that Γ_n need not contain Γ_{n-1}, but since the endpoint of $f(\tau_n')$ is contained in $B(a, r_{n-1})$ we have

$$\Gamma_n \setminus B(a, r_{n-1}) = \Gamma_{n-1} \setminus B(a, r_{n-1}) \supset \Gamma_{n-1} \setminus B(a, r_{n-2})$$

for $n \geq 2$.

We now combine the curves Γ_n defined inductively in the above way to a curve Γ^∞ by putting

$$\Gamma^\infty := \{a\} \cup \bigcup_{n=1}^{\infty} \left(\Gamma_n \setminus B(a, r_{n-1}) \right).$$

Then Γ^∞ is a simple (and in fact a-monotonic) curve that connects a to b, and it follows from the construction of the Γ_n that the preimage of Γ^∞ that contains c_j is a simple curve that connects z_j with c_j.

Finally we connect b to ∞ by an a-monotonic curve Σ with the property that every compact subarc of Σ is good. Such a curve exists by Proposition 2. It then follows that $\Gamma := \Gamma^\infty \cup \Sigma$ has the properties stated at the beginning of the proof.

This completes the proof of Theorem 3.

6. An example

We show that the function f given by (1) has infinitely many direct but no logarithmic singularity over 0. Let

$$g(z) := \sum_{k=1}^{\infty} \left(\frac{z}{2^k}\right)^{2^k}$$

so that $f(z) = \exp g(z)$. We fix ε with $0 < \varepsilon \leq \frac{1}{8}$ and put $r_n := (1+\varepsilon)2^{n+1}$ and $r'_n := (1-2\varepsilon)2^{n+2}$ for $n \in \mathbb{N}$. For $j \in \{0, 1, \dots, 2^n - 1\}$ we define the sets

$$A_{j,n} := \left\{ r \exp\left(\frac{2\pi i j}{2^n}\right) : r \geq r_n \right\}, \quad i = \sqrt{-1},$$

$$B_{j,n} := \left\{ r \exp\left(\frac{\pi i}{2^n} + \frac{2\pi i j}{2^n}\right) : r_n \leq r \leq r'_n \right\},$$

and

$$C^{\pm}_{j,n} := \left\{ r \exp\left(\frac{\pi i}{2^n} + \frac{2\pi i j}{2^n} \pm \frac{r - r'_n}{r_{n+1} - r'_n} \frac{\pi i}{2^{n+1}}\right) : r'_n \leq r \leq r_{n+1} \right\}.$$

We shall show that if n is large enough, then

$$\operatorname{Re} g(z) > 2^{2^n} \quad \text{for} \quad z \in A_{j,n} \tag{3}$$

while

$$\operatorname{Re} g(z) < -2^{2^n} \quad \text{for} \quad z \in B_{j,n} \cup C^{+}_{j,n} \cup C^{-}_{j,n}. \tag{4}$$

Note that $C^{-}_{j,n}$ connects $B_{j,n}$ to $B_{2j,n+1}$ while $C^{+}_{j,n}$ connects $B_{j,n}$ to $B_{2j+1,n+1}$. This implies that

$$T := [-ir_1, ir_1] \cup \bigcup_{n=1}^{\infty} \bigcup_{j=0}^{2^n-1} \left(B_{j,n} \cup C^{+}_{j,n} \cup C^{-}_{j,n}\right)$$

is an infinite binary tree; see Figure 1. By (4), every unbounded simple path on this tree starting at 0 is an asymptotic curve on which $\operatorname{Re} g(z) \to -\infty$. Choosing U_ρ as the component of

$$\{z : \operatorname{Re} g(z) < \log \rho\} = \{z : |f(z)| < \rho\}$$

which contains the "tail" of this curve we thus obtain a transcendental singularity of f^{-1} over 0, and this singularity is direct because f has no zeros. Using (3) we see that different curves define different singularities. Thus we obtain a set of direct singularities which has the power of the continuum.

Moreover, it follows from (3) and (4) and the above considerations that if U_ρ is a component of $\{z : |f(z)| < \rho\}$ containing the "tail" of some curve in T, then U_ρ also contains the "tail" of some other curve in T and thus there exists $\rho' < \rho$ such that U_ρ contains at least two components of $\{z : |f(z)| < \rho'\}$. This implies that the singularity defined by $\rho \mapsto U_\rho$ is not logarithmic.

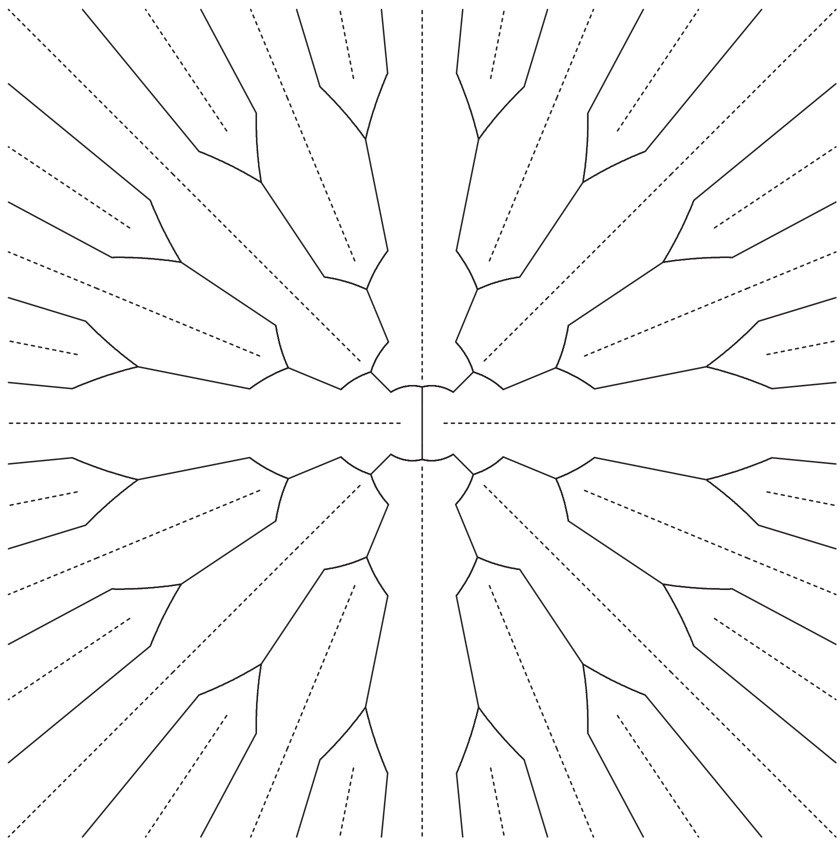

FIGURE 1. The part of the tree T lying in $\{z : |\operatorname{Re} z| \leq 80, |\operatorname{Im} z| \leq 80\}$, for $\varepsilon = 1/16$. The sets $A_{j,n}$ are drawn as dotted lines.

To prove (3) we note that if $z = r\exp(2\pi ij/2^n) \in A_{j,n}$ so that $r \geq r_n$, then

$$\operatorname{Re} g(z) \geq \sum_{k=n}^{\infty} \left(\frac{r}{2^k}\right)^{2^k} - \sum_{k=1}^{n-1} \left(\frac{r}{2^k}\right)^{2^k} \geq \left(\frac{r}{2^n}\right)^{2^n} - \sum_{k=1}^{n-1} \left(\frac{r}{2^k}\right)^{2^k}.$$

Put $s := r/2^n$ and

$$\Sigma_1 := \sum_{k=1}^{n-1} \left(\frac{r}{2^k}\right)^{2^k}.$$

Then

$$\Sigma_1 = \sum_{k=1}^{n-1} \left(s2^{n-k}\right)^{2^k} \leq s^{2^{n-1}} \sum_{k=1}^{n-1} 2^{(n-k)2^k}.$$

Now $(n-k)2^k \leq 2^{n-1}$ for $1 \leq k \leq n-1$ and $s < 2 + 2\varepsilon$ so that

$$\Sigma_1 \leq s^{2^{n-1}}(n-1)2^{2^{n-1}} = o\left(s^{2^n}\right)$$

as $n \to \infty$ and hence

$$\operatorname{Re} g(z) \geq \left(\frac{r}{2^n}\right)^{2^n} - \Sigma_1 = (1 - o(1))s^{2^n} > 2^{2^n}$$

for large n. To prove (4) for $z \in B_{j,n}$, let $z = r\exp\left(\pi i/2^n + 2\pi i j/2^n\right) \in B_{j,n}$, with $r_n \leq r \leq r'_n$. Then

$$\operatorname{Re} g(z) \leq -\left(\frac{r}{2^n}\right)^{2^n} + \Sigma_1 + \Sigma_2$$

with

$$\Sigma_2 := \sum_{k=n+1}^{\infty} \left(\frac{r}{2^k}\right)^{2^k} = \sum_{k=n+1}^{\infty} \left(s 2^{n-k}\right)^{2^k}.$$

Thus

$$\Sigma_2 \leq \left(\frac{s}{2}\right)^{2^n} + \sum_{k=n+2}^{\infty} \left(\frac{s}{4}\right)^{2^k}.$$

Since $s/4 \leq 1 - 2\varepsilon$ we find that

$$\Sigma_2 = o\left(s^{2^n}\right)$$

as $n \to \infty$ and thus

$$\operatorname{Re} g(z) \leq -(1 - o(1))s^{2^n} < -2^{2^n}$$

for $z \in B_{j,n}$, provided n is sufficiently large.

Finally we prove (4) for $z \in C^+_{j,n}$. So let

$$\begin{aligned} z &= r\exp\left(\frac{\pi i}{2^n} + \frac{2\pi i j}{2^n} + \frac{r - r'_n}{r_{n+1} - r'_n}\frac{\pi i}{2^{n+1}}\right) \\ &= r\exp\left(\frac{\pi i}{2^n} + \frac{2\pi i j}{2^n} + \frac{s - 4(1-2\varepsilon)}{12\varepsilon}\frac{\pi i}{2^{n+1}}\right) \\ &\in C^+_{j,n}, \end{aligned}$$

with $r'_n \leq r \leq r_{n+1}$, so that $4(1 - 2\varepsilon) \leq s \leq 4(1 + \varepsilon)$. We have

$$\operatorname{Re} g(z) \leq \operatorname{Re}\left(\frac{z}{2^n}\right)^{2^n} + \operatorname{Re}\left(\frac{z}{2^{n+1}}\right)^{2^{n+1}} + \Sigma_1 + \Sigma_3$$

with

$$\Sigma_3 := \sum_{k=n+2}^{\infty} \left(\frac{r}{2^k}\right)^{2^k} \leq \sum_{k=n+2}^{\infty} \left(\frac{s}{4}\right)^{2^k} = o(1)$$

since $s > 4$. Since $\Sigma_1 = o\left(s^{2^n}\right)$ we find that

$$\begin{aligned}\operatorname{Re} g(z) &\leq s^{2^n} \cos\left(\pi + \frac{s-4(1-2\varepsilon)}{12\varepsilon}\frac{\pi}{2}\right) \\ &\quad + \left(\frac{s}{2}\right)^{2^{n+1}} \cos\left(\frac{s-4(1-2\varepsilon)}{12\varepsilon}\pi\right) + o\left(s^{2^n}\right) \\ &= s^{2^n}\left(\cos\left(\pi + t\frac{\pi}{2}\right) + \left(\frac{s}{4}\right)^{2^n}\cos(t\pi) + o(1)\right)\end{aligned}$$

as $n \to \infty$, with $t := (s - 4(1-2\varepsilon))/12\varepsilon$. The range $4(1-2\varepsilon) \leq s \leq 4(1+\varepsilon)$ corresponds to $0 \leq t \leq 1$ and $s = 4(1-2\varepsilon) + 12\varepsilon t$. We define

$$\begin{aligned}h(t) &:= \cos\left(\pi + t\frac{\pi}{2}\right) + \left(\frac{s}{4}\right)^{2^n}\cos(t\pi) \\ &= \cos\left(\left(1+\frac{t}{2}\right)\pi\right) + (1-2\varepsilon+3\varepsilon t)^{2^n}\cos(t\pi).\end{aligned}$$

and put $\delta := -\cos(11\pi/8)/2 > 0$. For $0 \leq t \leq \frac{1}{2}$ we have

$$h(t) \leq \cos\left(\frac{5}{4}\pi\right) + \left(1 - \frac{\varepsilon}{2}\right)^{2^n} < -2\delta$$

if n is large enough. For $\frac{1}{2} \leq t \leq \frac{3}{4}$ we have $\cos(t\pi) < 0$ and thus

$$h(t) \leq \cos\left(\left(1+\frac{t}{2}\right)\pi\right) \leq \cos\left(\frac{11}{8}\pi\right) = -2\delta.$$

Finally, for $\frac{3}{4} \leq t \leq 1$ we have $\cos\left(\left(1+\frac{t}{2}\right)\pi\right) < 0$ and thus

$$h(t) \leq (1-2\varepsilon+3\varepsilon t)^{2^n}\cos(t\pi) \leq \left(1+\frac{1}{4}\right)^{2^n}\cos\left(\frac{3}{4}\pi\right) \leq -2\delta$$

if n is large. Overall we find that $h(t) \leq -2\delta$ for all t and thus

$$\operatorname{Re} g(z) \leq -\delta s^{2^n} < -2^{2^n}$$

for $z \in C^+_{j,n}$, provided n is large enough. The proof that

$$\operatorname{Re} g(z) < -2^{2^n}$$

for $z \in C^-_{j,n}$ is analogous. This completes the proof of (3) and (4). As already mentioned, this implies that every path going to ∞ in T corresponds to a direct singularity of f over 0 which is not logarithmic, and the set of such singularities has the power of the continuum. Also, we see that if $\rho \to U_\rho$ is a singularity over 0 such that $U_\rho \cap T \neq \emptyset$ for all $\rho > 0$, then this singularity is not logarithmic.

It remains to prove that there are no other singularities over 0. Suppose that $\rho \to U_\rho$ is a singularity over 0 such that $U_\rho \cap T = \emptyset$ for some $\rho > 0$. In order to obtain a contradiction we note that it follows as in the proof of (3) and (4) that if $r_n \leq |z| \leq r_n'$, then

$$g(z) = (1 + \eta(z)) \left(\frac{z}{2^n}\right)^{2^n}$$

where $\eta(z) \to 0$ as $n \to \infty$. For large n we thus have $|\eta(z)| \leq \varepsilon^2 \leq \frac{1}{2}$. Differentiating we obtain

$$\frac{g'(z)}{g(z)} - \frac{2^n}{z} = \frac{\eta'(z)}{1+\eta(z)}.$$

For $(1+2\varepsilon)2^{n+1} \leq |z| \leq (1-3\varepsilon)2^{n+2}$ we thus have

$$\begin{aligned} \left|\frac{g'(z)}{g(z)} - \frac{2^n}{z}\right| &\leq 2|\eta'(z)| \\ &= 2\left|\frac{1}{2\pi i}\int_{|\zeta - z| = \varepsilon 2^{n+1}} \frac{\eta(\zeta)}{(\zeta - z)^2} d\zeta\right| \\ &\leq 2\frac{1}{\varepsilon 2^{n+1}} \max_{|\zeta - z| = \varepsilon 2^{n+1}} |\eta(z)| \\ &\leq \frac{\varepsilon}{2^n} \end{aligned}$$

and hence

$$\left|\frac{zg'(z)}{g(z)} - 2^n\right| \leq \frac{\varepsilon|z|}{2^n} \leq 4\varepsilon(1-3\varepsilon) < \frac{1}{2}.$$

We deduce that

$$\begin{aligned} \frac{d \arg g(re^{i\theta})}{d\theta} &= \operatorname{Im}\left(\frac{d \log g(re^{i\theta})}{d\theta}\right) \\ &= \operatorname{Re}\left(\frac{re^{i\theta} g'(re^{i\theta})}{g(re^{i\theta})}\right) \\ &\geq 2^n - \frac{1}{2} \\ &> 0 \end{aligned}$$

for $(1+2\varepsilon)2^{n+1} \leq r \leq (1-3\varepsilon)2^{n+2}$ and large n. We conclude that $\arg g(re^{i\theta})$ is an increasing function of θ, and it increases by $2^n 2\pi$ as θ increases by 2π. Choose n and r as above so large that the circle $\{z : |z| = r\}$ intersects U_ρ, that (3) and (4) hold and that $-2^{2^n} < \log \rho$. From the behavior of $\arg g(re^{i\theta})$ we deduce that the circle $\{z : |z| = r\}$ contains at most 2^n arcs where $\operatorname{Re} g(re^{i\theta}) < \log \rho$. On the other hand, for $j \in \{0, 1, \ldots, 2^n - 1\}$ the points $r \exp(\pi i/2^n + 2\pi i j/2^n)$ are contained in such an arc by (4), and each of them is contained in a different one by (3). Hence there are precisely 2^n such arcs and each one contains one of the points $r \exp(\pi i/2^n + 2\pi i j/2^n)$. Thus each such arc intersects some $B_{j,n}$ and hence T. In particular, $U_\rho \cap \{z : |z| = r\}$

intersects T, contradicting the assumption that $U_\rho \cap T = \emptyset$. This completes the proof that f has no logarithmic singularities over 0.

Remark. It is much easier to find meromorphic functions with a direct singularity which is not logarithmic. For example, $f(z) = 1/(z \sin z)$ has two direct singularities over 0, but none of them is logarithmic, since their neighborhoods are multiply connected.

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