ON QUASIANALYTICITY AND GENERAL DISTRIBUTIONS

(Lecture 1)

A. Beurling

Let γ be a given compact Jordan arc, rectifiable or not. Let $C(\gamma)$ denote the set of complex-valued functions continuous on γ. Let D be a simply connected domain having γ as a boundary arc. We introduce the sequences of real numbers

$$0 = a_o < a_1 < a_2 < \cdots < a_n < \cdots \to \infty$$

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots .$$

Let

$$\sigma = \sum_{n=1}^{\infty} \frac{a_n - a_{n-1}}{\lambda_n} \quad \text{(finite or infinite).}$$

__Theorem 1:__

Let $f(z) \in C(\gamma)$, and let $\{f_n(z)\}_1^\infty$ be a sequence of functions analytic in D, continuous on γ, and satisfying:

(i) $\quad |f_n(z)| \leq e^{\lambda_n} \quad$ in D

(ii) $\quad |f_n(z) - f(z)| \leq e^{-a_n} \quad$ on γ.

Then the following two conclusions hold:

(a) If $f(z) \equiv 0$ on a set $E_o \subset \gamma$ of positive harmonic measure[1], and if $\sigma = \infty$, then $f(z) \equiv 0$ on γ.

[1] The harmonic measure $\omega(z)$ of E_o does not vanish (identically) in D.

(b) If $\sigma < \infty$, then corresponding to each subarc $\gamma_0 \subset \gamma$ there exists a function f and a sequence $\{f_n\}$ satisfying the stated hypotheses, such that $f(z) \equiv 0$ on γ_0 while $f(z) \not\equiv 0$ on γ.

<u>Remark</u>: If the sequence $\{\lambda_n\}$ is bounded, then $\sigma = \infty$, and the conclusion (a) follows from the classical theorem of F. and M. Riesz concerning functions analytic and bounded inside the unit circle. (Theorem 1 is invariant under conformal mapping.)

A partial summation (Abel summation) gives

$$\sigma = \sum_{n=1}^{\infty} \frac{a_n - a_{n-1}}{\lambda_n} = \sum_{n=1}^{\infty} a_n \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} \right)$$

in the sense that both series are either convergent and equal, or divergent. Let

$$a_n^* = \min(a_n, \lambda_n)$$

$$\sigma^* = \sum_{n=1}^{\infty} a_n^* \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} \right)$$

Observe that $a_n^* < a_{n+1}^*$, $n = 1, 2, \ldots$.

<u>Lemma</u>: $\sigma = \infty \iff \sigma^* = \infty$.

<u>Proof</u>: That $\sigma^* = \infty$ implies $\sigma = \infty$ is obvious. To prove the converse, let

$$\delta = \limsup_{n \to \infty} a_n / \lambda_n$$

$$\delta^* = \limsup_{n \to \infty} a_n^* / \lambda_n .$$

Then $\delta^* = \min(\delta, 1)$. There are two cases:

1) $\delta > 0 \Rightarrow \delta^* > 0 \Rightarrow \sigma^* = \infty$ since the tail of the series

$$\sum_{n=N}^{\infty} a_n^*\left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right) = \frac{a_N^*}{\lambda_N} + \sum_{n=N+1}^{\infty} \frac{a_n^* - a_{n-1}^*}{\lambda_n} \geq \frac{a_N^*}{\lambda_N} .$$

2) $\delta = 0 \Rightarrow a_n = a_n^*$ for all n sufficiently large

$\Rightarrow \sigma^* = \infty$ if $\sigma = \infty$.

Proof of Theorem 1:

Part a): A conformal mapping takes D onto the strip $0 < y < 1$ $(z = x+iy)$, carrying γ onto the entire real axis and E_0 onto a set E of positive linear measure. By hypothesis $|f_n(z)| \leq e^{\lambda_n}$ on the strip, and $f(x) \equiv 0$ on E. Without restriction we shall assume $|f(x)| < 1$ throughout the real axis.

We construct the Laplace transform.

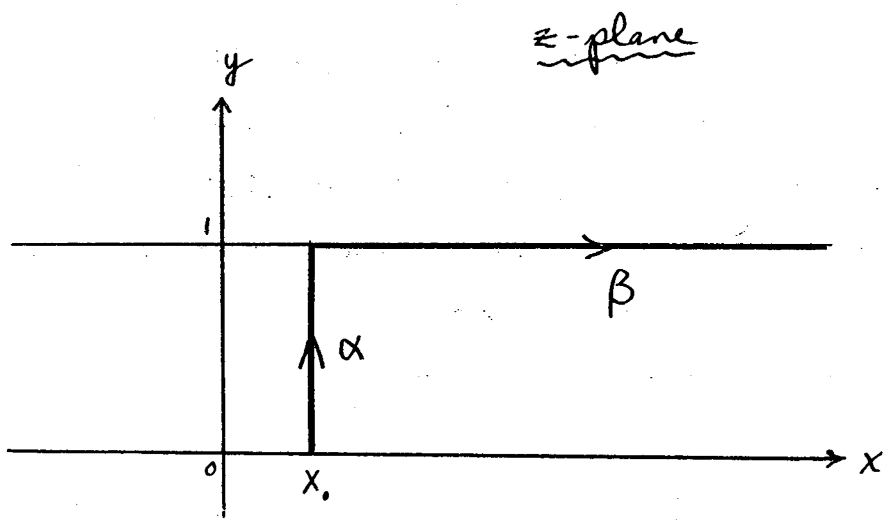
$$F(s) = \int_{x_0}^{\infty} f(x)e^{-xs}dx , \qquad s = \sigma + it ,$$

where $x_0 > 0$ is to be chosen later in a suitable way. $F(s)$ is analytic and bounded in $\mathrm{Re}\{s\} \geq 1$.

Our aim is to show $F(s) \equiv 0$. In the integral for $F(s)$, we shall replace $f(x)$ by the analytic function $f_n(z)$, committing an error. One finds

$$|F(1-it)| \leq e^{-a_n} + \left|\int_{x_0}^{\infty} f_n(x) e^{-x(1-it)} dx\right| .$$

For $t > 2\lambda_n$, we shall now estimate the integral on the right. To do so, we deform the contour of integration to two rectilinear parts α and β as shown:



Since $f_n(z)$ is bounded in the strip,

$$\int_{x_0}^{\infty} = \int_{\alpha} + \int_{\beta} .$$

(Apply Cauchy's theorem to the rectangle with vertices x_o, x_o+i, $R+i$, R; then let $R \to \infty$.)

We find

$$\left|\int_\beta\right| = \left|\int_{x_o}^{\infty} f_n(x+i)e^{-(x+i)(1-it)}dx\right| \leq e^{-\lambda_n} ,$$

since $t > 2\lambda_n$. In order to estimate the integral over α, we recall that $|f(x)| < 1$ and note that for $n \geq N$ sufficiently large

$$\log |f_n(z)e^{itz}| \leq \begin{cases} -a_n & \text{on } E \\ 0 & \text{elsewhere on real axis} \\ -\lambda_n & \text{on line } y = 1 . \end{cases}$$

It is easy to construct a function harmonic in the strip and having these boundary values. If $\omega(z)$ is the harmonic measure of E, such a function is

$$-a_n\, \omega(z) - \lambda_n\, y .$$

Therefore, by the harmonic majorant principle (that is, by the fact that log | | is subharmonic), one concludes

$$\log |f_n(z)e^{itz}| \leq -a_n\, \omega(z) - \lambda_n\, y , \qquad n \geq N.$$

Now choose x_o to be a point of density of E; that is, a point for which

$$\omega(x_o,y) \to 1 \quad \text{as} \quad y \to 0 .$$

By the maximum principle

$$\inf_{0<y<1} [\omega(x_o,y) + y] = \theta \,, \qquad 0<\theta<1,$$

where θ depends only on E and x_o. Thus

$$-a_n\,\omega(z) - \lambda_n y \le -\theta\, a_n^* \quad \text{on } \alpha\,,$$

so that

$$\left|\int_\alpha\right| \le \int_o^1 \left| f_n(x_o + iy)e^{it(x_o + iy)}\right| \le e^{-\theta a_n^*}\,, \qquad n \ge N\,.$$

For $n \ge N$ and $t > 2\lambda_n$ we have proved

$$|F(1-it)| \le e^{-a_n} + e^{-\lambda_n} + e^{-\theta a_n^*} \le 3\, e^{-\theta a_n^*}\,.$$

This gives

$$\int_{2\lambda_N}^{\infty} \log |F(1-it)| \frac{dt}{t^2} = \sum_{n=N}^{\infty} \int_{2\lambda_n}^{2\lambda_{n+1}} \log |F(1-it)| \frac{dt}{t^2}$$

$$\le -\frac{\theta}{2} \sum_{n=N}^{\infty} a_n^* \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right) + \text{const} = -\infty$$

by hypothesis. Thus $F(s) \equiv 0$, which implies $f(x) \equiv 0$ for $x \ge x_o$. The proof for $x < x_o$ is similar.

Part b): Again it is sufficient to prove the assertion for a particular geometric configuration. We choose D to be the strip $0<y<1$, γ to be the real axis, and γ_o to be the positive real axis.

Let $u(s)$ be harmonic in the right half-plane and have boundary values

$$u(it) \geq a_n + \log(1 + t^2) \quad \text{for} \quad \lambda_n \leq |t| \leq \lambda_{n+1} .$$

Such a function $u(s)$ exists because the hypothesis $\sigma < \infty$ ensures that the appropriate Poisson integral converges. Let $v(s)$ be a harmonic conjugate of u, and form

$$F(s) = e^{-u(s)-iv(s)} .$$

Define $f(x)$ as the inverse Laplace transform of $F(s)$:

$$f(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} F(s)e^{sx}\, ds .$$

Because

$$\int_{-\infty}^{\infty} |F(\sigma+it)|^2 dt < \text{const. for } \sigma > 0,$$

it follows from a theorem of Paley-Wiener that $f(x) = 0$ for $x > 0$. However, $f(x) \not\equiv 0$.

Now let the sequence $\{f_n(z)\}$ be defined by

$$f_n(z) = \frac{1}{2\pi i} \int_{-i\lambda_n}^{i\infty} F(s)e^{sz}\, ds .$$

Then for $z = x + iy$ $(0 < y < 1)$,

$$|f_n(z)| \leq \frac{1}{2\pi} \int_{-\lambda_n}^{\infty} \frac{e^{-yt}}{1 + t^2}\, dt \leq \frac{e^{y\lambda_n}}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{1 + t^2} \leq e^{\lambda_n} .$$

Furthermore, for $z = x$ real,

$$|f(x)-f_n(x)| \leq \frac{1}{2\pi} \int_{-\infty}^{-\lambda_n} |F(it)|dt \leq e^{-a_n} .$$

This completes the proof of Theorem 1.

For each $\lambda > 0$, we now define the class $S(\lambda,\gamma,D)$ of functions $f(z)$ analytic in D, continuous on γ, and satisfying the inequality

$$|f(z)| \leq e^{\lambda} \quad \text{in } D .$$

Each $S(\lambda,\gamma,D)$ is a subclass of $C(\gamma)$ consisting of what we shall call semi-analytic functions.

Given $f(z) \in C(\gamma)$, let

$$\inf_{g \in S(\lambda,\gamma,D)} \|f-g\| = e^{-A(\lambda,f,\gamma,D)} ,$$

where $\| \ \|$ is the uniform norm on γ. The non-negative number $A(\lambda,f,\gamma,D)$ will be called the approximation index of f with respect to the class $S(\lambda,\gamma,D)$. Certainly $A \to \infty$ as $\lambda \to \infty$, since each $f \in C(\gamma)$ may be uniformly approximated by a polynomial. Further properties are:

1) $A(\lambda,f,\gamma,D)$ increases monotonically as λ increases.

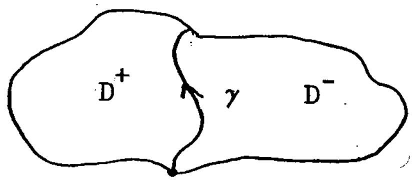
2) $\gamma_o \subset \gamma, \ D_o \subset D \Rightarrow A(\lambda,f,\gamma_o,D_o) \geq A(\lambda,f,\gamma,D)$.

3) $A(\lambda,f+g,\gamma,D) \geq \min\{A(\lambda,f,\gamma,D),\ A(\lambda,g,\gamma,D)\} - \log 2$.

4) $A(\lambda,f\text{-}g,\gamma,D) \geq \min\{A(\frac{\lambda}{2},f,\gamma,D),\ A(\frac{\lambda}{2},g,\gamma,D)\} - \text{const}$.

5) Conformal invariance.

The Jordan arc γ has a positive direction. It is necessary to distinguish domains D^+ and D^- which lie, respectively, on the positive and the negative sides of γ, as shown:



Let $\alpha(\lambda)$ $(0 < \lambda < \infty)$ be a given function which increases monotonically to ∞. We introduce the classes $C^{(\pm)}(\gamma, a(\lambda))$ of functions $f \in C(\gamma)$ for which there exists a D^+ (or D^-) and positive constants k_1, k_2, k_3 such that

$$A(\lambda, f, \gamma, D^{(\pm)}) \geq k_1 \, a(k_2\lambda) - k_3 .$$

Such a class C^+ or C^- is an algebra.

Theorem 1 described a class of functions $f \in C(\gamma)$ which can vanish throughout a set $\gamma_o \subset \gamma$ of positive harmonic measure only if identically zero on γ. Any subclass of $C(\gamma)$ with this property will be called <u>quasi-analytic</u> (QA) with respect to harmonic measure.

<u>Theorem 2:</u>

$C^{(\pm)}(\gamma, a(\lambda))$ is QA with respect to harmonic measure if and only if

$$\int_1^\infty \frac{da(\lambda)}{\lambda} = \infty .$$

ON QUASIANALYTICITY AND GENERAL DISTRIBUTIONS

A. Beurling   Lecture 2

(Notes Prepared by P. L. Duren)

Applications to Harmonic Analysis

1) Let $\mu(t)$ be a bounded measure on $-\infty < t < \infty$, and consider its Fourier-Stieltjes transform

$$f(x) = \int_{-\infty}^{\infty} e^{-itx}\, d\mu(t) \ .$$

Define

$$s(\lambda) = \begin{cases} \int_{\lambda}^{\infty} |d\mu(t)| \ , & \lambda > 0 \\ \int_{-\infty}^{\lambda} |d\mu(t)| \ , & \lambda < 0 \ . \end{cases}$$

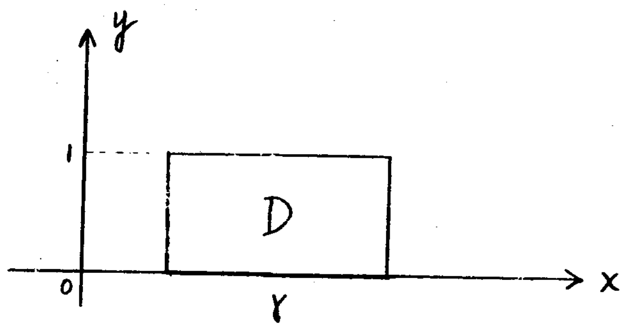
Corollary 2.1: Assume

(i) $\int_{1}^{\infty} \frac{\log s(\lambda) d\lambda}{\lambda^2} = -\infty \quad (\text{or } \int_{-\infty}^{-1} = -\infty)$

(ii) $f(x) = 0$ on a set E of positive measure.

Then $f(x) \equiv 0$ and $\mu \equiv 0$.

Proof: Let γ be any finite interval such that $\gamma \cap E$ is of positive measure. Construct the domain D as follows:



Assume $\int_1^\infty \frac{\log s(\lambda)d\lambda}{\lambda^2} = -\infty$. (The proof in the other case is similar.)
For $\lambda > 0$, define

$$f_\lambda(z) = \int_{-\infty}^{\lambda} e^{-itz}\, d\mu(t) , \qquad z \in D.$$

Then, for $0 \leq y < 1$,

$$|f_\lambda(x + iy)| \leq e^\lambda \|\mu\| \leq e^\lambda ,$$

assuming, as we may, $\|\mu\| \leq 1$. Also

$$|f(x)-f_\lambda(x)| \leq s(\lambda) ,$$

so by the definition of the approximation index

$$e^{-A(\lambda)} \leq s(\lambda) , \qquad A(\lambda) = A(\lambda,f,\gamma,D) .$$

Thus $A(\lambda) \geq -\log s(\lambda)$, and

$$\int_1^\infty \frac{A(\lambda)d\lambda}{\lambda^2} = \infty .$$

Theorem 2 may now be applied: $f(x) \equiv 0$ on γ, hence on the entire real axis. This implies $\mu \equiv 0$.

2) The preceding result may be generalized to Euclidean space R^p of p dimensions. Let $\mu(\xi)$ be a positive bounded Borel measure on R^p, and consider the Hilbert space L^2_μ with square norm

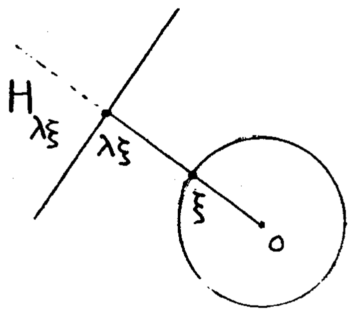
$$\|f\|^2 = \int_{R^p} |f(\xi)|^2 \, d\mu(\xi) .$$

Let $\widehat{L^2_\mu}$ denote the set of Fourier transforms

$$F(x) = \int_{R^p} e^{-ix\cdot\xi} f(\xi) d\mu(\xi) , \qquad f \in L^2_\mu .$$

We shall give a condition under which $\widehat{L^2_\mu}$ is QA with respect to positive measure.

For $\xi \in R^p$ on the unit sphere $(|\xi| = 1)$ and any number $\lambda \geq 1$, let $H_{\lambda\xi}$ denote the half-space consisting of all points $x \in R^p$ whose projection $x\cdot\xi \geq \lambda$. (See diagram below.)



Let

$$s(\lambda,\xi) = \int_{H_{\lambda\xi}} d\mu(\eta) \ ,$$

and let

$$m(\xi) = \int_1^\infty \log s(\lambda,\xi) \frac{d\lambda}{\lambda^2} \ .$$

Since μ is a bounded measure, $-\infty \leq m(\xi) < \text{const.}$

<u>Corollary 2.2</u>: Let $\xi_1, \xi_2, \ldots, \xi_p \in R^p$ be a set of linearly independent unit vectors, and suppose $m(\xi_k) = -\infty$, $k = 1, 2, \ldots, p$. Then $\widehat{L^2_\mu}$ is QA with respect to positive measure; that is, if $F(x) \in \widehat{L^2_\mu}$ vanishes on a set $E \subset R^p$ of positive measure, then $F(x) \equiv 0$ in R^p.

<u>Proof</u>: For $\xi = \xi_k$ any one of the given vectors, let

$$L_{x_o,\xi} = \{x | x = x_o + t\xi \ , \ -\infty < t < \infty\}$$

be any line parallel to ξ, and consider the restriction $\varphi(t) = F(x_o + t\xi)$ of $F(x)$ to this line:

$$\begin{aligned}\varphi(t) &= \int_{R^p} e^{-i(x_o+t\xi)\cdot\eta} f(\eta) d\mu(\eta) \\ &= \int_{R^p} e^{-it(\xi\cdot\eta)} [e^{-ix_o\cdot\eta} f(\eta)] d\mu(\eta) \\ &= \int_{-\infty}^{\infty} e^{-it\rho} \, d\nu(\rho) \ ,\end{aligned}$$

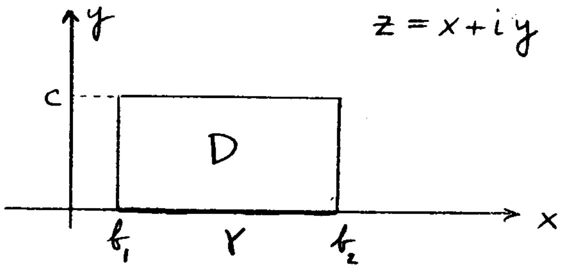
where the new measure $\nu(\rho)$ is obtained by projecting the measure $e^{-ix_o\cdot\eta} f(\eta)d\mu(\eta)$ on the line. By construction,

$$\int_{\lambda}^{\infty} |d\nu(\xi)| \leq \int_{H_{\lambda\xi}} |f(\eta)|d\mu(\eta) \leq \|f\| \sqrt{s(\lambda,\xi)} .$$

Thus the assumption $m(\xi) = -\infty$ implies that hypothesis (i) of Corollary 2.1 is fulfilled. We therefore conclude that $F(x) \equiv 0$ on the line $L_{x_o,\xi}$ if the set $E \cap L_{x_o,\xi}$ has positive linear measure. Here $\xi = \xi_k$ is any of the given linearly independent vectors, so the proof of Corollary 2.2 is completed by the following:

Geometrical Lemma: Let $\{\xi_k\}_1^p$ be p linearly independent unit vectors in R^p, and let $E \subset R^p$ be a closed set of positive measure. If the entire line $L_{x_o,\xi_k} \subset E$ whenever the linear measure of $E \cap L_{x_o,\xi_k}$ is positive, then $E = R^p$.

Theorems I and II dealt with approximation in the uniform norm. Our next object is to prove an analogous theorem for the L^p norm, $1 \leq p < \infty$. We have now to assume that γ is a compact interval; say $\gamma = [b_1,b_2]$ on the real axis. Let γ form one side of a rectangular domain D, as shown below. (The theorem which follows is equally valid if D lies in the lower half-plane.)



For each $\lambda > 0$, let $S^p(\lambda,\gamma,D)$ denote the class of functions $g(z)$ analytic in D and satisfying

$$\left[\int_{b_1}^{b_2} |g(x+iy)|^p \, dx\right]^{1/p} \leq e^{\lambda}, \qquad 0 < y < c .$$

Each such function must necessarily have a vertical limit

$$g(x) = \lim_{y \to 0} g(x+iy) \qquad \text{a.e. on } \gamma ,$$

and $g(x) \in L^p(\gamma)$.

For each $f(x) \in L^p(\gamma)$ we define the approximation index $A_p(\lambda) = A_p(\lambda,f,\gamma,D)$ by

$$\inf_{g \in S^p(\lambda,\gamma,D)} \|f-g\|_p = e^{-A_p(\lambda)} .$$

Theorem III:

Let $f(x) \in L^p(\gamma)$, and suppose

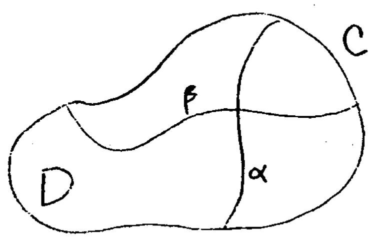
(i) $f(x) = 0$ on a set $E \subset \gamma$ of positive measure

(ii) $\displaystyle\int_1^{\infty} \frac{A_p(\lambda)\, d\lambda}{\lambda^2} = +\infty$.

Then $f(x) = 0$ a.e. on γ.

Proof: It is sufficient to prove the theorem for $p = 1$, because γ is compact and therefore the norm in L^1 is a minorant of the norm in L^p, $p > 1$. The major task in the proof is to show that $f(x)$ must vanish a.e. on an interval. Once this is known, the theorem becomes a simple consequence of Theorem I.

Preliminary to the proof we mention a few lemmas. Let D be a domain bounded by a rectifiable Jordan curve C. A line of symmetry in D is a curve which is the counter-image, under some conformal mapping of D onto the unit disk, of a circular arc orthogonal to the unit circle. Let α and β be orthogonal lines of symmetry in D.



For any function $f(z)$ analytic in D, let

$$A = \int_{\alpha} |f(z)||dz| \ ; \quad B = \int_{\beta} |f(z)||dz| \ ; \quad P = \int_{C} |f(z)||dz| \ .$$

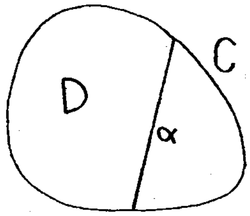
Lemma 1: $A^2 + B^2 \leq P^2/4$. (Proof in Lunds Univ. Mat. Sem., Suppl., 1952.)

Corollary: Let $f(z)$ be analytic in the rectangle $-a < x < a$, $0 < y < b$, and suppose

$$\int_{-a}^{a} |f(x + iy)| dx \leq M \quad \text{for} \quad 0 < y < b \ .$$

Then $\int_0^b |f(x+iy)|dy \leq M[2 + (b/(a-|x|))]$.

Lemma 2: Let D be a bounded convex domain, and let α be any chord. Then for $f(z)$ analytic in D



$$\int_\alpha |f(z)|\ |dz| \leq \int_C |f(z)|\ |dz| .$$

It is assumed in Theorem III (for $p = 1$) that

$$\int_1^\infty \frac{A(\lambda)d\lambda}{\lambda^2} = \infty , \quad \text{where } A(\lambda) = A_1(\lambda,f,\gamma,D) .$$

We may therefore select a sequence $\{\lambda_n\}_1^\infty$ such that

(1) $\lambda_{n+1} - \lambda_n \geq 1$

(2) $a_n = A(\lambda_n) - 1 > \sqrt{\lambda_n}$.

The divergence of the integral and the monotonic character of $A(\lambda)$ imply that the choice may be made such that

$$\sum_{n=1}^\infty a_n\left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right) = \infty .$$

Given $f \in L^1(\gamma)$, we may select a sequence of functions $f_n(z) \in S^1(\lambda_n,\gamma,D)$ such that

$$\|f-f_n\| \leq e^{-a_n} .$$

We are assuming $f = 0$ on a set E, $|E| > 0$ ($|E|$ denotes measure). Therefore

$$\int_E |f_n(x)|\,dx \le e^{-a_n} ,$$

so if

$$e_n = \{x \mid x \in E,\ |f_n(x)| \ge e^{-\frac{1}{2} a_n}\} ,$$

we have

$$|e_n| \le e^{-\frac{1}{2} a_n} .$$

For n_o sufficiently large,

$$\left|\bigcup_{n=n_o}^{\infty} e_n\right| \le \sum_{n=n_o}^{\infty} e^{-\frac{1}{2} a_n} < \frac{1}{2} |E| ,$$

convergence of the series being assured by the construction of $\{a_n\}$. Let

$$E_o = E - \bigcup_{n=n_o}^{\infty} e_n ; \qquad |E_o| > 0 .$$

Then

$$|f_n(x)| \le e^{-\frac{1}{2} a_n} \quad \text{for } x \in E_o,\ n \ge n_o .$$

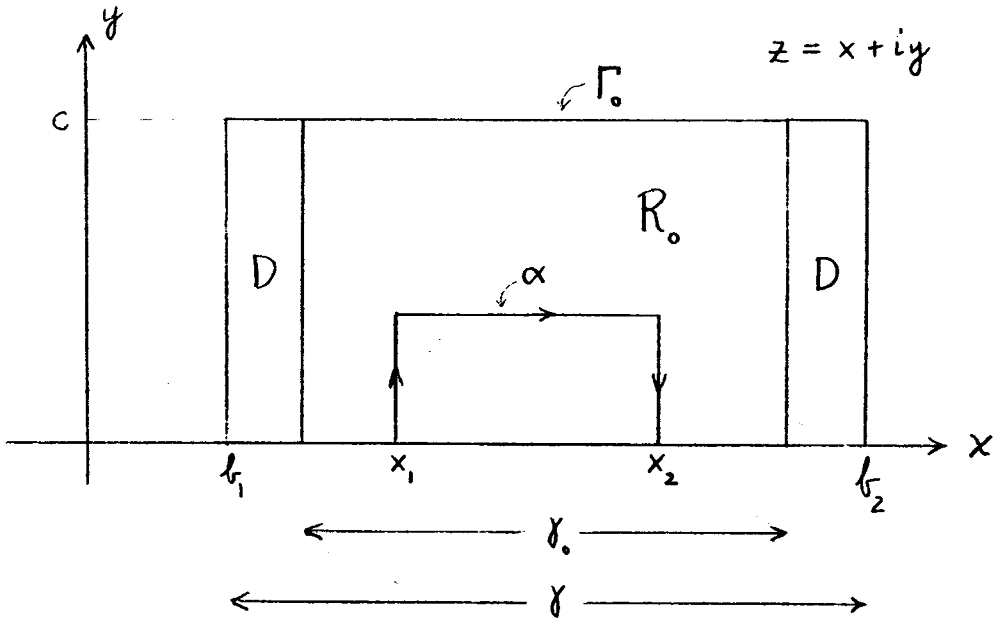
Now let $\gamma_o \subset \gamma$ be a subinterval concentric with γ and so large that

$$E_1 = E_o \cap \gamma_o$$

has positive measure. Let $R_o \subset D$ be the rectangle with base γ_o (see diagram), and let ∂R_o denote the boundary of R_o. Finally, let $x_1, x_2 \in E_1$ be density points; in particular,

$$\lim_{y \to 0} \omega(x_1,y) = \lim_{y \to 0} \omega(x_2,y) = 1 ,$$

where $\omega(z)$ is the harmonic measure of E_1 with respect to R_o.



Consider now the Fourier transform

$$F(t) = \int_{x_1}^{x_2} f(x)e^{itx}\, dx .$$

We replace $f(x)$ by the analytic function $f_n(z)$ and integrate over a path α which forms with $[x_1,x_2]$ a rectangle as shown above. We find

$$|F(t)| \leq e^{-a_n} + \left|\int_\alpha f_n(z)e^{itz}\, dz\right| .$$

The main difficulty of the proof is to estimate this integral over α. We shall eventually factor the integrand into the product of a bounded function and a function bounded in mean.

Let $G(z,\zeta)$ be the Green's function of R_o, and let

$$K(z,\zeta) = \frac{1}{2\pi} \frac{\partial}{\partial n_\zeta} G(z,\zeta) \qquad \text{(inner normal derivative)} .$$

By virtue of the harmonic majorant principle,

$$\log |f_n(z)| \leq \int_{\partial R_o} K(z,\zeta) \log |f_n(\zeta)| \, |d\zeta|$$

$$\leq \int_{\Gamma_o} + \int_{E_1'} + \int_{E_1} ,$$

where $\Gamma_o = \partial R_o - \gamma_o$ and $E_1' = \gamma_o - E_1$. In an obvious way we may construct functions $h_n(z)$ and $g_n(z)$ analytic in R_o such that

$$\log |h_n(z)| = \int_{\Gamma_o} \; ; \quad \log |g_n(z)| = \int_{E_1'} .$$

Because $|f_n| \leq e^{-\frac{1}{2} a_n}$ on E_1, we have

$$\int_{E_1} \leq \frac{-a_n}{2} \omega(z) , \qquad n \geq n_o .$$

Putting this together,

$$(*) \qquad \log |f_n(z)| \leq \log |h_n(z)| + \log |g_n(z)| - \frac{a_n}{2} \omega(z) , \qquad n \geq n_o .$$

For fixed $\zeta \in \Gamma_o$, $K(z,\zeta)$ is harmonic for $z \in R_o$ and vanishes on γ_o. From this we conclude that

$$\frac{K(z,\zeta)}{y} \leq C_1 < \infty \quad \text{if} \quad \zeta \in \Gamma_o , \quad z \in \alpha .$$

For $z \in \alpha$, we therefore have

$$\log |h_n(z)| \leq C_1 y \int_{\Gamma_o} \log |f_n(\zeta)| \; |d\zeta|$$

$$\leq |\Gamma_o| \; C_1 y \log \left\{ \frac{1}{|\Gamma_o|} \int_{\Gamma_o} |f_n(\zeta)| \; |d\zeta| \right\} ,$$

where the continuous form of the geometric-arithmetic mean inequality has been used. But by the corollary to Lemma 1,

$$\int_{\Gamma_o} |f_n(\zeta)| \; |d\zeta| \leq C_2 \, e^{\lambda_n} ,$$

so

$$\log |h_n(z)| \leq C_3 \, y \, \lambda_n , \qquad z \in \alpha, \quad n \geq n_o .$$

On the other hand,

$$|g_n| = \begin{cases} 1 & \text{on } \Gamma_o \\ 1 & \text{on } E_1 \\ |f_n| & \text{on } E_1' \end{cases} ,$$

so

$$\int_{\partial R_o} |g_n(z)|\ |dz| \leq C_4 .$$

This implies, by Lemma 2,

$$\int_{\alpha} |g_n(z)|\ |dz| \leq 3\ C_4 .$$

Exponentiation of (*) gives for real t

$$|f_n e^{itz}| \leq |h_n e^{-\frac{1}{2} a_n \omega} e^{-ty}|\ |g_n| , \qquad z \in R_o ,\quad n \geq n_o$$

$$\leq |g_n| \exp \{C_3 y \lambda_n - \frac{1}{2} a_n \omega - ty\} ,$$

$$\leq |g_n| \exp\{-C_3 y \lambda_n - \frac{1}{2} a_n \omega\} , \qquad t \geq 2\ C_3\ \lambda_n .$$

But because x_1 and x_2 are points of density,

$$\inf_{z \in \alpha} [C_3 y + \frac{1}{2} \omega] = \theta > 0 .$$

Consequently,

$$|\int_{\alpha} f_n(z) e^{itz} dz| \leq e^{-\theta a_n^*} \int_{\alpha} |g_n(z)|\ |dz| , \qquad \begin{array}{l} n \geq n_o , \\ t \geq 2\ C_3\ \lambda_n , \end{array}$$

where $a_n^* = \min\{a_n, \lambda_n\}$. Hence for large n and t

$$|F(t)| \leq e^{-a_n} + (3C_4) e^{-\theta a_n^*} \leq e^{-\theta' a_n^*} \qquad (\theta' > 0) ,$$

so the divergence of the sum (recall the lemma of Lecture 1)

$$\sum_{n=1}^{\infty} a_n^*\left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right)$$

implies

$$\int_1^{\infty} \frac{\log |F(t)|\, dt}{t^2} = -\infty .$$

But $F(t)$ is an entire function of exponential type, so $F(t) \equiv 0$. We therefore conclude

$$f(x) = 0 \quad \text{a.e. on} \quad x_1 \leq x \leq x_2 .$$

With the knowledge that f vanishes a.e. on an interval, the theorem may now be deduced from Theorem I. Let the continuous function $\varphi_\epsilon(x) \geq 0$ have support in $(-\epsilon,\epsilon)$, and suppose $\int \varphi_\epsilon(x)dx = 1$. Let $\gamma_\epsilon = [b_1 + \epsilon,\ b_2 - \epsilon]$, and consider the convolution

$$f_\epsilon(x) = f * \varphi_\epsilon = \int_{-\infty}^{\infty} f(t)\varphi_\epsilon(x-t)dt , \qquad x \in \gamma_\epsilon .$$

Then $f_\epsilon(x)$ is continuous on γ_ϵ and vanishes on $[x_1 + \epsilon,\ x_2 - \epsilon]$. The functions

$$f_{n,\epsilon}(z) = f_n * \varphi_\epsilon = \int_{-\infty}^{\infty} f(z-t)\varphi_\epsilon(t)dt$$

are analytic in the rectangle $\{b_1 + \epsilon < x < b_2 - \epsilon,\ 0 < y < C\}$, and for $x \in \gamma_\epsilon$

$$|f_{n,\epsilon}(x)| \leq (\text{const})e^{\lambda_n} \; ; \quad |f_\epsilon(x)-f_{n,\epsilon}(x)| \leq (\text{const})e^{-a_n} .$$

The constants may depend on ϵ, but not on n. By Theorem I,

$$f_\epsilon(x) \equiv 0 \, , \quad x \in \gamma_\epsilon \, , \quad \epsilon > 0 \, .$$

But $\lim_{\epsilon \to 0} f_\epsilon(x) = f(x)$ a.e. on γ, so Theorem III is proved.

Application (p = 2): It is well known that each function

$$f(z) = \sum_{n=o}^{\infty} c_n z^n \in H^2 \qquad (|z| < 1)$$

has for almost every θ a radial limit

$$f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$

which belongs to $L^2(0,2\pi)$ and does not vanish on a set of positive measure unless $f(z) \equiv 0$. By means of Theorem III, we shall now prove that a larger subclass of $L^2(0,2\pi)$, boundary values of analytic functions or not, has this quasi-analytic property.

Let $f(x) \in L^2(0,2\pi)$ and let

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx} \text{ be its formal Fourier series.}$$

Since $\sum |c_n|^2 < \infty$, we may define

$$s_n = \begin{cases} \left[\sum_{\nu > n} |c_\nu|^2 \right]^{1/2}, & n > 0 \\ \left[\sum_{\nu < n} |c_\nu|^2 \right]^{1/2}, & n < 0 . \end{cases}$$

<u>Corollary 3.1</u>: Let $f(x) \in L^2(0,2\pi)$, and suppose

(i) $f(x) = 0$ on a set $E \subset [0,2\pi]$ of positive measure,

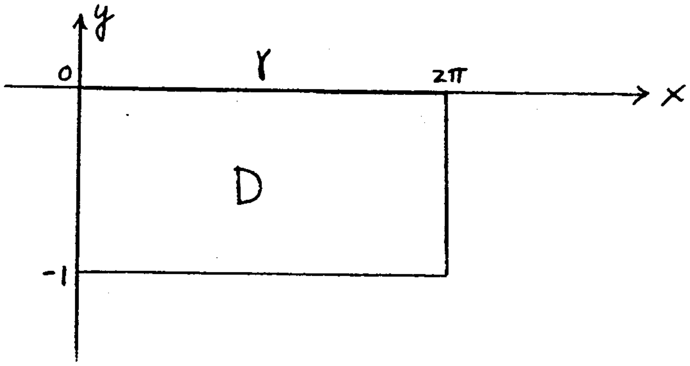
(ii) $\sum_{n=1}^{\infty} \frac{\log s_n}{n^2} = -\infty \quad (\text{or} \sum_{-\infty}^{-1} = -\infty)$.

Then $f(x) \equiv 0$ a.e. on $[0,2\pi]$.

<u>Proof</u>: Assume $\sum_{n=1}^{\infty} = -\infty$, the other case being similar. Define the approximating functions

$$f_n(x+iy) = \sum_{-\infty}^{\infty} c_\nu \, e^{i\nu(x+iy)}, \qquad -1 < y < 0 .$$

These functions $f_n(z)$ are analytic in D, as shown:



For $-1 < y < 0$,

$$\int_0^{2\pi} |f_n(x+iy)|^2 dx = \sum_{-\infty}^{n} |c_\nu|^2 e^{-2\nu y} \leq \|f\|^2 e^{2n} ,$$

so

$$\|f_n(x+iy)\| \leq e^n .$$

(We assume $\|f\| < 1$.) Furthermore,

$$\|f(x)-f_n(x)\| = \left[\sum_{n+1}^{\infty} |c_\nu|^2 \right]^{1/2} = S_n ,$$

so

$$e^{-A(n)} \leq S_n , \quad \text{where} \quad A(n) = A_2(n,f,\gamma,D).$$

Thus $A(n) \geq -\log S_n$, so by hypothesis (ii)

$$\sum_{n=1}^{\infty} \frac{A(n)}{n^2} = +\infty ,$$

Since $A(\lambda)$ is an increasing function, the divergence of this sum is equivalent to the divergence of the integral

$$\int_1^{\infty} \frac{A(\lambda)d\lambda}{\lambda^2} = +\infty ,$$

so Theorem III may be invoked to conclude $f(x) = 0$ a.e. in $0 \leq x \leq 2\pi$.

ON QUASIANALYTICITY AND GENERAL DISTRIBUTIONS

A. Beurling Lecture 3

(Notes Prepared by P. L. Duren)

The following theorem is classical:

Theorem A: Let $f(z)$ be an entire function of exponential type, and suppose

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(x)|dx}{1 + x^2} < \infty .$$

Then

$$\int_{-\infty}^{\infty} \frac{\log^- |f(x)|dx}{1 + x^2} > -\infty$$

unless $f \equiv 0$. A simple consequence is

Theorem B: Let $\mu(x)$ be a bounded measure on $-\infty < x < \infty$, and let

$$\hat{\mu}(t) = \int_{-\infty}^{\infty} e^{-itx} \, d\mu(x) \qquad (-\infty < t < \infty)$$

be its Fourier-Stieltjes transform. If $\hat{\mu}(t)$ has compact support, then μ is absolutely continuous, and $\frac{d\mu}{dx} = f(x)$ is an entire function of exponential type satisfying

$$\int_{-\infty}^{\infty} \log^- \left|\frac{d\mu}{dx}\right| \frac{dx}{1 + x^2} > -\infty$$

unless $\mu \equiv 0$.

Our Corollary 2.1 is similar to Theorem B, but the hypothesis that $\hat{\mu}$ have compact support is replaced by the much weaker assumption that it vanish on a set of positive measure. In Theorem IV we shall relax also the hypothesis that $\mu(x)$ be a bounded measure.

Given a measure $\mu(\xi)$, bounded or not, let

$$e^{\sigma(x)} = e^{\sigma(x,\mu)} = \int_{-\infty}^{\infty} e^{-|x-\xi|} \, |d\mu(\xi)| \, .$$

The integral may diverge, but if it converges at one point x_0, then $\sigma(x) < \infty$ for all x. Indeed, differentiation under the integral sign gives the Lipschitz condition

$$|\sigma(x) - \sigma(x_0)| \leq |x - x_0| \, .$$

<u>Theorem IV</u>: Let $\mu(x)$ be an arbitrary measure on $-\infty < x < \infty$, and assume

(i) $$\int_{-\infty}^{\infty} \frac{\sigma^+(x)}{1 + x^2} \, dx < \infty$$

(ii) $\hat{\mu} = 0$ on some interval (*).

Then

(iii) $$\int_{-\infty}^{\infty} \frac{\sigma^-(x)dx}{1 + x^2} > -\infty$$

unless $\mu \equiv 0$.

(*) For unbounded μ the integral defining $\hat{\mu}$ may diverge, but a certain summation process may still be applied to give meaning to (ii). This will be made precise in part 2 of the proof.

Proof:

1) We assume first that μ is bounded, and later reduce the unbounded case to the bounded case.

If μ is bounded, it is harmless to assume $\|\mu\| < 1$. We further suppose without loss of generality that

$$\hat{\mu}(t) = 0 \quad \text{on} \quad -1 \leq t \leq 1 .$$

If, in fact, $\mu(x)$ is replaced by $\mu_1(x) = \mu(ax + b)$, we may choose a, b such that $\hat{\mu}_1 = 0$ on $[-1,1]$, while the convergence or divergence of the integrals involving σ will not be affected. The function

$$F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\mu(\xi)}{\xi - z} \qquad (z = x + iy)$$

is analytic in $y > 0$ and in $y < 0$. We shall prove that if (iii) diverges and $\hat{\mu} = 0$ on $[-1,1]$, then $F \equiv 0$ and $\mu \equiv 0$ follows. Another expression for $F(z)$ is

$$F(z) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{0}^{\infty} \hat{\mu}(t)e^{itz}\, dt , & y > 0 \\ -\dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{0} \hat{\mu}(t)e^{itz}\, dt , & y < 0 . \end{cases}$$

Now let $k(t)$ be any smooth function such that

$$k(t) = \begin{cases} 0 , & t \leq -1 \\ 1 , & t \geq +1 . \end{cases}$$

Recalling that $\hat{\mu}(t) = 0$ in $[-1,1]$, we may write

$$F(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} k(t)\hat{\mu}(t)e^{itz}\, dt , \qquad y > 0 .$$

We introduce the kernel

$$K(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} k(t)e^{itz}\, dt , \qquad y > 0 ,$$

in terms of which

$$F(z) = \int_{-\infty}^{\infty} K(z-\xi)d\mu(\xi) , \qquad y > 0 .$$

A particular sequence of functions $k_n(t)$ will now be chosen. Let

$$Q_n(t) = \frac{1}{2} \frac{(2n+1)!!}{(2n)!!} (1-t^2)^n ,$$

where $(2n)!! = 2\cdot4\cdot6\cdots(2n)$ and $(2n+1)!! = 1\cdot3\cdot5\cdots(2n+1)$. By a standard formula of elementary calculus,

$$\int_{-1}^{1} Q_n(t)dt = 1 .$$

Now let $k_n(t) = 0$ for $t \leq -1$, $= 1$ for $t \geq 1$, and

$$k_n(t) = \int_{-1}^{t} Q_n(t)dt , \qquad -1 < t < 1.$$

The $n^{\underline{th}}$ Legendre polynomial $P_n(t)$ of the interval $[-1,1]$ may be expressed by Rodrigues' formula

$$P_n(t) = \frac{(-1)^n}{2^n\, n!} \frac{d^n}{dt^n} (1-t^2)^n .$$

Also

$$\int_{-1}^{1} [P_n(t)]^2 dt = \frac{2}{2n+1} .$$

For each fixed z we shall now estimate the kernels

$$K_n(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} k_n(t) e^{itz}\, dt \qquad (y > 0)$$

for $z = x+i$. Trivially

$$|K_n(x+i)| < 1 ,$$

but for large $|x|$ we can do better. Integration by parts $(n+1)$ times yields

$$|K_n(x+i)| \leq \frac{1}{2\pi} \frac{1}{|x+i|^{n+1}} \int_{-1}^{1} \left|\frac{d^n}{dt^n} Q_n\right| e^{-t}\, dt$$

$$\leq \frac{C}{|x|^{n+1}}\, 2^n\, n! \qquad (C = \text{some const.})$$

Stirling's formula therefore gives

$$|K_n(x+i)| \leq C\, e^{-n} , \qquad |x| \geq 2n .$$

For convenience, let $\ell(x) = -\sigma(x)$; then $\ell(x) \geq 0$ because $\|\mu\| < 1$. We note that

$$e^{-\ell(x)} = \int_{-\infty}^{\infty} e^{-|x-\xi|}\,|d\mu(\xi)| \geq e^{-\frac{\ell(x)}{2}} \int |d\mu(\xi)| \;, \quad |x-\xi| < \frac{\ell(x)}{2}$$

so

$$\int_{|x-\xi| < \frac{\ell(x)}{2}} |d\mu(\xi)| \leq e^{-\frac{\ell(x)}{2}} \,.$$

For each n we have

$$F(x+i) = \int_{-\infty}^{\infty} K_n(x+i-\xi)\, d\mu(\xi) = \int_{|x-\xi| < \frac{\ell(x)}{2}} + \int_{|x-\xi| \geq \frac{\ell(x)}{2}}$$

$$|F(x+i)| \leq e^{-\frac{\ell(x)}{2}} + \left| \int_{|x-\xi| \geq \frac{\ell(x)}{2}} \right| .$$

If we now choose $n = [\frac{\ell(x)}{4}]$, then $\ell/2 \geq 2n$, so

$$|F(x+i)| \leq e^{-\frac{\ell(x)}{2}} + C\, e^{-[\frac{\ell(x)}{4}]} \leq C\, e^{-\frac{\ell(x)}{4}} \,.$$

$$|F(x+i)| \leq C\, e^{\frac{\sigma^{-}(x)}{4}} \qquad [\sigma^{-}(x) = \sigma(x)].$$

Hence the assumption (iii) $= -\infty$ implies

$$\int_{-\infty}^{\infty} \log |F(x+i)| \frac{dx}{1+x^2} = -\infty \;,$$

and since $F(x+iy)$ is bounded in the half-plane $y \geq 1$, it vanishes identically there. Thus $F(z) \equiv 0$ in $y > 0$. Similarly, $F(z) \equiv 0$ in the lower half-plane. This proves the theorem for bounded measures.

2) The assumption that μ is bounded will now be replaced by

$$\text{(i)} \qquad \int_{-\infty}^{\infty} \frac{\sigma^+(x)}{1+x^2}\, dx < \infty \, .$$

The Lipschitz condition $|\sigma^+(x)-\sigma^+(x_o)| \leq |x-x_o|$ shows

$$\int_{x_o}^{x_o+\sigma^+(x_o)} \sigma^+(x)dx \geq \tfrac{1}{2}[\sigma^+(x_o)]^2 \, .$$

Hence the convergence of (i) implies $[\sigma^+(x_o)/x_o]^2 \to 0$ as $x_o \to \infty$; that is,

$$\sigma^+(x) = o(x) \qquad (x \to \infty) \, .$$

Thus

$$\int_{x-1}^{x+1} |d\mu(x)| \leq e^{\sigma^+(x)+1} = e^{o(x)} \, ,$$

and, in particular,

$$\hat{\mu}_\epsilon(t) = \int_{-\infty}^{\infty} e^{-\epsilon|x|-itx}\, d\mu(x) \qquad (\epsilon > 0)$$

is well defined for real t. We shall say, by definition, that $\hat{\mu}$ vanishes on an open interval ω if $\hat{\mu}_\epsilon \to 0$ as $\epsilon \to 0$ uniformly on each closed subinterval of ω.

Given a finite open interval ω, let Γ_{ω,σ^+} denote the set of continuous functions $f(x)$ satisfying

$$|f(x)| \le \frac{e^{-\sigma^+(x)}}{1+x^2}\ , \qquad \hat{f} = 0 \ \text{ off } \ \omega\ .$$

According to a result in Malliavin's lectures, we have:

<u>Lemma</u>: If $\sigma^+(x) \in \text{Lip } 1$ satisfies (i), then for every ω the set Γ_{ω,σ^+} contains at least one $f \not\equiv 0$.

Let $\hat{\mu} = 0$ on $\omega = (a,b)$, and let $\omega_1 \subset [a+\delta_o, b-\delta_o]$ $(\delta_o > 0)$ be a closed subinterval. For $f \in \Gamma_{\omega_1} = \Gamma_{\omega_1,\sigma^+}$ we find

$$\int_{n-1}^{n+1} |f|\ |d\mu| \le \frac{e^{-\sigma^+(n)}}{1+n^2} \int_{n-1}^{n+1} |d\mu| \le \frac{e}{1+n^2}\ ,$$

so the measure $\overline{f(x)}\, d\mu(x)$ is <u>bounded</u>. As $\epsilon \to 0$,

$$\overline{f(x)}\, e^{-\epsilon|x|} d\mu(x) \to \overline{f(x)}\, d\mu(x) \qquad \text{(strong convergence)}.$$

Therefore,

$$\int_{-\infty}^{\infty} \overline{f(x)e^{itx}}\, e^{-\epsilon|x|} d\mu(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{f}(s-t)}\, \hat{\mu}_\epsilon(s) ds$$

$$\to \int \overline{f(x)e^{itx}}\, d\mu(x) \qquad \text{as } \epsilon \to 0\ .$$

On the other hand, because $\hat{f} = 0$ off ω_1

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \overline{\hat{f}(s-t)}\, \hat{\mu}_\epsilon(s)ds = 0 , \qquad |t| < \delta_o .$$

Putting the two together, we see that the bounded measure $\overline{f(x)}\, d\mu(x)$ has a Fourier-Stieltjes transform equal to zero on each interval $(-\delta,\delta)$, $0 < \delta < \delta_o$. Furthermore, assuming (as we may) $|f| \le 1$, we easily find $\sigma(x,\bar{f}d\mu) \le \sigma(x,d\mu)$, so the convergence of (i) and the divergence of (iii) are undisturbed. Part 1 of the proof therefore shows $\bar{f}(x)d\mu(x) \equiv 0$. But, by definition, $\hat{f}$ has compact support, so f is an entire function (of exponential type), which may be assumed to have no real zeros. This completes the proof of Theorem IV.

Applications:

1) Theorem IV may be used to prove old and new gap theorems, because the hypotheses will be fulfilled, in particular, if $\mu(x)$ is bounded and has sufficiently lacunary support. Specifically,

Corollary 4.1: Let $\mu(x)$ be a bounded measure which vanishes throughout the disjoint intervals $[x_n, x_n + \ell_n]$ $(n = 1, 2, \ldots)$, where $\sum_{n=1}^{\infty} \ell_n^2/x_n^2 = \infty$. Let $\hat{\mu} = 0$ on some interval. Then $\mu \equiv 0$.

Proof: Assume without loss of generality $\|\mu\| \le 1$. Then $\sigma(x) = \sigma^-(x) \le 0$. It follows from the definition of σ that

$$\sigma(x) = - \text{distance } (x,S) ,$$

where S is the support of μ. Thus we conclude from Theorem IV that $\mu \equiv 0$ because

$$\sum \int_{x_n}^{x_n+\ell_n} \sigma(x) \frac{dx}{x^2} \text{ diverges if } \sum \frac{\ell_n^2}{x_n^2} \text{ diverges.}$$

2) Consider now a torus T consisting of equivalence classes of points $x = (x_1, x_2)$ congruent (mod 2π). It is known that for all integers n_1 and n_2 the characters

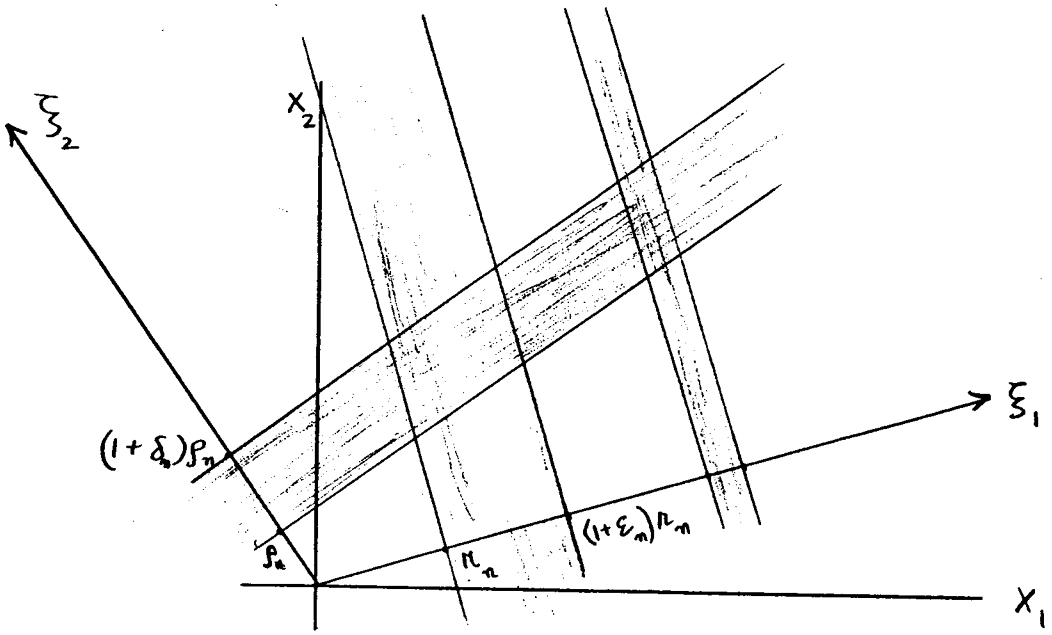
$$e^{inx} = e^{i(n_1x_1 + n_2x_2)}, \qquad n = (n_1, n_2),$$

form a complete orthonormal set of functions in the Hilbert space $L^2(T)$. The set is no longer complete if a single character is removed. Nevertheless, we shall give a general condition under which a proper subset of the characters is complete in $L^2(T-\omega)$, where ω is an arbitrarily small open set.

Let ξ_1 and ξ_2 be any fixed pair of linearly independent vectors, and let $\{\epsilon_n\}_1^\infty$, $\{\delta_n\}_1^\infty$ be sequences of positive numbers such that

$$\sum \epsilon_n^2 = \infty, \quad \sum \delta_n^2 = \infty.$$

Let S_1 denote the union of all infinite strips orthogonal to ξ_1 and meeting the ray $x = t\xi_1$ in the disjoint segments $(r_n, (1+\epsilon_n)r_n)$. Similarly, let S_2 be the union of the strips meeting $x = t\xi_2$ orthogonally in the disjoint segments $(\rho_n, (1+\delta_n)\rho_n)$. Set $S = S_1 \cup S_2$ (see diagram).



<u>Corollary 4.2</u>: For each open set $\omega \subset T$, the functions e^{inx} $(n \in S)$ span $L^2(T-\omega)$.

<u>Proof</u>: Let $f(x) \in L^2(T)$ be $= 0$ a.e. on ω, and suppose $f \sim \sum a_n e^{inx}$, where $a_n = 0$ for $n \in S$. It must be shown that $f \equiv 0$ on T.

Let $\varphi_\epsilon(x) \in C^\infty$ vanish outside the circle $|x| < \epsilon$. Then

$$\varphi_\epsilon \sim \sum c_n e^{inx}, \qquad \sum |c_n| < \infty .$$

and the convolution

$$f_\epsilon = f * \varphi_\epsilon = \sum a_n c_n e^{inx} = \sum b_n e^{inx}, \qquad \sum |b_n| < \infty .$$

Let ω_ϵ be the set of all $x \in \omega$ whose distance from the boundary is $< \epsilon$. Because $f \equiv 0$ on ω, $f_\epsilon \equiv 0$ on ω_ϵ. Furthermore, for $x = x_0 + t\xi_1$ on any line parallel to ξ_1 we may write

$$f_\epsilon(x_o + t\xi_1) = \sum b_n e^{in\cdot x_o} e^{it(n\cdot\xi_1)}$$

$$= \int_{-\infty}^{\infty} e^{itr} \, d\nu(r) ,$$

where $\nu(r) \equiv 0$ on the intervals $r_n < r < r_n + \epsilon_n r_n$. In view of the assumption $\sum \epsilon_n^2 = \infty$, Corollary 4.1 may now be invoked to show that $f_\epsilon(x_o + t\xi_1) \equiv 0$ on any line whose intersection with ω_ϵ contains an interval of positive length. Thus $f_\epsilon(x) \equiv 0$ throughout a strip of positive width. Since each line parallel to ξ_2 meets this strip in an interval of positive length, the same argument (using $\sum \delta_n^2 = \infty$) shows $f_\epsilon(x)$ vanishes everywhere. Letting $\epsilon \to 0$, we conclude $f(x) = 0$ a.e.

Remark: If the slope of ξ_1 is irrational, then the characters e^{inx} $(n \in S_1)$ alone span $L^2(T-\omega)$, because on the torus a line with this slope is everywhere dense. These applications are of some interest in connection with results of Arens and Helson-Lowdenslager.

QUASI-ANALYTICITY AND GENERAL DISTRIBUTIONS

1. Distributions

If X is a locally compact space, we may form on X a vector space $\mathcal{A}$ consisting of continuous numerical functions with compact support and with a topology of the kind introduced by L. Schwartz. The distributions will by definition be elements of the dual space $\mathcal{A}'$. Even if X is compact, a localization principle for the distribution will require the existence in $\mathcal{A}$ of functions vanishing outside given compact sets $\subset X$. The topology of $\mathcal{A}$ does therefore not allow the existence of both localizable distributions and quasi-analytic properties. This is the reason why the two notions are connected. We also want to point out that considerable advantages are to be gained by making $\mathcal{A}$ an algebra under point-wise multiplication.

The notion of distribution to be considered in the following lectures is based on ideas previously set forth in the work of L. Schwartz[1] (concerning the topology) and in two papers by the speaker[2] (concerning the algebras A_ω and the harmonic analysis). The development of the theory for Euclidean spaces and its application to analytic functions will also require some elementary though essential lemmas, which will be stated and proved at the end of these notes.

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[1] Theorie des Distributions, I, II, Hermann, Paris.

[2] Scand. Congr. 1938, Harmonic Analysis Conf., Nancy, 1947.

We begin with a general definition valid on locally compact and sigma-compact Abelian groups. Let G be such a group and $\hat{G}$ its dual. Denote by x elements $\in G$ and by ξ elements $\in \hat{G}$. Let the Haar measures dx and $d\xi$ on G and $\hat{G}$ respectively be so normalized with respect to the character-function (denoted $e^{i\xi x}$) that the Fourier constant is 1. On $\hat{G}$ we introduce functions $\omega(\xi)$ subject to two conditions, the first being

$$(\alpha) \qquad \begin{aligned} &\omega(\xi) \text{ is continuous on } \hat{G}, \\ &0 = \omega(0) \leq \omega(\xi+\eta) \leq \omega(\xi) + \omega(\eta), \qquad \xi, \eta \in \hat{G}. \end{aligned}$$

Let A_ω be the set of functions on G with a representation

$$\varphi(x) = \int \hat{\varphi}(\xi) e^{i\xi x}\, d\xi$$

where $\hat{\varphi}(\xi) e^{\omega(\xi)} \in L^1(\hat{G})$. Define

$$\|\varphi\| = \int |\hat{\varphi}(\xi)| e^{\omega(\xi)} d\xi \;.$$

The subadditive property of ω implies that A_ω with norm $\|\varphi\|$ is a Banach algebra under pointwise multiplication and we shall have $\|\varphi\psi\| \leq \|\varphi\|\,\|\psi\|$. Conversely, if ω is continuous, the preceding inequality implies $\omega(\xi+\eta) \leq \omega(\xi) + \omega(\eta)$. On replacing ω by $\lambda\omega$, λ = pos. parameter, we obtain a family of algebras $A_{\lambda\omega}$ with norms

$$\|\varphi\|_\lambda = \int |\hat{\varphi}(\xi)| e^{\lambda\omega(\xi)} d\xi \;.$$

We observe that $\lambda_1 < \lambda_2$ implies $A_{\lambda_1\omega} \supset A_{\lambda_2\omega}$, $\|\varphi\|_{\lambda_1} \leq \|\varphi\|_{\lambda_2}$. Now let $\mathcal{A}_\omega = \mathcal{A}_\omega(G)$ be defined as the set of all functions with compact support contained in $\bigcap_{\lambda > 0} A_{\lambda\omega}$. By definition, $\mathcal{A}_\omega$ is linear and closed under pointwise multiplication. At this instance we introduce the second condition on ω:

(β) $\mathcal{A}_\omega$ possesses local units; i.e., for each compact $K \in G$ and for each open $W \supset K$ there exists a function $\rho \in \mathcal{A}_\omega$ such that $\rho = 1$ on K, $\rho = 0$ off W'.

To a compact K we associate the subset $\mathcal{A}_\omega(K) = \{\varphi | \varphi \in \mathcal{A}_\omega(G), \text{supp. } \varphi \subset K\}$.

<u>Topology of</u> $\mathcal{A}_\omega(K)$: A sequence $\{\varphi_n\}_1^\infty \subset \mathcal{A}_\omega(K)$ is convergent if and only if it is a Cauchy sequence in each $A_{\lambda\omega}$, $\lambda > 0$. Following Schwartz we define a <u>topology on</u> $\mathcal{A}_\omega(G)$ by choosing a fixed sequence $\{K_n\}_1^\infty$ of compact sets, $K_n \subset K_{n+1}$, $G = \bigcup_1^\infty K_n$, and form on $\mathcal{A}_\omega(G)$ the inductive limit of the topologies on $\mathcal{A}_\omega(K_n)$ (which will be independent of the choice of the K_n). By a theorem of Dieudonné-Schwartz, each bounded subset of $\mathcal{A}_\omega$ is contained in a set of the form

$$B = \{\varphi | \varphi \in \mathcal{A}_\omega(K) \ , \ \|\varphi\|_\lambda \leq a_\lambda \ , \ \lambda = 1, 2, \ldots\}$$

where K is compact and a_λ are positive constants.

We recall the following property concerning <u>the topology of</u> $\mathcal{A}'_\omega$: a sequence $\{T_n\}_1^\infty \subset \mathcal{A}'_\omega$ is convergent if and only if $T_n(\varphi)$ converges pointwise on $\mathcal{A}_\omega$ and if the convergence is uniform on each set of type B.

For G = a Euclidean space, the function $\omega(\xi) = \log(1+|\xi|)$ will obviously satisfy (α) and (β) and we recognize that the spaces $\mathcal{S}$ and $\mathcal{S}'$ of Schwartz will coincide with $\mathcal{A}_\omega$ and $\mathcal{A}'_\omega$ respectively.

The problem to decide whether a given ω on $\hat{G}$ is admissible, i.e., condition (β) satisfied, can of course not be solved unless the structure of $\hat{G}$ is known. Our main objective is to solve this problem for Euclidean spaces. The solution will also hold if G is a torus, but this case will not be considered in detail.

<u>Theorem I</u>: Let $\omega(\xi)$ satisfy (α) on R^N, $N \geq 1$. Condition (β) is then satisfied if and only if

$$J_N(\omega) \equiv \int_{|\xi|\geq 1} \omega(\xi) \frac{d\xi}{|\xi|^{N+1}} < \infty .$$

<u>Proof</u>: Writing $\xi = r\,\xi^*$, $r = |\xi|$, $\xi^* \in S$ (unit sphere in R^N) we may consider $d\xi$ as a product-measure $r^{N-1}\,dr\,d\theta(\xi^*)$ where $d\theta$ is a positive measure on S. Define

$$p(\xi) = \int_1^\infty \omega(r\xi)\,\frac{dr}{r^2}, \qquad \xi \neq 0.$$

Then

$$J_N(\omega) = \int_S p(\xi)\,d\theta(\xi) .$$

Let us next prove this statement: If $J_N = \infty$, then $p(\xi) = \infty$ on some hemisphere S_o of S. If $J_N < \infty$, then $p(\xi)$ is bounded on S.

It follows by condition (α) that for $\xi_1, \xi_2, \xi_1 + \xi_2 \neq 0$,

$$p(\xi_1 + \xi_2) \leq p(\xi_1) + p(\xi_2) ,$$

and for $\xi \neq 0$,

$$p(\xi) < \infty \Longrightarrow p(\lambda\xi) < \infty , \qquad \lambda > 0.$$

Define $K = \{\xi \,|\, p(\xi) < \infty, \ \xi \neq 0\}$. If therefore $\{\xi_\nu\}_1^n$ belong to K, the same is true of each $\xi \neq 0$ of the form $\xi = \sum_1^n \lambda_\nu \xi_\nu$, $\lambda_\nu > 0$, and K is consequently a convex cone (with vertex removed). But a convex cone in R^N does either coincide with the whole space or it is contained in a closed half space. Under the last mentioned alternative $p = \infty$ on some open hemisphere S_o. Under the first alternative p is finite on S. The stated boundedness on S can readily be deduced from the subadditivity of p but this point is not essential for the following proof.

Consider first the case $J_N = \infty$. Let φ belong to some algebra $A_{\lambda\omega}$, say for $\lambda = 1$, and have compact support. Consequently, $\hat{\varphi}$ is is the restriction to R^N of an entire function of exponential type. Define for $\xi \in S$

$$q(\xi) = \int_1^\infty |\hat{\varphi}(r\xi)| e^{\omega(r\xi)} r^{N-1} \, dr .$$

Then

$$\int_S q(\xi) \, d\theta(\xi) = \int_{|\eta| \geq 1} |\hat{\varphi}(\eta)| e^{\omega(\eta)} \, d\eta \leq \|\varphi\|_1 ,$$

and it follows that $q < \infty$ a.e. on S. Hence, for almost all $\xi \in S$, $q(\xi) < \infty$ and $\max(p(\xi), p(-\xi)) = \infty$. We have (formally)

$$\int_1^\infty \log |\hat{\varphi}(r\xi)| \frac{dr}{r^2} = \int_1^\infty \log \{|\hat{\varphi}(r\xi)| e^{\omega(r\xi)} r^{N+1}\} \frac{dr}{r^2}$$

$$- \int_1^\infty (\omega(r\xi) + \log r^{N+1}) \frac{dr}{r^2} .$$

On applying the inequality between geometrical and arithmetical means to the first integral at the right, we find that

$$\int_1^\infty \log |\hat{\varphi}(r\xi)| \frac{dr}{r^2} \leq \log q(\xi) - p(\xi) ,$$

where the right-hand side is $= -\infty$ for a.a. $\xi \in S_0$. But $f(r) = \hat{\varphi}(r\xi)$ is the restriction to the real axis of an entire function of exponential type, bounded for real r. Hence, by the classical theorem, $\hat{\varphi}(r\xi)$ vanishes identically in r for a.a. ξ. The obvious conclusion is $\hat{\varphi} \equiv 0$. $J_N = \infty$ thus implies that $\mathcal{A}_\omega$ is void.

We shall show now that $\mathcal{A}_\omega$ possesses local units if $J_N < \infty$. For this purpose let K be a compact set $\subset R^N$ and W an open set $\supset K$. Define $K_\epsilon = \{x | \text{dist.}(x,K) \leq \epsilon\}$ and choose ϵ so small that $K_{2\epsilon} \subset W$. Let us assume that there exists an $f \in \bigcap_{\lambda > 0} A_{\lambda\omega}$ and with support contained in the ball $|x| \leq \epsilon$ and such that $\int f \, dx = 1$. If φ_ϵ denotes the characteristic function of K_ϵ, the convolution $\varphi = \varphi_\epsilon * f$ will have all the requested properties. It is therefore sufficient to show the existence of f. By Lemma I we conclude that there are functions $\Omega(r)$, concave for $r > 0$, and with the properties:

$$\lim_{|\xi|\to\infty} \frac{\omega(\xi)}{\Omega(|\xi|)} = 0 , \quad \Omega(r) > \sqrt{r} , \quad \int_1^\infty \Omega(r) \frac{dr}{r^2} < \infty .$$

It is a well-known fact concerning entire functions of exponential type that there are continuous functions $F(x)$ on each interval $[-\delta,\delta]$, $\delta > 0$, such that

$$\int_{-\delta}^{\delta} F(x)dx \neq 0$$

$$\left| \int_{-\delta}^{\delta} F(x)e^{-i\xi x}\, dx \right| \leq e^{-\Omega(|\xi|)} , \qquad \xi \text{ real},$$

where $\Omega(r)$ is concave and has the summability indicated. If $\{\alpha_\nu\}_1^N$ are Cartesian coordinates in R^N and

$$k = \left\{ \int_{-\delta}^{\delta} F(x)dx \right\}^{-N} ,$$

then

$$f(x) = k \prod_1^N F(x_\nu)$$

will vanish off the cube $\bigcap_1^N \{x \mid |x_\nu| \leq \delta\}$ and $\int f(x)dx = 1$. By the choice of Ω and by the inequality

$$|\hat{f}(\xi)| = |\hat{f}(\xi_1, \xi_2, \ldots , \xi_N)| \leq \text{const.}\ e^{-\sum_1^N \Omega(|\xi_\nu|)}$$

we conclude that

$$\|f\|_\lambda \leq \text{const.} \prod_{\nu=1}^N \int_{-\infty}^{\infty} e^{\lambda\omega_\nu(\xi_\nu) - \Omega(|\xi_\nu|)} d\xi_\nu < \infty , \qquad \lambda > 0$$

where $\omega_\nu(\xi_\nu)$ is the restriction of ω to the ξ_ν-axis. Hence, $f \in \mathcal{A}_\omega$ and this finishes the proof of Theorem I.

* * * * * *

The previous results allow us to derive two important conclusions in the Euclidean case: 1) If $\mathcal{A}_\omega$ is not void then local units exist. 2) If ω_1 and ω_2 are admissible (i.e., $\mathcal{A}_{\omega_1}, \mathcal{A}_{\omega_2}$ not void) then the same is true of $\omega = \max(\omega_1, \omega_2)$. Consequently, if $T_1 \in \mathcal{A}'_{\omega_1}$, $T_2 \in \mathcal{A}'_{\omega_2}$, then T_1, T_2 and $T_1 + T_2$ will belong to $\mathcal{A}'_\omega$.

* * * * * *

A theory of distributions $\mathcal{A}'_\omega(G)$ may now be developed along familiar lines. In this general area we shall only consider one problem.

<u>Theorem II</u>. Let T belong to $\mathcal{A}'_\omega(G)$ and have compact support K. Let $\rho \in \mathcal{A}_\omega(G)$ and be a local unit for K. Define $\check{T}(\xi) = T(\rho(x)e^{i\xi x})$. Then there exist constants a, b, independent of ξ and such that

$$|\check{T}(\xi)| \leq e^{a\omega(\xi)+b}, \qquad \xi \in \hat{G}.$$

<u>Proof</u>: It follows by the definition of the Banach algebras $A_{\lambda\omega}$ that

$$\|\rho(x)e^{i\xi x}\|_\lambda \leq \|\rho\|_\lambda \, e^{\lambda\omega(\xi)},$$

and ρ multiplied by a character therefore remains in $\mathcal{A}_\omega$. Another conclusion is that the definition of $\check{T}$ does not depend on the particular unit for K used.

If the theorem were false there would exist a sequence $\{\xi_\nu\}_1^\infty$ such that

$$|\check{T}(\xi_\nu)| = \exp\{a_\nu(\omega(\xi_\nu)+1)\}, \qquad a_\nu \to +\infty .$$

Let $\{b_\nu\}_1^\infty$ have the properties $b_\nu \to +\infty$, $b_\nu/a_\nu \to 0$. Then

$$k(x) = \max_{\nu \geq 1} \{(\lambda - b_\nu)\omega(\xi_\nu) - \omega(\xi_\nu)\}$$

is finite for $\lambda > 0$. Define

$$\varphi_\nu(x) = e^{-b_\nu(\omega(\xi_\nu)+1)} \rho(x) e^{i\xi_\nu x}, \qquad \nu \geq 1 .$$

Thus,

$$\|\varphi_\nu\|_\lambda \leq \|\rho\|_\lambda \exp\{(\lambda - b_\nu)\omega(\xi_\nu) - \omega(\xi_\nu)\}$$

$$\leq \|\rho\|_\lambda \exp\{k(x)\} .$$

This proves that $\{\varphi_\nu\}_1^\infty$ is contained in a bounded set B. Hence

$$\text{const.} \geq |T(\xi_\nu)| = \exp\{(a_\nu - b_\nu)(\omega(\xi_\nu)+1)\} \to \infty .$$

This contradiction proves the theorem.

2. Applications to Analytic Functions

Time allows us to consider only the simplest case: functions $f(z)$ of one variable analytic on a disk. We recall this consequence of the theory of Schwartz: If $f(z)$ is holomorphic in $|z| < 1$, the functions $f(re^{i\theta})$ converge to a distribution as $r \uparrow 1$ if and only if there exists a finite k such that

$$\mathcal{M}(r) \leq \frac{\text{const.}}{(1-r)^k} \qquad r < 1 ,$$

$\mathcal{M}(r)$ being the maximum modulus of f on $|z| = r$. The corresponding result for general distribution is as follows:

Theorem III: For $r \uparrow 1$, $f(re^{i\theta})$ converges to a distribution T (belonging to some $\mathcal{A}'_\omega$) if and only if

(1) $$\int_0^1 \log^+ \log^+ \mathcal{M}(r) dr < \infty .$$

Proof: Assume

$$f(z) = \sum_0^\infty c_n z^n , \qquad |z| < 1 .$$

According to Cauchy,

$$|c_n| \leq \inf_{0 \leq r < 1} \frac{\mathcal{M}(r)}{r^n} .$$

On defining $r = e^{-y}$, $h(y) = \log \mathcal{M}(e^{-y})$, we obtain

$$|c_n| \leq e^{\omega(n)}$$

where

$$\omega(x) = \inf_{y > 0} [h(y) + xy] .$$

Condition (1) is equivalent to

(2) $$\int_0^1 \log h(y)dy < \infty .$$

The function $h(y)$ is convex on $(0,\infty)$ according to the three circle theorem of Hadamard. Hence, by Lemma II (stated later), the integral (2) and

$$\int_1^\infty \omega(x) \frac{dx}{x^2}$$

converge at the same time, and $\omega(x)$ is concave on $(0,\infty)$. If $|f(0)| = 1$, which we may assume, then $\omega(0) = 0$, and $\omega(x)$ defined on R by the condition $\omega(-x) = \omega(x)$ will therefore belong to $\alpha(R)$, and hence to $\alpha(I)$, (I = the group of integers). We claim that $f(re^{i\theta})$ converges in $\mathcal{A}_\omega'(\Gamma)$ (Γ = the circle group). Let

$$\varphi(\theta) = \sum_{-\infty}^{\infty} \hat{\varphi}(n)e^{in\theta} \in \mathcal{A}_\omega(\Gamma) .$$

Then

$$\frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})\, \varphi(\theta)d\theta = \sum_0^\infty c_n\, \hat{\varphi}(-n)r^n$$

where the series is majorized by

$$\sum_{-\infty}^{\infty} |\hat{\varphi}(n)| e^{\omega(n)} = \|\varphi\|_{\lambda} \qquad (\lambda = 1)$$

and the stated convergence follows.

Assume now that $f(re^{i\theta})$ converges in $\mathcal{A}'_{\omega}(\Gamma)$ for some $\omega \in \alpha(I)$. It is readily seen that ω can be extended from I into R such that $\omega \in \alpha(R)$. An application of Theorem II to the present case yields the existence of constants a, b such that

$$|c_n| \leq e^{a\omega(n)+b} .$$

Hence

$$\mathcal{M}(e^{-y}) \leq e^b \sum_{0}^{\infty} e^{a\omega(n)-ny} .$$

On defining $\omega_1(x) = a\,\omega(2x)$ we obtain

$$\mathcal{M}(e^{-y}) \leq \frac{e^b}{1-e^{-(y/2)}} e^{h(y)}$$

where

$$h(y) = \sup_{x>0} (\omega_1(x)-xy) .$$

By Lemma II, $h(y)$ will satisfy (2) provided $\omega_1(x)$ has a concave majorant on $(0,\infty)$ which is summable on $(1,\infty)$ for the measure $x^{-2}dx$. By Lemma I this is true. Hence (2) and (1) follow.

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We shall finally state the localized form of Theorem III and its application to analytic continuation in one variable. Let $f(z)$ be analytic in the rectangle $Q = \{z = x + iy \mid -a < x < a,\ 0 < y < c\}$, and let K denote any interval $[-b,b]$, $b < a$. Then $f(x + iy)$, $y \downarrow 0$, tends to a distribution $T_1 \in \mathcal{A}'_\omega(K)$ (for each such K) if and only if the following majorization condition is satisfied:

$$(3) \qquad |f(x + iy)| \leq e^{e^{m(y,b)}}, \qquad -b \leq x \leq b, \quad 0 < y < \delta = \frac{c}{2},$$

where $m(y,b)$ is decreasing in y and $m(y,b) \in L^1(0,\delta)$, for each $b < a$. Furthermore, if g is analytic in $Q^* = \{z \mid \bar{z} \in Q\}$, then f and g are analytic continuations of each other across $(-a,a)$ if and only if $f(x + iy)$ and $g(x - iy)$ satisfy condition (3) and the distributions T_1 and $T_2 \in \mathcal{A}'_\omega(K)$ defined by f and g respectively coincide (for each K). It is to be noted that ω in the preceding statement may depend on K.

We also want to outline an application of Theorem IV (of Lecture 3) to analytic continuation. Let

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n$$

be analytic in an annulus $1-2\delta < |z| < 1$, and

$$g(z) = \sum_{-\infty}^{\infty} b_n z^n$$

analytic for $1 < |z| < 1 + 2\delta$, $\delta > 0$.

Define

$$\mathcal{M}(r) = \max_{|z| = r} |f(z)| \qquad 1 - \delta < r < 1 ,$$

$$\mathcal{M}(r) = \max_{|z| = r} |g(z)| \qquad 1 < r < 1 + \delta ,$$

and suppose

(4) $$\int_{1-\delta}^{1+\delta} \log^+ \log^+ \mathcal{M}(r) dr < \infty .$$

Assume further that a_n vanishes on non-overlapping intervals $[n_\nu, n_\nu + \ell_\nu]$, $n_\nu > 0$, such that

(5) $$\sum_1^\infty \frac{\ell_\nu^2}{n_\nu^2} = \infty .$$

If f can be analytically continued into g across one point $z_o = e^{i\theta_o}$, then the same holds true for each point on $|z| = 1$ and we shall have

$$f(z) = g(z) = \sum_o^\infty b_n z^n + \sum_{-\infty}^{-1} a_n z^n$$

for $1-2\delta < |z| < 1+2\delta$.

Proof: By the proof of Theorem II we conclude that (4) implies the existence of a symmetric ω with $J_1(\omega) < \infty$ such that

(6) $$|a_n|, |b_{-n}| \leq e^{\omega(n)} , \qquad n \geq n_o .$$

The algebra A_ω for the circle does therefore contain functions

$$\text{(7)} \qquad \varphi(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta}, \quad \sum_{-\infty}^{\infty} |c_n| e^{\omega(n)} < \infty$$

with support contained in a given interval $[-\epsilon,\epsilon]$. We may also assume that $c_n \neq 0$ for all n. The functions

$$f_\epsilon(z) = \frac{1}{2\pi} \int_0^{2\pi} f(ze^{-i\theta})\, \varphi(\theta)\, d\theta = \sum_{-\infty}^{\infty} a_n c_n z^n ,$$

$$g_\epsilon(z) = \frac{1}{2\pi} \int_0^{2\pi} g(ze^{-i\theta})\, \varphi(\theta)\, d\theta = \sum_{-\infty}^{\infty} b_n c_n z^n ,$$

will remain holomorphic in the two annuli, and f_ϵ can still be continued analytically into g_ϵ across the point $e^{i\theta_0}$ if ϵ is sufficiently small. Let μ be the measure on R with support on the integers:

$$\mu\{n\} = c_n(a_n - b_n) .$$

By virtue of (6) and (7), μ is bounded and we shall have

$$\int_{-\infty}^{\infty} e^{itx}\, d\mu(x) = 0$$

on some interval containing the point θ_0. Define as in Theorem IV $\sigma(x)$ by

$$e^{\sigma(x)} = \int_{-\infty}^{\infty} e^{-|x-\xi|}\, |d\mu(\xi)| = \sum_{-\infty}^{\infty} e^{-|x-n|}\, |c_n(a_n - b_n)| .$$

Let S be the set of positive integers where $a_n \neq 0$. For all n sufficiently large

$$|b_n| < e^{-\eta n} , \qquad \delta < \eta < 2\delta .$$

For all ν sufficiently large, we shall therefore have

$$\sigma(x) < -\delta \text{ dist } (x,S) , \qquad x \in [n_\nu, n_\nu + \ell_\nu] .$$

This inequality together with (7) implies that

$$\int_1^\infty \frac{\sigma(x)}{x^2} dx = -\infty .$$

Hence, by Theorem IV, μ vanishes identically and $a_n - b_n = 0$ follows.

3. The Lemmas

In the sequel we denote by $\alpha(\hat{G})$ the set of functions ω subject to the condition (α) on $\hat{G}$.

Lemma I. Let $\omega \in \alpha(R^N)$, $N \geq 1$, and assume $J_N(\omega) < \infty$. Under these conditions there exists a function $\Omega(r)$ concave for $r > 0$ and such that

$$\max_{|\xi| \leq r} \omega(\xi) \leq \Omega(r) , \quad \int_1^\infty \Omega(r) \frac{dr}{r^2} < \infty .$$

Proof. We shall first prove the lemma in the case $N = 1$. Without loss of generality, we may assume that $\omega(-x) = \omega(x)$, because $\max(\omega(x), \omega(-x))$ will still belong to $\alpha(R)$ and the convergence condition $J_1 < \infty$ will not be obstructed. Define $\omega_1(x) = \max_{|\xi| \leq |x|} \omega(\xi)$. We shall prove that $J_1(\omega_1) < \infty$. Let (a,b), $a > 0$, be one of the intervals that form the open set where $\omega < \omega_1$. Define $\ell = \min(a, b-a)$,

$$E = \{x | a < x < a + \ell,\ \omega(x) < \tfrac{1}{3}\omega(a)\} ,$$

$$E' = \text{complement of } E \text{ with respect to } (a, a+\ell) ,$$

$$E^* = \{x | x = a + x_1 - x_2 > a,\ \ x_1, x_2 \in E\} .$$

For $x \in E^*$, $\omega(x) \geq \omega(a) - \omega(-x_1) - \omega(x_2) > \frac{\omega(a)}{3}$. Thus, $E^* \subset E'$. By definition of E^*, $|E^*| \geq |E|$. Therefore, $|E'| \geq |E|$, and we conclude that $|E'| \geq \frac{\ell}{2}$. On comparing the integrals

$$A_1 = \int_a^b \omega_1(x) \frac{dx}{x^2} = \omega(a)\left(\frac{1}{a} - \frac{1}{b}\right)$$

and

$$A = \int_a^b \omega(x) \frac{dx}{x^2} \geq \int_{E'} \omega(x) \frac{dx}{x^2} \geq \frac{\omega(a)}{3} \left(\frac{1}{a + \frac{1}{2}\ell} - \frac{1}{a+\ell} \right)$$

we find

$$\frac{A_1}{A} \leq \frac{6(b-a)(a + \frac{\ell}{2})(a+\ell)}{a\, b\, \ell} \leq 9 ,$$

and $J_1(\omega_1) < \infty$ follows. We next define $\omega_2(x)$ as the least concave majorant of $\omega_1(x)$ over $(0,\infty)$. It is easily shown that $\omega_1 \in \alpha(R)$. Let again (a,b) be one of the intervals forming the set where $\omega_1 < \omega_2$. Set

$$k = \frac{\omega_1(b) - \omega_1(a)}{b - a} .$$

Then,

$$\omega_2(x) = \omega_1(a) + k(x-a) \qquad\qquad x \in (a,b)$$

$$\omega_2(b) = \omega_1(b) \le \omega_1(b-x) + \omega_1(x) \le \omega_2(b-x) + \omega_1(x) .$$

Consequently,

$$\omega_1(x) \ge \omega_2(b) - \omega_2(b-x) = kx , \qquad\qquad x \in (a,b)$$

$$\omega_1(x) \ge \omega_1(a) \qquad\qquad , \qquad\qquad x \ge a .$$

Thus,

$$\omega_1(x) \ge \max\,(\omega_1(a),\ kx) \qquad , \qquad\qquad x \in (a,b) .$$

We may now normalize by assuming $a = \omega_1(a) = 1$, in which case we shall have $0 \le k \le 1$. We find

$$A_2 = \int_1^b \omega_2 \frac{dx}{x^2} = (1-k)(1 - \tfrac{1}{b}) + k \log b ,$$

$$A_1 = \int_1^b \omega_1 \frac{dx}{x} \ge (1-k) + k \log b + k \log k .$$

If $b \leq e$ (the Napier number) it follows that $\omega_2/\omega_1 \leq e$ on $(1,b)$. If $b > e$,

$$\frac{A_2}{A_1} \leq \frac{1 + k \log b/e}{1 + k \log b/e + k \log k} \leq \frac{e}{e-1} < e .$$

Thus $J_1(\omega_2) < \infty$ and the lemma is proved for $N = 1$.

If $N > 1$ we introduce in R^N an orthogonal coordinate system and denote by $\omega_\nu(r)$ the restriction of $\omega(\xi)$ to the ξ_ν-axis. Let $\bar{\omega}_\nu(r)$ denote the maximum of $\omega_\nu(r)$ and $\omega_\nu(-r)$. Then $\bar{\omega}_\nu \in \alpha(R)$ and

$$\int_1^\infty \bar{\omega}_\nu(r) \frac{dr}{r^2} < \infty .$$

The same properties hold true for

$$\omega_o(r) = \max_{1 \leq \nu \leq N} \bar{\omega}_\nu(r) .$$

On applying the lemma for $N = 1$ to $\omega_o(r)$ we find that there exists a concave function $\Omega_o(r) \geq \omega_o(r)$ with the stated summability. For each $\xi \in R^N$, $|\xi| = r$, we shall have

$$\omega(\xi) \leq N\, \Omega_o(r).$$

Consequently, $\Omega(r) \equiv N\, \Omega_o(r)$ has the properties stated in the lemma.

As a preparation for the next lemma, we introduce these definitions. For functions $\omega(x)$ continuous and ≥ 0 for $x \geq 0$ we define an operator S as follows:

$$h(y) = S\omega = \sup_{x \geq 0} (\omega(x) - xy), \qquad y > 0.$$

As the upper envelope of a family of linear functions with negative slope $h(y)$ is ≥ 0, convex and non-increasing for $y > 0$. For functions $h(y) \geq 0$ and non-increasing for $y > 0$ define $V(h)$ as

$$\omega(x) = V(h) = \inf_{y > 0} (h(y) + xy), \qquad x \geq 0.$$

It is readily seen that $VS\omega$ equals the least concave majorant of ω on $(0,\infty)$. Similarly, SVh is the largest convex minorant of $h(y)$ on $(0,\infty)$.

<u>Lemma</u> <u>II</u>. a) Let $\omega(x) \in \alpha(R)$ and let $h(y) = S\omega$. Then the integrals

(4) $$\int_1^\infty \omega(x) \frac{dx}{x^2}$$

and

(5) $$\int_0^1 \log h(y)\, dy$$

are either both convergent or both divergent.

b) Let $h(y)$ be ≥ 0 and non-increasing on $(0,\infty)$. Then the previous statement concerning (4) and (5) holds for h and for $\omega = Vh$.

Proof: By virtue of the proof of Lemma I we conclude that Lemma II is established if part a) is true for a concave ω. Without loss of generality we assume that ω has a continuous derivative ω' and that $\omega(\xi_o) = 1$ for some $\xi_o > 0$. Set $y_o = \omega'(\xi_o)$ and define $\xi = \xi(y)$ for $0 < y \leq y_o$ by the relation $\omega'(\xi) = y$. Then

$$h(y) = S\omega = \omega(\xi) - \xi y \leq \omega(\xi) ,$$

$$\int_y^{y_o} \log h(y)dy \leq \int_y^{y_o} \log \omega(\xi)dy$$

$$\leq \int_y^{y_o} y \, d[\log \omega(\xi)] = \int_1^{\xi(y)} \frac{\omega'^2(\xi)}{\omega(\xi)} d\xi .$$

Since ω is concave and ≥ 0, $\xi\omega'(\xi) \leq \omega(\xi)$. Hence

$$\int_o^{y_o} \log h(y)dy \leq \int_1^{\infty} \omega(\xi) \frac{d\xi}{\xi^2} .$$

In order to prove a reversed inequality we recall that $\omega(x) = Vh$ if ω is concave. Hence, for any $y > 0$

$$\omega(x) \leq h(y) + xy = y(h_1(y) + x)$$

where $h_1(y) = h(y)/y$. Let $y = y(x)$ be defined by the relation $h_1(y) = x$. Assume $h_1(y_o) = 1$ for some $y_o > 0$. Then for $x \geq 1$, $\omega(x) \leq 2x\, y(x)$ and

$$\int_1^x \omega(x)\,\frac{dx}{x^2} \le 2\int_1^x \frac{y(x)}{x}\,dx = 2\int_y^{y_o} y\,d[-\log h_1(y)]$$

$$= 2y\log h_1(y) + 2\int_y^{y_o} \log h_1(y)dy\ .$$

If, therefore, (5) converges, the same is true for h_1 and we conclude that

$$\int_1^\infty \omega(x)\,\frac{dx}{x^2} \le 2\int_0^{y_o} \log h_1(y)dy\ .$$

Lecture 1.

p. 1, line 6: read: $\lambda_n < \cdots \to \infty$

p. 3, line 3: read: $\sum_{n=N}^{\infty} a_n^*(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}) \geq \frac{a_N^*}{\lambda_N}$

p. 8, line 10: Delete "non-negative".

p. 8, next to last line: $f-g$ should read fg .

p. 9, line 11: For γ_o read E_o.

p. 9, The proof of Theorem II has disappeared but can easily be reconstructed by the reader.

Lecture 2.

p. 8, line 1 read: $M(1 + \frac{b}{2(a-|x|)})$

p. 13, next to last line: read: for $t \in [2\, C_3\, \lambda_n,\ 2\, C_3\, \lambda_{n+1}]$

p. 16, next to last line: For $\sum_{-\infty}^{\infty}$ read $\sum_{-\infty}^{n}$.

Lecture 3.

p. 2, The main distinction between Theorem III and Theorem IV is that $\hat{\mu}$ is supposed to vanish on a set of positive measure in Theorem III, whereas in Theorem IV $\hat{\mu}$ is $= 0$ on some open set.

THE STRUCTURE OF SOLUTIONS OF SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

by

Leon Ehrenpreis

(Outline of Lectures)

Lecture I

The great simplicity of the theory of linear ordinary differential equations with constant coefficients is due, for a great part, to the fact that every solution of such an equation can be expressed as a linear combination of the exponential polynomial solutions. One of the main objects of these lectures is to give a generalization of this property to systems of linear partial differential equations with constant coefficients.

Let $R(C)$ denote real (complex) Euclidean space of dimension n. $x = (x_1, \ldots, x_n)$ $[z = (z_1, \ldots, z_n)]$ is the coordinate on $R(C)$. By ∂ we denote a linear partial differential operator with constant coefficients, and we write $\partial = P(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ where P is a polynomial.

The following Theorem I is approximately correct and will be made precise later:

<u>Theorem I</u> (Imprecise Version): Let $f(x)$ be a function or distribution which satisfies $\partial_j f = 0$ for $j = 1, 2, \ldots, r$. Then there exists a measure $dv(z)$ whose support is contained in the complex algebraic variety V of common zeros of $P_j(z) = 0$ such that

$$f(x) = \int e^{ix\cdot z}\, dv(z)$$

where the integral converges in a suitable sense.

Another main object will be to prove theorems on inhomogeneous equations, a prototype of which is

<u>Theorem II</u> (Incomplete version): Let Ω be a convex set in R and let $g_1, \ldots, g_r$ be C^∞ on Ω. A necessary and sufficient condition that there should exist an $f \in C^\infty(\Omega)$ with $\partial_j f = g_j$ for $j = 1, \ldots, r$ is that the g_j should satisfy the obvious compatibility conditions:

$$\Sigma\, d_j g_j = 0$$

whenever d_j are linear constant coefficient partial differential operators with

$$\Sigma\, d_j \partial_j = 0 .$$

<u>Analytically Uniform Spaces</u>. A locally convex, Hausdorf topological vector space W of functions or distributions on R is called <u>analytically uniform</u> if

(a) W is reflexive.

(b) For each z, $e^{ix\cdot z} \in W$ and the map $z \to e^{ix\cdot z}$ of $C \to W$ is analytic.

(c) The linear combinations of $e^{ix\cdot z}$ are dense in W.

(d) Denote by W' the dual of W. For each $g \in W'$ we define the Fourier transform

$$G(z) = g \cdot e^{ix \cdot z}$$

for $z \in C$. (We shall sometimes write $\hat{g}(z)$ for $G(z)$.) Denote by $\hat{W}'$ the set of Fourier transforms of W' with the topology to make the Fourier transform a topological isomorphism. By (c), $\hat{W}'$ is a space of entire functions. Then we require that there should exist a family K of continuous positive function $k(z)$ such that

d_1. For any $G \in \hat{W}'$,

$$|G(z)|/k(z) \to 0 \quad \text{as} \quad |z| \to \infty .$$

d_2. The semi-norms

$$\|G\|_k = \sup_{z \in C} |G(z)|/k(z)$$

define the topology of $\hat{W}'$.

$K = \{k\}$ is called an analytic uniform structure for W.

Proposition 1. The following spaces are analytically uniform

1. H of entire functions of x (considered as complex variables).

2. E of C^∞ functions on R.

3. D' of distributions on R.

The same is true for the analogous spaces on a convex set in R.

For the proofs see American Jour. of Math., vol. 78, pp. 685-715.

Example. The space of real analytic functions on R with its usual topology is _not_ analytically uniform.

We have the following _Representation Theorem_.

Theorem 1. For every $T \in W$ there is a bounded measure $d\mu$ on C and a $k \in K$ such that (symbolically)

$$T(x) = \int e^{ix \cdot z} \, d\mu(z)/k(z) .$$

This means, for any $g \in W'$,

$$g \cdot T = \int G(z) \, d\mu(z)/k(z) .$$

This is proven by the Hahn-Banach theorem.

Definition. A set $\sigma \subset C$ is called W _sufficient_ if for every $k \in K$ there is a $k' \in K$ and an $A > 0$ such that for every $G \in \hat{W}'$ we have

$$\|G\|_k \leq A \sup_{z \in \sigma} |G(z)|/k'(z) .$$

Theorem 1^*. The support of μ in Theorem 1 can be taken to be any W sufficient set.

Example. For the spaces H, E $(n = 1)$ we may take σ as the real and imaginary axes. (Proof by harmonic majorant; see a paper to appear in Trans. AMS.)

Lecture 2

A much simpler application of Fourier transforms gives the following extension of a theorem of Hartogs (which contains Bochner's extension):

Theorem 2. Let $\partial_1, \ldots, \partial_r$ be such that $P_1, \ldots, P_r$ have no common factor. Let $\Omega_1 \subset\subset \Omega_2$ be convex sets in R. Let $f \in C^\infty(\Omega_2 - \bar{\Omega}_1)$ satisfy $\partial_j f = 0$ for $j = 1, \ldots, r$. Then there exists an $f_1 \in C^\infty(\Omega_2)$ with $\partial_j f_1 = 0$ for $j = 1, \ldots, r$ and $f_1 = f$ on $\Omega_2 - \Omega_1$.

The proof will appear in Bull. AMS, Research Announcements.

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Algebraic-Geometric Background

Definition. An algebraic variety V is the set of common zeros of a set of polynomials on C. V is irreducible if it is not the union of two algebraic varieties. Let F(z) be a function on V such that for each $z^o \in V$ there is a neighborhood $N(z^o)$ in C and a function G(z) holomorphic on N such that $F(z) = G(z)$ for $z \in N \cap V$; then F is called holomorphic on V. By a theorem of Oka there is an entire function H(z) whose restriction to V is F. By H(V) we denote the ring of holomorphic functions on V.

Definition. A multiplicity variety V is a finite set of pairs (V_j, d_j) where V_j is an algebraic variety and d_j is a constant coefficient linear differential operator. For each j let $F_j \in H(V_j)$. Assume that whenever

$$z_o \in V_{j_1} \cap V_{j_2} \cap \ldots \cap V_{j_s}$$

there exists a neighborhood $N(z_o)$ and holomorphic functions G_{j_s} on N such that

(a) $G_{j_s}(z) = F_{j_s}(z)$ for $z \in N \cap V_{j_s}$.

(b) Whenever $\sum D_{j_s} d_{j_s} = 0$ where D_{j_s} are linear constant coefficient differential operators then also

$$\sum D_{j_s} G_{j_s}(z) = 0 \quad \text{for} \quad z \in N .$$

Then the collection $\{F_j\} = \underline{F}$ is called a <u>holomorphic function</u> on $\underline{V}$.

We call $H(\underline{V})$ the space of holomorphic function on $\underline{V}$.

Next let W be an analytically uniform space with analytic structure K. Let V be an algebraic variety. We call $\hat{W}'(V)$ the space of $F \in H(V)$ for which

$$|F(z)|/k(z) \to 0$$

for $z \in V$, $|z| \to \infty$. $\hat{W}'(V)$ is given the natural topology. We define $\hat{W}'(\underline{V})$ as the space of $\underline{F} = \{F_j\}$ such that each $F_j \in \hat{W}'(V_j)$. The topology of $\hat{W}'(\underline{V})$ is the natural one.

There is a natural continuous map $\varphi_{\underline{V}}: \hat{W}' \to \hat{W}'(\underline{V})$ which is defined by

$$F(z) \to (d_1 F | V_1, \ldots, d_\ell F | V_\ell)$$

where $d_j F | V_j$ is the restriction of $d_j F$ to V_j.

Suppose every polynomial defines (by multiplication) a continuous map of $\hat{W}'$ into itself. Let $\underline{P} = (P_1, \ldots, P_r)$ be a set of polynomials. Let $\underline{P}\,\hat{W}'$ be the module of elements $\sum P_j G_j$ where $G_j \in \hat{W}'$.

In a future lecture we shall introduce further hypotheses on W which we shall call <u>localizability</u>.

<u>Theorem III</u>. Let W be analytically uniform and localizable. Let $\underline{P}$ be as above. Then $\underline{P}\,\hat{W}'$ is closed.

Let $\partial_1, \ldots, \partial_r$, $P_1, \ldots, P_r$ have the usual meaning.

<u>Theorem IV</u>. There exists a multiplicity variety $\underline{V}$ such that whenever W is analytically uniform and localizable, $\varphi_{\underline{V}}$ defines a topological isomorphism of the quotient space $\hat{W}'/\underline{P}\,\hat{W}'$ onto $\hat{W}'(\underline{V})$.

Each V_j occurring in V is a subvariety of the algebraic variety defined by the set of P_t.

By Hahn-Banch, Theorem III implies

<u>Theorem I</u>. Let W be as in Theorem III, and $\partial_1, \ldots, \partial_r$ as usual. Then there exists a finite set of pairs (V_j, Q_j) where V_j is an algebraic subvariety of the set of common zeros of the P_s and Q_j is a polynomial with the following property: For any $T \in W$ which satisfies $\partial_j T = 0$ for $j = 1, \ldots, r$ we can find bounded measures μ_t with support on V_t and $k_t \in K$ such that the symbolic representation

$$T(x) = \sum_t Q_t(x) \int e^{ix\cdot z}\, d\mu_t(z)/k_t(z) .$$

Here $\underline{V} = \{(V_j, d_j)\}$ of Theorem IV and Q_j is the polynomial corresponding to d_j.

Lecture 3

Leon Ehrenpreis

Theorem IV of the previous lecture will be called the <u>quotient structure theorem</u>.

There are three conditions for localizability. The first two are:

(a) Any entire function which is $O(k(z))$ for all $k \in K$ is in $\hat{W}'$.

(b) For any $N > 0$ if we replace the analytic uniform structure $K = \{k\}$ by $K_N = \{k_N(z)\}$ where

$$k_N(z) = \max_{|z'-z| \leqq N} |k(z)|(1+|z'|)^N$$

then K_N is again an analytic uniform structure for W.

Condition (c) will be introduced later.

For the proof of Theorems III and IV, we consider first the following simplifications: $K = \{k(z)\}$ where the $k(z)$ are functions of $|z|$ only and are monotonically increasing; $\underline{P}$ consists of a single polynomial which we denote by P. Theorem III for this case is readily verified.

By a linear change of coordinates we may assume that

$$P(z_1,\ldots,z_n) = z_n^m + \sum_{j=o}^{m-1} P^j(z_1,\ldots,z_{n-1})z_n^j$$

where P^j is a polynomial of degree at most $m-j$. Let $P = P_1^{t_1}P_2^{t_2}\ldots P_s^{t_s}$ be the prime power decomposition of P and let V_j be the variety of zeros of P_j. Then we assert that for this case we can choose

$$\underline{V} = \{(V_1,\mathrm{id}),(V_1,\partial/\partial z_n),\ldots,(V_1,\partial^{t_1-1}/\partial z_n^{t_1-1}),\ldots,(V_s,\mathrm{id}),\ldots,(V_s,\partial z_n^{t_s-1})\}$$

where id is the identity.

We must construct the inverse of $\varphi_{\underline{V}}$. Let $\underline{F} \in \hat{W}'(\underline{V})$. It follows from classical results (Oka) that there is an entire function G such that $\varphi_{\underline{V}} G = \underline{F}$. However, we must show that G can be chosen to be in $\hat{W}'$ with bounds depending only on $\underline{F}$. Write

(1) $$G(z_1,\ldots,z_n) = \sum_{j=o}^{\infty} G^j(z_1,\ldots,z_{n-1})z_n^j$$

where G^j are entire. By replacing z_n^m by $-\sum P^j z_n^j + P$ in (1) we deduce

(2) $$G(z) = \sum_{j=o}^{m-1} \tilde{G}^j(z_1,\ldots,z_{n-1})z_n^j + L(z)\, P(z) .$$

A simple computation shows that $\tilde{G}$ and L are entire. (2) says that

(3) $$G \equiv \sum \tilde{G}^j\, z_n^j \qquad \mathrm{mod}\ PH(C) .$$

Now, we can determine the $\tilde{G}^j$ explicitly from (2) by applying the Lagrange interpolation formula. This leads to an expression

(4) $$\tilde{G}^j(z_1,\ldots,z_{n-1}) = \frac{D_j(z_1,\ldots,z_{n-1})}{D(z_1,\ldots,z_{n-1})}$$

where D^2 is a polynomial which is not identically zero and D_j is a sum of terms of the form

(5) $$A(z)\,\partial^u/\partial z_n^u\, G(z_1,\ldots,z_{n-1}, z_n)$$

for $z_n \in V_j$, $0 \leqq u \leqq t_j-1$ and where A is an algebraic function.

By property (b) of localizability $A(z)$ plays no role. By our choice of z_n we see that there is a constant B so that for $z \in V_j$

(6) $$z_n \leqq B[1+|(z_1,\ldots,z_{n-1})|] .$$

Using (6) we deduce that for some $B' \geqq 1$ and all $k \in K$

(7) $$\tilde{G}^j(z_1,\ldots,z_{n-1}) = o[k(B'[|(z_1,\ldots,z_{n-1})|])] .$$

Thus if the set $\{k(B'z)\}$ is an analytic uniform structure for W then

(8) $$\tilde{G}^j \in \hat{W}'(z_n = 0) .$$

It follows from (2) that $\tilde{G} = \sum \tilde{G}^j z_n^j \in \hat{W}'$ is an extension of $\underline{F}$ to C; that is,

$$\varphi_{\underline{V}}\, \tilde{G} = \underline{F} .$$

We have also another representation for the quotient space $\hat{W}'/\underline{P}\hat{W}'$, namely

(9) $$\hat{W}'/\underline{P}\hat{W}' = \hat{W}'(z_n=0) \oplus \cdots \oplus \hat{W}'(z_n=0) \quad (m \text{ factors})$$

$$= \hat{W}'(z_n=0)^m .$$

The isomorphism is given by writing any $G \in \hat{W}'$ in the form (1) and then using the reduction process to deduce (2).

It is readily verified that the process of passing from $\underline{F}$ to $\sum \tilde{G}^j z_n^j$ is one-one, and continuous and provides an inverse for $\varphi_{\underline{V}}$ on $\hat{W}'/\underline{P}\hat{W}'$.

Next we assume, with the same hypotheses on K that $\underline{P} = (P_1, P_2)$. To study the quotient space

$$\hat{W}'/\underline{P}\hat{W}'$$

we study the quotient

$$(\hat{W}'/P_1\hat{W}')/P_2\hat{W}' \; .$$

For this purpose we use the representation (9). Then multiplication by P_2 on $\hat{W}'$ defines on $\hat{W}'(z_n=0)^m$ a matrix

$$\not{P}_2 = (p_2^{ij})_{i,j=1,\ldots,m}$$

where p_2^{ij} are polynomials.

Thus to study $\hat{W}'/\underline{P}\hat{W}'$ we have roughly the same problem as before, except that

(1) The number of variables is $n-1$;

(2) We have to form the quotient of $W'(z_n=0)^m$ by the <u>matrix</u> of polynomials (p_2^{ij}).

Now (1) suggests that an induction argument will work. (2) is indeed a complication and requires a difficult algebraic theory. This theory leads to a prescription of $\underline{V}$; it will not be given here.

Next we pass to the case of general W; i.e., the functions $k(z)$ are no longer assumed to depend on $|z|$. We apply the above procedure locally and then need a piecing together process:

Let $z^o \in C$ and let $N > 0$. By $S(z^o,N)$ we denote the cube

$$|R\, z_j - R\, z_j^o| < N$$

$$|I\, z_j - I\, z_j^o| < N$$

where R and I denote the real and imaginary parts.

If $\underline{V}$ is a multiplicity variety then the intersection of $\underline{V}$ with $S(z^o,N)$ is defined in an obvious way. If $\underline{F} \in H(\underline{V})$ and $G \in H(C^n)$, we say G extends $\underline{F}$ if $\varphi_{\underline{V}}\, G = \underline{F}$. Similarly we speak of an extension of $\underline{F}$ over $S(z^o,N)$. The main local extension result we have is

<u>Theorem 3</u>. Let $(P_1,\ldots,P_r)$ as usual, let $\underline{V}$ be the multiplicity variety associated, and let $N > 0$. Choose $\underline{F} \in \hat{W}'(\underline{V})$. Then for each z^o we can find a function $G(z^o,N,z)$ which is holomorphic on $S(z^o,N)$ and which extends $\underline{F}$ from $\underline{V} \cap S(z^o,N)$ to $S(z^o,N)$. For any $k \in K$ we have

$$\sup_{z \in S(z^o,N)} |G(z^o,N,z)|/k(z^o) \to 0 \quad \text{as} \quad |z^o| \to \infty .$$

In addition to Theorem 3 we shall need another result which is proven in the same way as Theorem 3:

<u>Theorem 4</u>. Let $(P_1,\ldots,P_r)$ be polynomials and let $N > 0$. Then there exist positive constants A, N' which depend only on $P_1,\ldots,P_r$ and N with the following properties: Let F be holomorphic on $S(z^o,N')$ and satisfy

$$F(z) = \sum G_j(z)\, P_j(z) \qquad z \in S(z^o,N')$$

where G_j are holomorphic on $S(z^o,N')$. Then there exist $\tilde{G}_j$ which are holomorphic on $S(z^o,N)$ such that

$$F(z) = \sum \tilde{G}_j(z)\, P_j(z) \qquad z \in S(z^o,N)$$

$$|\tilde{G}_j(z)| \leq A \max_{z' \in S(z^o,N')} [|F(z')|(1+|z'|)^A] .$$

The importance of Theorems 3 and 4 is that, due to the conditions for localizability, all local questions can be treated satisfactorily. We shall now explain how to pass from the local to the global.

In general, the N' of Theorem 4 is $> N$; however, for the purpose of simplifying the exposition we shall assume that N' can be chosen equal to N. The modifications if $N' > N$ are easy.

We shall use α, α', etc. to denote lattice points in C. Let $N > 1$ and suppose that for each α we are given a function $f(\alpha,z)$ which is holomorphic on $S(\alpha,N)$. Then we call f a <u>cochain</u> (for N).

We say that f is a <u>nice</u> <u>cochain</u> if for every $k \in K$

$$\sup_{z \in S(\alpha,N)} |f(\alpha,z)|/k(z) \to 0 \qquad \text{as} \quad |\alpha| \to \infty .$$

the nice cochains form a space $\hat{W}'(N)$ with a natural topology. We say that $f \in \hat{W}'(N,\underline{P})$ if whenever $S(\alpha,N) \cap S(\alpha' N)$ is non-empty

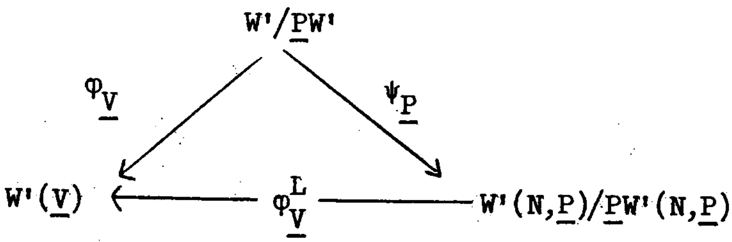
$$f(\alpha,z) - f(\alpha',z) = \sum G_j(\alpha,\alpha',z)P_j(z)$$

for $z \in S(\alpha,N) \cap S(\alpha',N)$ where G_j are homomorphic on $S(\alpha,N) \cap S(\alpha',N)$. Finally, we define the quotient space $\hat{W}'(N,\underline{P})/\underline{P}\hat{W}'(N,\underline{P})$. All these spaces have natural topologies defined by the functions $k \in K$.

The restriction map $\varphi_{\underline{V}}$ defines a local restriction map $\varphi^L_{\underline{V}}: \hat{W}'(N,\underline{P}) \to \hat{W}'(\underline{V})$ which by construction is seen to be a continuous map of $\hat{W}'(N,\underline{P})/\underline{P}\hat{W}'(N,\underline{P}) \to \hat{W}'(\underline{V})$. There is also a natural continuous map

$$\psi_{\underline{P}}: \hat{W}'/\underline{P}\hat{W}' \to \hat{W}'(N,\underline{P})/\underline{P}\hat{W}'(N,\underline{P}) .$$

<u>Theorem</u> <u>V</u>. In the diagram (for N large enough)



all maps are topological isomorphisms <u>onto</u> and the above diagram and all variations obtained by replacing any map by its inverse are commutative.

Clearly, Theorem V contains Theorem IV.

Theorem 3 shows that $\varphi_{\underline{V}}^{L}$ has a continuous inverse. Thus the proof of Theorem V will be complete if we can produce a continuous inverse for $\psi_{\underline{P}}$. This is constructed by "piecing together" one real dimension at a time by a method which is analogous to the one used by Oka when there are no bounds. It is here that the third condition for localizability is needed.

By using the fact that $\psi_{\underline{P}}$ is a topological isomorphism onto we deduce

Theorem II (complete version). Let W be localizable and analytically uniform; let $g_1, \ldots, g_r \in W$. A necessary and sufficient condition that there should exist an $f \in W$ satisfying $\partial_j f = g_j$ for $j = 1, \ldots, r$ is that the g_j should satisfy the obvious compatibility conditions:

$$\sum d_j g_j = 0$$

whenever d_j are linear constant coefficient partial differential operators with

$$\sum d_j \partial_j = 0 \ .$$

We shall now give several applications of our theory.

Application to Pólya's Theorem.

Let $n = 1$ and let us consider x as a complex variable. Let Ω be a convex set in the x space and denote by $H(\Omega)$ the space of functions holomorphic on Ω. The result of Pólya gives a description of $\hat{H}'(\Omega)$ as follows: Write $z = |z|e^{i\theta}$. Then there is a function $\varphi(\theta)$ depending on Ω such that an entire function $F(z)$ belongs to $\hat{H}'(\Omega)$ if and only if there is a $B < 1$ and an $A > 0$ such that

$$|F(z)| \leq A\, e^{B|z|\varphi(\theta)}.$$

The method of Pólya can be applied to the case $n > 1$ but, apparently, only to those convex sets Ω which are products of convex sets in the complex x_j planes. We shall show how to extend Pólya's theorem to the general case:

Let $n = 2m$, and let Ω be a convex set in R. (We return to the usual notation that x_j are real variables.) Let $E(\Omega)$ denote the space of C^∞ functions on Ω with its usual topology. Then the Paley-Wiener-Schwartz theorem states that there is a function ψ of n real variables which is determined by Ω such that an entire function F belongs to $\hat{E}'(\Omega)$ if and only if there is a $B < 1$ and an $A > 0$ so that

(1) $$|F(z)| \leq A(1+|z|)^A\, e^{B\psi(Iz_1,\ldots,Iz_n)}$$

where I denotes the imaginary part.

We note that if we consider R^n as complex space C^m with the coordinates $x_j + ix_{m+j}$ then $H(\Omega)$ is the subspace of $E(\Omega)$ defined by the Cauchy-Riemann equations: $\partial_j f = 0$ where $\partial_j = \partial/\partial x_j + i\partial/\partial x_{m+j}$.

By Fourier transform and Theorem IV we claim that $\hat{H}'(\Omega)$ consists of those entire F on C^m for which there is a $B' < 1$ and an $A' > 0$ so that

$$F(\omega) \leq A' e^{B'\psi(I\omega_1,\ldots,I\omega_m,-R\omega_1,\ldots,-R\omega_m)} \tag{2}$$

For, the multiplicity variety $\underline{V}$ is the algebraic variety V of common zeros of $P_j(z) = z_j + iz_{m+j}$ for $j = 1,\ldots,m$. If we write

$$z_j = \xi_j + i\eta_j$$

then on V we have

$$\xi_j = -\eta_{m+j}\ ,\quad \eta_j = \xi_{m+j}\ .$$

Thus

$$\psi(\eta_1,\ldots,\eta_n) = \psi(\eta_1,\ldots,\eta_m,\ -\xi_1,\ldots,-\xi_m)$$

and the result follows by calling $\omega_j = \xi_j + i\eta_j$.

Application to Hyperbolicity and Ellipticity

Let W and W_1 be localizable analytically uniform (l.a.u) spaces, and let $\underline{\partial} = \partial_1,\ldots,\partial_r$. Denote by $W(\underline{\partial})$ (or $W_1(\underline{\partial})$) the space of $f \in W$ (or $f \in W_1$) satisfying $\partial_j f = 0$ for $j = 1,\ldots,r$. We are often interested in the question as to when $W(\underline{\partial}) = W_1(\underline{\partial})$. By Theorem IV this is the same as the question as to whether $\hat{W}'(\underline{V})$ and $\hat{W}_1'(\underline{V})$ are the same, where $\underline{V}$ is the multiplicity variety associated with $\underline{P}$. To resolve this, we have only to show that the functions $\{k\}$ and $\{k_1\}$ are the same on V, the algebraic variety of common zeros of $P_1, \ldots, P_r$ where $\{k\}$ and $\{k_1\}$ are analytic uniform structures for W and W_1 respectively.

For example, we take W = space of distributions on R and W_1 the space of C^∞ functions on R. The system $\underline{\partial}$ is called <u>hypoelliptic</u> if $W(\underline{\partial}) = W_1(\underline{\partial})$. By inspection of the analytic uniform structures we deduce

<u>Theorem 5</u>. $\underline{\partial}$ is hypoelliptic if and only if

$$\liminf_{z \in V,\ |z| \to \infty} |Iz|/\log(1+|z|) = \infty .$$

This theorem (for $r = 1$) was found first by Hörmander and later independently by the author. The case $r > 1$ can be reduced to that of $r = 1$ by a theorem of Lech.

As a second example, let us divide the variables x_j into two classes: $x_1, \ldots, x_\ell$ and $x_{\ell+1}, \ldots, x_n$. Let $b > 0$ and let W_1 be the space of functions which are C^∞ in $|x_1| < b, \ldots, |x_\ell| < b$ and all $x_{\ell+1}, \ldots, x_n$. W is the space of C^∞ functions on R. We say that $\underline{\partial}$ is <u>hyperbolic in</u> $(x_1, \ldots, x_\ell)$ if $W(\underline{\partial}) = W_1(\underline{\partial})$. We have

<u>Theorem 6</u>. A necessary and sufficient condition that $\underline{\partial}$ by hyperbolic in $(x_1, \ldots, x_\ell)$ is that there exists a $c > 0$ so that

$$|Iz_j| \le c(1+|Iz_{\ell+1}| + \cdots + |Iz_n|) \quad \text{for } j = 1, \ldots, \ell$$

for $z \in V$.

Extension to Convolution Systems

We can obtain some of the above results for systems of convolution equations, that is, where the ∂_j are replaced by certain convolution operators. In particular, we have the following

Theorem 7. Let ∂ be a linear differential difference operator with constant coefficients. Then every $f \in W$ which satisfies $\partial f = 0$ has a Fourier representation

$$f(x) = \int e^{ix \cdot z} \, d\mu(z)/k(z)$$

where $k \in K$ and μ is a bounded measure on C whose support is contained in the set

$$V^* = \{z | \operatorname{dist}(z,V) \leq 1\} .$$

V is the set of zeros of $P(z) = 0$, where P is the Fourier transform of ∂.

Hörmander

CHAPTER VIII

DIFFERENTIAL OPERATORS WITH NON-SINGULAR CHARACTERISTICS

8.1 Necessary conditions for the main estimates. Let $P(x,D)$ be a differential operator of order m defined in an open set $\Omega \subset R_n$. In this chapter we are concerned with proving and applying estimates of the form

$$(8.1.1) \qquad \tau \sum_{|\alpha| \leq m-1} \int \binom{m-1}{\alpha} |D^\alpha u|^2 e^{2\tau\varphi}\, dx \leq K_1 \int |P(x,D)u|^2 e^{2\tau\varphi}\, dx\,,$$

$$u \in C_0^\infty(\Omega)\,, \quad \tau > \tau_0\,,$$

and also weaker estimates of the form

$$(8.1.2) \qquad \tau \sum_{|\alpha| = m-1} \int \binom{m-1}{\alpha} |D^\alpha u|^2 e^{2\tau\varphi}\, dx \leq K_2 \int |P(x,D)u|^2 e^{2\tau\varphi}\, dx +$$

$$+ K_3 \sum_{|\alpha| \leq m-2} \tau^{2(m-|\alpha|)-1} \binom{m-2}{\alpha} \int |D^\alpha u|^2 e^{2\tau\varphi}\, dx, \quad u \in C_0^\infty(\Omega), \ \tau > \tau_0.$$

(The polynomial coefficients have been introduced here for convenience later on.) From inequalities of the form (8.1.1) we shall obtain uniqueness theorems for the Cauchy problem (see section 8.9) and from the estimates (8.1.2) we deduce existence and regularity theorems for solutions of the differential equation $P(x,D)u = f$ (see sections 8.7 and 8.8). In this section, we shall only prove necessary conditions for the validity of (8.1.1) or (8.1.2). We begin by studying two simple but very useful examples.

Lemma 8.1.1. The estimate

$$(8.1.3) \qquad \int |u|^2 e^{at^2}\, dt \le C\int |du/dt|^2 e^{at^2}\, dt, \quad u \in C_0^\infty(R_1)$$

where a and C are real constants, $C > 0$, is valid if and only if

$$(8.1.4) \qquad 2aC \ge 1 .$$

Proof. If $a < 0$ and we take $u_\epsilon(t) = v(\epsilon t)$ where $v \in C_0^\infty(R_1)$ and $v(0) = 1$, it follows when $\epsilon \to 0$ that

$$\int |u_\epsilon|^2 e^{at^2}\, dt \to \int e^{at^2}\, dt; \ \int |du_\epsilon/dt|^2 e^{at^2}\, dt \to 0 ,$$

which contradicts (8.1.3). If $a = 0$, the left-hand side of (8.1.3) would instead be proportional to $1/\epsilon$ and the right-hand side proportional to ϵ, which again shows that (8.1.3) cannot hold. Now if $a > 0$ we set $v(t) = u(t)\, e^{at^2/2}$ and obtain by means of an integration by parts

$$\int |u'(t)|^2 e^{at^2}\, dt = \int |v'(t)-atv(t)|^2\, dt = \int |v'(t)+atv(t)|^2\, dt +$$

$$+ 2a \int |v(t)|^2\, dt \ge 2a \int |u(t)|^2 e^{at^2}\, dt .$$

Since we can come arbitrarily close to the sign of equality if we let u approach the function e^{-at^2}, for example by taking $u_\epsilon(t) = e^{-at^2} v(\epsilon t)$ with v as above, the lemma is proved.

Lemma 8.1.2. The estimate

$$(8.1.5) \quad \iint |u|^2 e^{at^2 + 2bts + cs^2} \, dsdt \le$$

$$\le C\iint |\partial u/\partial s + i\partial u/\partial t|^2 e^{at^2 + 2bts + cs^2} \, dsdt, \; u \in C_o^\infty(R_2),$$

where a, b, c and C are real constants, $C > 0$, is valid if and only if

$$(8.1.6) \qquad 2(a+c)C \ge 1 .$$

Proof. Writing $\alpha = \frac{1}{2}(a+c)$ and $\beta = \frac{1}{2}(a-c)$ we make the substitution

$$u(s,t) = v(s,t)e^{\frac{1}{2}(\beta+ib)(s+it)^2} ,$$

which reduces (8.1.5) to the estimate

$$(8.1.5)' \quad \iint |v|^2 e^{\alpha(t^2+s^2)} \, ds\, dt \le C\iint |\partial v/\partial s+i\partial v/\partial t|^2 e^{\alpha(t^2+s^2)} \, ds\, dt ,$$

$$v \in C_o^\infty(R_2) .$$

As in the proof of Lemma 8.1.1, it follows that α must be > 0. Writing

$$v(s,t)e^{\frac{1}{2}\alpha(s^2+t^2)} = w(s,t) ,$$

we also obtain as there

$$\iint |\partial v/\partial s + i\partial v/\partial t|^2 e^{\alpha(t^2+s^2)} dsdt = \iint |\partial w/\partial s + i\partial w/\partial t - \alpha(s+it)w|^2 dsdt =$$

$$= \iint |\partial w/\partial s - i\partial w/\partial t + \alpha(s-it)w|^2 dsdt + 4\alpha \iint |w|^2 dsdt .$$

The sufficiency of (8.1.6) now follows immediately and to see that (8.1.6) is necessary we only have to let v approach $e^{-\alpha(s^2+t^2)}$. The proof is complete.

Combining the two previous lemmas we can now prove the following one.

Lemma 8.1.3. Let $A(x) = \sum_{j,k=1}^{n} a_{jk}x_j x_k$, where $a_{jk} = a_{kj}$, be a real quadratic form and $b = (b_1, \ldots, b_n)$ be a vector in C_n. Then the inequality

$$(8.1.7) \qquad \int |u|^2 e^A \, dx \leq C \int |\sum_1^n b_j D_j u|^2 e^A \, dx \, , \quad u \in C_o^\infty(R_n) \, ,$$

when C is a constant > 0, is valid if and only if

$$(8.1.8) \qquad 2C \sum_{j,k=1}^{n} a_{jk} b_j \bar{b}_k \geq 1 \, .$$

Proof: In view of the invariance of the result we may if b is proportional to a real vector assume that $b = (1, 0, \ldots, 0)$ and otherwise we may suppose that $b = (1, i, 0, \ldots, 0)$. Assuming, for example, that we have the second case, we choose $u(x) = v(x_1, x_2)w(x_3/\epsilon, \ldots, x_n/\epsilon)$ in (8.1.7), where v and w are in C_o^∞. Letting $\epsilon \to 0$ after dividing by ϵ^n, we obtain

$$(8.1.9) \qquad \iint |v|^2 e^{A(x_1, x_2, 0, \ldots, 0)} dx_1 dx_2 \leq$$

$$\leq C \iint |\partial v/\partial x_1 + i \partial v/\partial x_2|^2 e^{A(x_1, x_2, 0, \ldots, 0)} dx_1 dx_2 \, .$$

Hence it follows from Lemma 8.1.2 that $2C(a_{11} + a_{22}) \geq 1$, which is the same as (8.1.8). Similarly, if $b = (1, 0, \dots, 0)$, we also obtain using Lemma 8.1.1 that (8.1.8) is necessary for (8.1.7) to hold. Since in these coordinate systems the sufficiency also follows immediately from Lemmas 8.1.1 and 8.1.2, the proof is complete.

We shall now prove conditions which are necessary for (8.1.1) or (8.1.2) to hold. In doing so, we assume that the coefficients of $P(x,D)$ are bounded, that the coefficients in the principal part $P_m(x,D)$ are in $C^1(\Omega)$ and that φ is real valued and belongs to $C^2(\Omega)$. Keeping the notations of Chapter VI, we shall write

$$P_m^{(j)}(x,\xi) = \partial P_m(x,\xi)/\partial\xi_j \ , \quad P_{m,j}(x,\xi) = \partial P_m(x,\xi)/\partial x_j$$

and similarly for higher order derivatives when they occur.

<u>Theorem 8.1.1</u>. Let $N = \operatorname{grad} \varphi(x)$ where $x \in \Omega$ and let $\zeta = \xi + i\sigma N$ with $\xi \in R_n$ and $0 \neq \sigma \in R_1$ satisfy the characteristic equation

(8.1.10) $$P_m(x,\zeta) = 0 \ .$$

If (8.1.1) is valid it then follows that

(8.1.11) $$|\zeta|^{2(m-1)} \leq 2K_1 \left\{ \sum_{j,k=1} \frac{\partial^2\varphi}{\partial x_j \partial x_k} P_m^{(j)}(x,\zeta) \overline{P_m^{(k)}(x,\zeta)} + \right.$$

$$\left. + (2i\sigma)^{-1} \sum_1^n (P_{m,k}(x,\zeta)\overline{P_m^{(k)}(x,\zeta)} - P_m^{(k)}(x,\zeta)\overline{P_{m,k}(x,\zeta)}) \right\} ;$$

and if (8.1.2) holds it follows that

$$(8.1.12)\quad |\zeta|^{2(m-1)}-K_3\sigma^2(|\zeta|^2+\sigma^2)^{m-2} \le 2K_2\Big\{\sum_{j,k=1}^{n} \frac{\partial^2\varphi}{\partial x_j\partial x_k} P_m^{(j)}(x,\zeta)\,\overline{P_m^{(k)}(x,\zeta)} +$$

$$+ (2i\sigma)^{-1}\sum_1^n (P_{m,k}(x,\zeta)\overline{P_m^{(k)}(x,\zeta)} - P_m^{(k)}(x,\zeta)\,\overline{P_m^{(k)}(x,\zeta)})\Big\} ,$$

when the left-hand side is non-negative.

<u>Proof</u>: It is no restriction to assume that $x = 0$ and that $\varphi(0) = 0$. Take a function $w \in C^\infty$ such that

$$w(x) = < x,\zeta > + O(|x|^2),\ x \to 0 ,$$

and set with $\psi \in C_o^\infty(R_n)$

$$u_\tau(x) = \exp(i\tau w(x)/\sigma)\psi(x\sqrt{\tau}) .$$

Note that

$$\varphi(x) - \operatorname{Im} w(x)/\sigma = A(x) + o(|x|^2)$$

where A is a quadratic form, and that the definition of u_τ is chosen so that $\tau A(x)$ is kept under control in the support of u_τ. From Leibniz' formula we obtain

$$P(x,D)u_\tau(x) = (\tau/\sigma)^{m-1} \exp(i\tau w/\sigma)\left\{(\tau/\sigma)P_m(x,\text{grad}w)\psi(x\sqrt{\tau}) + \right.$$

$$\left. + \sum_1^n P_m^{(j)}(x,\text{grad}w)\sqrt{\tau}\,(D_j\psi)(x\sqrt{\tau}) + O(1)\right\} ,$$

where $O(1)$ denotes a function which is uniformly bounded when $\tau \to \infty$. From (8.1.10) it follows that for some $a \in C_n$ we have

$$P_m(x, \text{grad } w) = \langle x,a \rangle + o(|x|) \quad \text{when} \quad x \to 0 .$$

Passing to the limit after introducing $x\sqrt{\tau}$ as a new variable, we now obtain

$$\tau^{n/2}(\sigma/\tau)^{2(m-1)}\,\tau^{-1} \int |P(x,D)u_\tau|^2 e^{2\tau\varphi}\,dx \to$$

$$\to \int |\langle x,a \rangle \psi/\sigma + \sum_{j=1}^n P_m^{(j)}(0,\zeta)\,D_j\psi|^2 e^{2A}\,dx, \quad \tau \to \infty .$$

Similarly, we find that

$$\tau^{n/2}(\sigma/\tau)^{2(m-1)} \sum_{|\alpha| = m-1} \int \binom{m-1}{\alpha} |D^\alpha u_\tau|^2 e^{2\tau\varphi}\,dx \to |\zeta|^{2(m-1)} \int |\psi|^2 e^{2A}\,dx$$

when $\tau \to \infty$. If (8.1.1) holds, we hence obtain the inequality

$$(8.1.13) \quad |\zeta|^{2(m-1)} \int |\psi|^2 e^{2A} dx \le K_1 \int |\langle x,a \rangle \psi/\sigma + \sum_{j=1}^n P_m^{(j)}(0,\zeta) D_j\psi|^2 e^{2A} dx ,$$

$$\psi \in C_0^\infty(R_n) .$$

In the same way, it follows from (8.1.2) that

$$(8.1.14)\quad |\zeta|^{2(m-1)}\int|\psi|^2\, e^{2A}dx \le K_2 \int|<x,a>\psi/\sigma + \sum_{j=1}^{n} P_m^{(j)}(0,\zeta)D_j\psi|^2\, e^{2A}dx$$

$$+ K_3(|\zeta|^2+\sigma^2)^{(m-2)}\,\sigma^2\int|\psi|^2\, e^{2A}dx\ ,\quad \psi\in C_o^\infty(R_n)\ .$$

Let us first analyze (8.1.13). If $P_m^{(j)}(0,\zeta) = 0$ for all j, we immediately find that (8.1.13) cannot hold by just taking any ψ with support in the neighbourhood of the origin where $K_1|<x,a>|^2 < |\zeta|^{2(m-1)}\sigma^2$. Hence $P_m^{(j)}(0,\zeta) \neq 0$ for some j. We can therefore choose w so that $a = 0$. For this means that $D_kP_m(x, \text{grad } w) = 0$ for $k = 1, \dots, n$ when $x = 0$, that is, with $w_{jk} = \partial^2 w/\partial x_j\partial x_k$,

$$(8.1.15)\qquad \sum_{j=1}^{n} P_m^{(j)}(0,\zeta)w_{jk}(0) + P_{m,k}(0,\zeta) = 0,\ k = 1, \dots, n\ ,$$

and if some $P_m^{(j)}(0,\zeta) \neq 0$, this can obviously be satisfied by a suitable choice of $w_{jk}(0)$. (See also the proof of Lemma 6.1.3.) With w chosen in this way, it follows from (8.1.13) in view of Lemma 8.1.3 that

$$(8.1.16)\quad |\zeta|^{2(m-1)} \le 2K_1 \sum_{j,k=1}^{n} [\varphi_{jk}(0) - \operatorname{Im} w_{jk}(0)/\sigma]\, P_m^{(j)}(0,\zeta)\, \overline{P_m^{(k)}(0,\zeta)}\ .$$

Now we have, in view of (8.1.15),

$$\sum_{j,k=1}^{n} w_{jk}(0)\, P_m^{(j)}(0,\zeta)\, \overline{P_m^{(k)}(0,\zeta)} = -\sum_{k=1}^{n} P_{m,k}(0,\zeta)\, \overline{P_m^{(k)}(0,\zeta)}$$

and using this to eliminate w in the right-hand side of (8.1.16) we obtain (8.1.11) for $x = 0$.

Next assume that (8.1.14) holds. When $K_3 \frac{\sigma^2}{|\zeta|^2}\left(1 + \frac{\sigma^2}{|\zeta|^2}\right)^{m-2} < 1$, it follows as before that $P_m^{(j)}(0,\zeta) \neq 0$ for some j and the above argument again proves that (8.1.12) is valid. The proof is complete.

When the coefficients are constant, the right-hand side of (8.1.11) and (8.1.12) simplifies to $\Sigma\, \partial^2\varphi/\partial x_j \partial x_k\, P_m^{(j)}(\xi)\, \overline{P_m^{(k)}(\xi)}$. Thus these inequalities are in fact conditions of uniform convexity and subharmonicity of φ along certain directions and two-dimensional planes. Also note the obvious analogy with the E. E. Levi condition for pseudo-convexity in the theory of several complex variables. A further discussion of the geometric significance will be given in section 8.4.

It is rather clear from the proof of Theorem 8.1.1 that the right-hand side of (8.1.11) and (8.1.12) is invariant for changes of coordinates. However, we shall give an explicit proof since the invariance will be used in an essential way later on. To do so we let w be a function and σ a real number $\neq 0$ such that $\varphi - \operatorname{Im} w/\sigma$ vanishes of the second order at a point x. Then the second derivatives

$$\partial^2\varphi/\partial x_j \partial x_k - \frac{1}{\sigma} \operatorname{Im} \partial^2 w/\partial x_j \partial x_k$$

at that point form a symmetric covariant tensor. Furthermore, we have already noted (see section 5.3) that $P_m(x, \operatorname{grad} w)$ is invariantly defined as the coefficient of t^m in the polynomial $e^{-itw}P(x,D)e^{itw}$ of t. Hence $P_m^{(j)}(x, \operatorname{grad} w)$ is a contravariant vector, for if ψ is another function, it follows that

$$\sum_1^n P_m^{(j)}(x, \operatorname{grad} w)\partial\psi/\partial x_j = \frac{d}{d\epsilon} P_m[x, \operatorname{grad}(w+\epsilon\psi)]_{/\epsilon = 0}$$

is an invariant. Combination of these two facts shows that

$$\sum_{j,k=1}^{n} (\partial^2\varphi/\partial x_j\partial x_k - \frac{1}{\sigma}\operatorname{Im}\partial^2 w/\partial x_j\partial x_k)\, P_m^{(j)}(x, \text{grad } w)\, \overline{P_m^{(k)}(x, \text{grad } w)}$$

is an invariant. Now the invariance of $P_m(x, \text{grad } w)$ shows that

$$\frac{\partial}{\partial x_k} P_m(x, \text{grad } w) = \sum_1^n P_m^{(j)}(x, \text{grad } w)\, \partial^2 w/\partial x_j\partial x_k + P_{m,k}(x, \text{grad } w)$$

is a covariant vector, hence the scalar product with the contravariant vector with components $\overline{P_m^{(k)}(x, \text{grad } w)}$ is invariant. It follows that

$$\sum_{j,k=1}^{n} \partial^2\varphi/\partial x_j\partial x_k\, P_m^{(j)}(x, \text{grad } w)\, \overline{P_m^{(k)}(x, \text{grad } w)} +$$

$$+ \operatorname{Im}\sum_1^n P_{m,k}(x, \text{grad } w)\, \overline{P_m^{(k)}(x, \text{grad } w)}/\sigma$$

is invariant, and since this is precisely the right-hand side of (8.1.11) and (8.1.12) if $\text{grad } w = \zeta$, our assertion is proved. Also note that substituting $a(x)P(x,D)$ or $P(x,D)a(x)$ for $P(x,D)$, where $a(x)$ is a non-vanishing function in C^1 or C^m, only means multiplying the right-hand side of (8.1.11) and (8.1.12) by the positive quantity $|a|^2$ when $P_m(x,\zeta) = 0$.

8.2 Differential quadratic forms. The essential point in our proof of inequalities of the form (8.1.1) and (8.1.2) is an integration by parts in the integral $\int |P(x,D)u|^2 e^{2\tau\varphi} dx$. As a preparation we shall in this section make a systematic study of partial integration in such integrals.

First consider a "sesqui-linear" form in the derivatives of a function u,

$$(8.2.1) \qquad \sum_{\alpha,\beta} a_{\alpha\beta} D^\alpha u \overline{D^\beta u}$$

where $a_{\alpha\beta}$ are constants and the sum is finite. Such an expression we call a <u>differential quadratic form</u> with constant coefficients, and we associate the form with the polynomial

$$(8.2.2) \qquad F(\zeta,\bar{\zeta}) = \sum a_{\alpha\beta} \zeta^\alpha \bar{\zeta}^\beta , \quad \zeta \in C_n .$$

This is a complex valued polynomial for the underlying real structure in C_n and it is obvious that the correspondence between the form (8.2.1) and the polynomial (8.2.2) is one to one. Since for $u(x) = e^{i\langle x,\zeta\rangle}$ the form (8.2.1) becomes $e^{-2\langle x, \operatorname{Im} \zeta\rangle} F(\zeta,\bar{\zeta})$, the correspondence is also invariant for linear changes of coordinates. (The variable ζ is transformed as a dual variable of x.)

From now on we may thus use the notation

$$F(D,\bar{D})u\bar{u} = \sum a_{\alpha\beta} D^\alpha u \overline{D^\beta u}$$

if F is defined by (8.2.2). The form will be said to be of (double) order $(\mu;m)$ if in (8.2.2) we have $|\alpha| + |\beta| \le \mu$ and $|\alpha| \le m$, $|\beta| \le m$

when $a_{\alpha,\beta} \neq 0$. Sometimes we refer to μ as the total order and to m as the separate order of F. We may, of course, always assume that $m \leq \mu \leq 2m$.

Let $G^k(D,\bar{D})u\bar{u}$, $k = 1, \ldots, n$, be differential quadratic forms and let $F(D,\bar{D})u\bar{u}$ be the divergence of the vector with these components, that is,

(8.2.3) $$F(D,D)u\bar{u} = \sum_1^n \frac{\partial}{\partial x_k} G^k(D,\bar{D})u\bar{u} .$$

Since Leibniz' rule gives

$$\frac{\partial}{\partial x_k}(u\bar{u}) = i[(D_k u)\bar{u} - u\,\overline{(D_k u)}] ,$$

the identity (8.2.3) is equivalent to the algebraic identity

(8.2.4) $$F(\zeta,\bar{\zeta}) = i \sum_1^n (\zeta_k - \bar{\zeta}_k)\, G^k(\zeta,\bar{\zeta}) .$$

From (8.2.4) it follows that

(8.2.5) $$F(\xi,\xi) = 0 , \quad \xi \in R_n ,$$

if F can be represented as a divergence (8.2.3). We shall now prove the sufficiency of (8.2.5). First we only note that the interest of the values of $F(\zeta,\bar{\zeta})$ for real ζ is also shown by the formula

$$\int F(D,\bar{D})u\bar{u}dx = (2\pi)^{-n} \int F(\xi,\xi)|\hat{u}(\xi)|^2 d\xi , \quad u \in C_o^\infty(R_n) ,$$

which follows from Parseval's formula.

Lemma 8.2.1. If (8.2.5) is fulfilled, it follows that there exist differential quadratic forms $G^k(D,\overline{D})u\overline{u}$ such that (8.2.3) is valid. We then have

$$G^k(\xi,\xi) = -\frac{1}{2}\frac{\partial}{\partial\eta_k} F(\xi+i\eta,\ \xi-i\eta)/_{\eta=0}, \quad \xi \in R_n . \tag{8.2.6}$$

If F is of double order $(\mu;m)$ it is always possible to choose G^k of double order $(\mu-1;m)$ and if $\mu < 2m$ one can even choose G^k of order $(\mu-1;m-1)$.

Proof. The expansion of $F(\xi+i\eta,\ \xi-i\eta)$ in powers of ξ and η contains no terms independent of η if (8.2.5) is fulfilled; hence we can find polynomials $g^k(\xi,\eta)$ such that

$$F(\xi+i\eta,\ \xi-i\eta) = \sum \eta_k\, g^k(\xi,\eta) . \tag{8.2.7}$$

Writing $\zeta = \xi + i\eta$ we have $\xi = \frac{1}{2}(\zeta + \overline{\zeta})$ and $i\eta = \frac{1}{2}(\zeta - \overline{\zeta})$. Thus (8.2.7) gives an identity of the form (8.2.4). Furthermore, if we take $\zeta = \xi + i\eta$ in (8.2.4), differentiate with respect to η_k and put $\eta = 0$ afterwards, we immediately obtain (8.2.6).

To prove the statement about the order of G^k we have to argue more carefully, however. Let us say that two polynomials $F_1(\zeta,\overline{\zeta})$ and $F_2(\zeta,\overline{\zeta})$ of order $(\mu;m)$ are congruent and write $F_1 \equiv F_2$ if $F = F_1 - F_2$ can be written in the form (8.2.4) with G^k of order $(\mu-1;\ m')$ where $m' = m-1$ if $\mu < 2m$ and $m' = m$ if $\mu = 2m$. We now claim that

$$(8.2.8) \qquad \zeta^{\alpha'} \bar{\zeta}^{\beta'} \equiv \zeta^{\alpha''} \bar{\zeta}^{\beta''}$$

if $\alpha' + \beta' = \alpha'' + \beta''$ and both sides are of order $(\mu;m)$, that is, $|\alpha'| + |\beta'| = |\alpha''| + |\beta''| \leq \mu$ and the lengths $|\alpha'|, \dots, |\beta''|$ are all $\leq m$. First let $\mu < 2m$. Then either $|\alpha'|$ or $|\beta'|$ is $< m$. If $|\alpha'| < m$ we can use the identity

$$\bar{\zeta}_j = \zeta_j - (\zeta_j - \bar{\zeta}_j)$$

and if $|\beta'| < m$ we can use the identity

$$\zeta_j = \bar{\zeta}_j + (\zeta_j - \bar{\zeta}_j)$$

to show that the congruence class of $\zeta^{\alpha'} \bar{\zeta}^{\beta'}$ does not change if one factor in $\bar{\zeta}^{\beta'}$ (resp. $\zeta^{\alpha'}$) is replaced by its complex conjugate. Repeated use of this procedure proves (8.2.8) when $\mu < 2m$. If $\mu = 2m$ this proof remains valid unless $|\alpha'| + |\alpha''| = |\beta'| + |\beta''| = \mu$, hence $|\alpha'| = |\alpha''| = |\beta'| = |\beta''| = m$. We can then use the identity

$$\zeta_j\bar{\zeta}_k - \zeta_k\bar{\zeta}_j = (\zeta_j - \bar{\zeta}_j)\bar{\zeta}_k - (\zeta_k - \bar{\zeta}_k)\bar{\zeta}_j$$

instead to replace one factor is $\zeta^{\alpha'}$ and one in $\bar{\zeta}^{\beta'}$ simultaneously by their complex conjugates without changing the congruence class of $\zeta^{\alpha'} \bar{\zeta}^{\beta'}$. This proves (8.2.8).

From (8.2.8) it follows that every F of order $(\mu;m)$ is congruent to a sum of the form

$$F_1(\zeta,\bar{\zeta}) = \sum a_{\alpha\beta}\, \zeta^\alpha\, \bar{\zeta}^\beta$$

of order $(\mu;m)$ where there never occur two different non-zero terms with the same multi-index sum $\alpha + \beta$. But if $F(\xi,\xi) = 0$ we have $F_1(\xi,\xi) = 0$ also, hence all $a_{\alpha\beta}$ must be 0. The proof is complete.

We shall now discuss differential quadratic forms with variable coefficients

$$\sum a_{\alpha\beta}(x)\, D^\alpha u\, \overline{D^\beta u} \tag{8.2.9}$$

which we again denote by $F(x,D,\bar{D})u\bar{u}$ where

$$F(x,\zeta,\bar{\zeta}) = \sum a_{\alpha\beta}(x)\, \zeta^\alpha\, \bar{\zeta}^\beta .$$

<u>Lemma 8.2.2</u>. Let $F(x,D,\bar{D})u\bar{u}$ be a differential quadratic form of degree $(\mu;m)$ with coefficients in $C^\nu(\Omega)$, $\nu \geq 1$, and assume that

$$F(x,\xi,\xi) = 0,\ x \in \Omega,\ \xi \in R_n . \tag{8.2.10}$$

Then there is a differential quadratic form $G(x,D,\bar{D})u\bar{u}$ of lower total order with coefficients in $C^{\nu-1}(\Omega)$ such that

$$\int F(x,D,\bar{D})u\bar{u}\, dx = \int G(x,D,\bar{D})u\bar{u}\, dx,\ u \in C_0^\infty(\Omega) . \tag{8.2.11}$$

G may always be chosen of order $(\mu-1;\ m)$ and if $2m > \mu$ we may choose G of order $(\mu-1;\ m-1)$. Furthermore, we have

$$(8.2.12) \qquad G(x,\xi,\xi) = \frac{1}{2}\sum_{1}^{n} \frac{\partial^2}{\partial x_k \partial \eta_k} F(x, \xi+i\eta, \xi-i\eta)/_{\eta=0} .$$

Proof. Let $F_1, \ldots, F_N$ be a basis in the finite dimensional vector space of all differential quadratic forms of order $(\mu;m)$ with constant coefficients, which satisfy (8.2.5). Then we can find differential quadratic forms G_j^k, $j = 1, \ldots, N$; $k = 1, \ldots, n$, of order $(\mu-1, m-1)$ if $\mu<2m$ and of order $(\mu-1, m)$ if $\mu = 2m$, so that

$$(8.2.13) \qquad F_j(D,\bar{D})\, u\bar{u} = \sum_{1}^{n} \frac{\partial}{\partial x_k} G_j^k(D,\bar{D})\, u\bar{u}, \quad j = 1, \ldots, N .$$

In view of (8.2.10) we may write with uniquely determined coefficients $a_j \in C^{\nu}(\Omega)$

$$F(x,D,\bar{D})\, u\bar{u} = \sum_{1}^{N} a_j(x)\, F_j(D,\bar{D})\, u\bar{u} .$$

Using (8.2.13) we thus obtain after an integration by parts that (8.2.11) is valid with

$$G(x,D,\bar{D})\, u\bar{u} = \sum_{j=1}^{N} \sum_{k=1}^{n} (-\partial a_j/\partial x_k)\, G_j^k(D,\bar{D})\, u\bar{u} .$$

Since (8.2.12) follows from (8.2.6), this completes the proof.

8.3 <u>Estimates for elliptic operators</u>. In this case we are only interested in proving an estimate of the form (8.1.1) for the consequences of an estimate (8.1.2) are then weaker than the results obtained in Chapter VII. Thus we leave it to the reader to verify by a slight modification of the proof of Theorem 8.3.1 that (8.1.12) implies (8.1.2) (with different constants K_2, K_3), but we shall prove in detail

<u>Theorem 8.3.1</u>. Let Ω be a bounded open set in R_n, φ a real valued function in $C^\infty(\bar{\Omega})$ with $\operatorname{grad}\varphi(x) \neq 0$ when $x \in \bar{\Omega}$, and $P(x,D)$ a differential operator of order m with bounded measurable coefficients in Ω and the coefficients of the principal part $P_m(x,D)$ in $C^1(\bar{\Omega})$. Assume further that P_m is elliptic in $\bar{\Omega}$, that is,

(8.3.1) $$P_m(x,\xi) \neq 0 \quad \text{if} \quad x \in \bar{\Omega},\ 0 \neq \xi \in R_n ,$$

and that

(8.3.2) $$\sum_{j,k=1}^{n} \partial^2\varphi/\partial x_j \partial x_k \, P_m^{(j)}(x,\zeta)\,\overline{P_m^{(k)}(x,\zeta)} + (2i\tau)^{-1}\sum_{k=1}^{n}[P_{m,k}(x,\zeta)\,\overline{P_m^{(k)}(x,\zeta)} - P_m^{(k)}(x,\zeta)\,\overline{P_{m,k}(x,\zeta)}] > 0$$

if $\zeta = \xi + i\tau \operatorname{grad}\varphi(x)$, with $x \in \bar{\Omega}$, $\xi \in R_n$ and $\tau \in R_1$, and the characteristic equation $P_m(x,\zeta) = 0$ is satisfied. Then there is a constant K such that

(8.3.3) $$\sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} \int |D^\alpha u|^2 e^{2\tau\varphi}dx \leq K\tau \int |P(x,D)u|^2 e^{2\tau\varphi}dx, \quad u \in C_\varphi^\infty(\Omega),$$

when τ is sufficiently large.

As a first step towards the proof of Theorem 8.3.1 we prove that the validity of (8.3.3) is entirely a local property. (Incidentally, this shows that Theorem 8.3.1 is also valid if Ω is an open set in a manifold instead of an open set in R_n.)

Lemma 8.3.1. Let Ω be a bounded open set and assume that every $x \in \bar{\Omega}$ has an open neighbourhood ω_x such that (8.3.3) is valid for some constant K_x when $u \in C_0^\infty(\Omega \cap \omega_x)$ and τ is sufficiently large. Then one can find K so that (8.3.3) is valid for all $u \in C_0^\infty(\Omega)$ when τ is large enough.

Proof. Choose a finite number of points x_j, $j = 1, \ldots, J$, in Ω such that the neighbourhoods ω_{x_j} cover the compact set $\bar{\Omega}$, and choose $\varphi_j \in C_0^\infty(\omega_{x_j})$ so that $\sum \varphi_j = 1$ in $\bar{\Omega}$. If $u \in C_0^\infty(\Omega)$ we then have $u = \sum u_j$ where $u_j = \varphi_j u \in C_0^\infty(\Omega \cap \omega_{x_j})$. Cauchy-Schwarz' inequality gives

$$|D^\alpha u|^2 \leq J \sum_1^J |D^\alpha u_j|^2 .$$

If (8.3.3) is valid for all functions in $C_0^\infty(\Omega \cap \omega_{x_j})$, $j = 1, \ldots, J$, we thus obtain with constants C and C'

$$\sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} \int |D^\alpha u|^2 e^{2\tau\varphi} dx \leq J \sum_j \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} \int |D^\alpha u_j|^2 e^{2\tau\varphi} dx \leq$$

$$\leq C\tau \sum_j \int |P(x,D)u_j|^2 e^{2\tau\varphi} dx \leq C'\tau \int (|P(x,D)u|^2 + \sum_{|\alpha| < m} |D^\alpha u|^2) e^{2\tau\varphi} dx ,$$

where the last estimate follows from Leibniz' formula

$$P(x,D)u_j = \varphi_j P(x,D)u + \sum_{|\alpha| \neq 0} [P^{(\alpha)}(x,D)u] \, D^\alpha \varphi_j / \alpha! \, .$$

When $\tau > \max(1, 2C')$, it follows that (8.3.3) is valid with $K = 2C'$.

We next show, which is still simpler, that lower order terms are irrelevant for the validity of (8.3.3).

<u>Lemma 8.3.2</u>. Let $r(x,D)$ be a differential operator of order $< m$, with bounded measurable coefficients. If the estimate (8.3.3) is valid for large τ, it will still be true for large τ with a larger constant K if $P(x,D)$ is replaced by $P(x,D) + r(x,D)$.

<u>Proof</u>. If we make the estimate

$$|P(x,D)u|^2 \leq 2|[P(x,D) + r(x,D)]u|^2 + C \sum_{|\alpha| < m} |D^\alpha u|^2$$

in the right-hand side of (8.3.3), the statement follows as in the proof of Lemma 8.3.1.

<u>Proof of Theorem 8.3.1</u>. It suffices to prove that (8.3.3) is valid locally when no lower order terms are present in $P(x,D)$. In doing so we assume that $0 \in \overline{\Omega}$ and have to prove that (8.3.3) holds for all $u \in C_o^\infty(\Omega_\delta)$, if $\Omega_\delta = \{x;\ x \in \Omega,\ |x| < \delta\}$ and δ is sufficiently small. Since by assumption $\operatorname{grad} \varphi(0) \neq 0$ and $\varphi \in C^\infty$ we can change coordinates in a neighbourhood of 0 so that φ becomes a linear function, $\varphi(x) = \langle x,N \rangle$, for we have already proved in section 8.1 that (8.3.2) is invariant for changes of coordinates and the invariance of (8.3.3) is obvious. With $v(x) = u(x)\, e^{\tau \langle x,N \rangle} = u(x)\, e^{-i \langle x, i\tau N \rangle}$ we now have

$$\int |P_m(x,D)u|^2 e^{2\tau <x,N>} dx = \int |P_m(x,D+i\tau N)v|^2 dx \geq$$

$$\geq \int (|P_m(x,D+i\tau N)v|^2 - |\overline{P}_m(x,D-i\tau N)v|^2) dx = \int F_\tau(x,D,\overline{D}) v\bar{v}\, dx ,$$

where

$$F_\tau(x,\zeta,\overline{\zeta}) = P_m(x,\ \zeta+i\tau N)\ \overline{P_m(x,\ \zeta+i\tau N)} - \overline{P}_m(x,\ \zeta-i\tau N)\ \overline{\overline{P}_m(x,\ \zeta-i\tau N)} .$$

The reason why we have made this trivial estimate is that $F_\tau(x,\xi,\xi) = 0$ for real ξ so that we can make an integration by parts using Lemma 8.2.2. Thus there exists a form

$$G_\tau(x,D,\overline{D}) = \sum_{0}^{2m-1} \tau^j G^{(j)}(x,D,\overline{D})$$

with continuous coefficients, such that $G^{(j)}$ is of order $(2m-j-1;m)$ and

(8.3.4) $$\int G_\tau(x,D,\overline{D}) v\bar{v}\, dx = \int F_\tau(x,D,\overline{D}) v\bar{v}\, dx \leq \int |P_m(x,D)u|^2 e^{2\tau <x,N>} dx .$$

Furthermore, (8.2.12) gives

(8.3.5) $$G_\tau(x,\xi,\xi) = i \sum_{1}^{n} [P_m^{(k)}(x,\ \xi+i\tau N)\ \overline{P_{m,k}(x,\ \xi+i\tau N)} -$$

$$- P_{m,k}(x,\xi+i\tau N)\overline{P_m^{(k)}(x,\xi+i\tau N)}] + 2\ \mathrm{Im}\ P_m(x,\xi+i\tau N) \sum_{1}^{n} \overline{P_{m,k}^{(k)}(x,\xi+i\tau N)}.$$

We shall now show that there are positive constants C_1 and C_2 such that

(8.3.6) $$|\xi+i\tau N|^{2m} \leq C_1 \tau\ G_\tau(0,\xi,\xi) + C_2 |P_m(0,\xi+i\tau N)|^2 ,\quad \tau \geq 0,\ \xi \in R_n .$$

In view of the homogeneity it is sufficient to prove (8.3.6) in the compact set M defined by $|\xi + i\tau N| = 1, \tau \geq 0$. In the subset M_0 of M where $P_m(0, \xi + i\tau N) = 0$ it follows from (8.3.1) that τ has a positive lower bound and (8.3.5) shows that (8.3.2) means precisely that $G_\tau(0,\xi,\xi) > 0$ there. Hence we can find C_1 so that (8.3.6) is valid with strict inequality when $(\xi,\tau) \in M_0$. For reasons of continuity (8.3.6) must remain valid in a neighbourhood V of M_0 and since $|P_m(0, \xi + i\tau N)|$ is bounded from below on the complement of V in M, we conclude that (8.3.6) holds if C_2 is chosen sufficiently large.

Multiplying (8.3.6) by $|\hat{v}(\xi)|^2$ and integrating, we now obtain

(8.3.7)
$$(2\pi)^{-n} \int |\hat{v}(\xi)|^2 |\xi+i\tau N|^{2m} d\xi \leq C_1 \tau \int G_\tau(0,D,\bar{D})v\bar{v}\, dx + $$
$$+ C_2 \int |P_m(0,D+i\tau N)v|^2\, dx .$$

If ϵ is any given positive number and δ is sufficiently small, the continuity of the coefficients of G_τ and of P_m, Schwarz' inequality and the fact that terms in G_τ containing a factor τ^j are of order $(2m-1-j;\, m)$ now gives

(8.3.8)
$$(2\pi)^{-n} \int |\hat{v}(\xi)|^2 |\xi+i\tau N|^{2m} d\xi \leq C_1 \tau \int G_\tau(x,D,\bar{D})v\bar{v}\, dx +$$
$$+ C_2 \int |P_m(x,D+i\tau N)v|^2 dx + \epsilon(2\pi)^{-n} \int |\hat{v}(\xi)|^2 |\xi+i\tau N|^{2m} d\xi ,$$
$$u \in C_0^\infty(\Omega_\delta) ,$$

in view of the trivial estimates

$$(|N|^2\tau^2)^{m-|\alpha|} \int |D^\alpha v|^2 \, dx \leq \int |\hat{v}(\xi)|^2 \, |\xi + i\tau N|^{2m} \, d\xi, \quad |\alpha| \leq m ,$$

and

$$\tau^{2(m-|\alpha|)} \int |D^\alpha u|^2 e^{2\tau < x,N >} dx = (2\pi)^{-n} \tau^{2(m-|\alpha|)} \int |(\xi+i\tau N)^\alpha|^2 |\hat{v}(\xi)|^2 \, d\xi$$

$$\leq |N|^{-2(m-|\alpha|)} (2\pi)^{-n} \int |\xi+i\tau N|^{2m} \, |\hat{v}(\xi)|^2 \, d\xi, \quad |\alpha| \leq m .$$

If $\epsilon = \frac{1}{2}$ the estimate (8.3.3) follows when $u \in C_o^\infty(\Omega_\delta)$. The proof is complete.

8.4 Estimates for operators with real coefficients. For non elliptic operators it is necessaryfor us to study the limiting case of the conditions (8.1.11) and (8.1.12) when $\sigma \to 0$, before passing to the proof of estimates.

Theorem 8.4.1. Let P_m have real coefficients and assume that (8.1.1ν) is valid for $\nu = 1$ or 2. If $x \in \Omega$ and $0 \neq \xi \in R_n$ is a solution of the characteristic equation $P_m(x, \xi) = 0$ such that

$$\sum_1^n P_m^{(j)}(x, \xi)\, \partial\varphi/\partial x_j = 0 \tag{8.4.1}$$

but $P_m^{(j)}(x, \xi) \neq 0$ for some j, it then follows that

$$|\xi|^{2(m-1)} \leq 2K_\nu \Big\{ \sum_{j,k=1}^n \partial^2\varphi/\partial x_j \partial x_k \; P_m^{(j)}(x, \xi)\, P_m^{(k)}(x, \xi) + \sum_{j,k=1}^n \Big(P_{m,j}^{(k)}(x, \xi)\, P_m^{(j)}(x, \xi) - P_{m,j}(x, \xi)\, P_m^{(jk)}(x,\xi) \Big) \partial\varphi/\partial x_k \Big\} . \tag{8.4.2}$$

Proof. Choose a real η such that

$$\sum_1^n P_m^{(j)}(x, \xi)\, \eta_j \neq 0 \tag{8.4.3}$$

and consider the equation

$$P_m(x, \xi + z\eta + \tau N) = 0 . \tag{8.4.4}$$

From (8.4.3) and the implicit function theorem it follows that there is a unique analytic function $z = z(\tau)$ in a neighbourhood of $\tau = 0$ which vanishes for $\tau = 0$ and satisfies (8.1.22). Since (8.4.1) is valid we have either $z(\tau) = 0$ identically or else

$$z(\tau) = c\tau^k + O(\tau^{k+1})$$

where $c \neq 0$ and $k \geq 2$. In both cases it follows that we can let $\tau \to 0$ on a curve with non real tangent at $\tau = 0$ so that $z(\tau) \to 0$ through <u>real</u> values. We may thus apply (8.1.1ν) with ξ replaced by $\xi + z(\tau)\eta + \operatorname{Re} \tau N$ and σ replaced by $\operatorname{Im} \tau$. When $\tau \to 0$ the inequality (8.4.2) follows, for when the coefficients of P_m are real the right hand side of (8.1.1ν) is a polynomial in ξ and σ which for $\sigma = 0$ reduces to the right hand side of (8.4.2).

<u>Remark</u>. If (8.1.1) or (8.1.2) is valid, the derivatives $P_m^{(\alpha)}(x, \xi)$ cannot vanish for all α with $|\alpha| \geq 2$ if ξ is real and $\neq 0$. In fact, assuming that $x = 0$ and taking

$$u(x) = e^{i\sigma\langle x,\xi\rangle - \tau\langle x,N\rangle} \psi(\mu x)$$

where $\psi \in C_0^\infty(R_n)$ and $\sigma = \tau^a$, $\mu = \tau^b$ with $a > 5/4$, $b < 1$, $a-b < \frac{1}{2}$ we otherwise obtain a contradiction. The simple proof, following that of Theorem 8.1.1, may be left to the reader. If $P_m^{(j)}(x, \xi) = 0$ for all j and we assume instead of (8.4.1) that

$$\sum_{j,k=1}^{n} P_m^{(jk)}(x, \xi)\, \partial\varphi/\partial x_j\, \partial\varphi/\partial x_k = 0$$

it is thus possible to apply the same reasoning as in the proof of Theorem 8.4.1 to prove that (8.4.2) must be valid.

We proved in section 8.1 that the right hand side of (8.1.11) is invariant. Hence it follows in particular that the right hand side of (8.4.2) is invariant for real ξ. We shall now discuss the geometrical meaning of the positivity of the right hand side of (8.4.2). Thus consider a real solution of the characteristic equation $P_m(x, \xi) = 0$ and the corresponding element of bicharacteristic strip given by the equations

$$dx_j = P_m^{(j)}(x, \xi)\, dt, \quad d\xi_j = - P_{m,j}(x, \xi)\, dt$$

where t is a parameter along the strip. The condition (8.4.1) means that $d\varphi = 0$ along the bicharacteristic. Differentiating again we obtain

$$d^2 \varphi/dt^2 = \sum_{j,k=1}^{n} \partial^2\varphi/\partial x_j \partial x_k \; P_m^{(j)}(x, \xi)\, P_m^{(k)}(x, \xi) +$$

$$+ \sum_{j,k=1}^{n} \partial\varphi/\partial x_k \left(P_{m,j}^{(k)}(x, \xi)\, dx_j/dt + P_m^{(jk)}(x, \xi)\, d\xi_j/dt\right).$$

of, after using the equation of the bicharacteristic once more

$$d^2\varphi/dt^2 = \sum_{j,k=1}^{n} \partial^2\varphi/\partial x_j \partial x_k \; P_m^{(j)}(x, \xi)\, P_m^{(k)}(x, \xi) +$$

$$+ \sum_{j,k=1}^{n} \partial\varphi/\partial x_k \,(P_{m,j}^{(k)}(x, \xi)\, P_m^{(j)}(x, \xi) - P_m^{(jk)}(x, \xi) P_{m,j}(x,\xi)).$$

Thus the positivity of the right-hand side in (8.4.2) when (8.4.1) is valid means exactly that φ is strictly convex along any bicharacteristic at any point where the bicharacteristic is tangential to a level surface of φ. (Thus the restriction of φ to a bicharacteristic has no stationary point other than minimum points.)

We shall now prove that the necessary conditions which we have found for the validity of the estimates (8.1.1) and (8.1.2) are also sufficient, if the condition in Theorem 8.4.1 is very mildly strengthened by requiring (8.4.2) to hold for every real solution of the equation $P_m(x, \xi) = 0$ satisfying (8.4.1) even if $P_m^{(j)}(x, \xi)$ should be equal to 0 for all j.

<u>Theorem 8.4.2</u>. Let Ω be a bounded open set, φ a real valued function in $C^\infty(\bar{\Omega})$ with grad $\varphi(x) \neq 0$ when $x \in \bar{\Omega}$, and $P(x, D)$ a differential operator of order m with bounded measurable coefficients such that the principal part $P_m(x, D)$ has real coefficients belonging to $C^1(\bar{\Omega})$. Assume further that

$$(8.4.5) \qquad \sum_{j,k=1}^{m} \partial^2\varphi/\partial x_j \partial x_k \; P_m^{(j)}(x, \xi)\, P_m^{(k)}(x, \xi) +$$

$$+ \sum_{j,k=1}^{m} (P_{m,j}^{(k)}(x, \xi)\, P_m^{(j)}(x, \xi) - P_{m,j}(x, \xi)\, P_m^{(jk)}(x,\xi)) \frac{\partial\varphi}{\partial x_k} > 0$$

if $x \in \overline{\Omega}$ and $0 \neq \xi \in R_n$ satisfy the characteristic equation $P_m(x, \xi) = 0$ and

(8.4.6) $$\sum_1^n P_m^{(j)}(x, \xi)\, \partial\varphi/\partial x_j = 0 .$$

Then there is a constant K such that when τ is sufficiently large

(8.4.7) $$\sum_{|\alpha|<m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau\varphi} dx \leq K \int (|P(x,D)u|^2 + \tau^{2m-1}|u|^2) e^{2\tau\varphi} dx, \quad u \in C_0^\infty(\Omega) .$$

<u>Theorem 8.4.3</u>. Assume that, in addition to the hypotheses of Theorem 8.4.2, we have

(8.4.8) $$\sum_{j,k=1}^n \partial^2\varphi/\partial x_j \partial x_k\, P_m^{(j)}(x, \xi)\, \overline{P_m^{(k)}(x, \zeta)} + (2i\tau)^{-1} \sum_1^n (P_{m,k}(x,\zeta)\, \overline{P_m^{(k)}(x,\zeta)} - P_m^{(k)}(x,\zeta)\, \overline{P_{m,k}(x,\zeta)}) > 0$$

if $\zeta = \xi + i\,\tau \operatorname{grad} \varphi(x)$, with $x \in \overline{\Omega}$, $\xi \in R_n$ and $0 \neq \tau \in R_1$, satisfies the characteristic equation $P_m(x, \zeta) = 0$. Then there is a constant K such that for sufficiently large τ

(8.4.9) $$\sum_{|\alpha|<m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau\varphi} dx \leq K \int |P(x,D)u|^2\, e^{2\tau\varphi}\, dx, \quad u \in C_0^\infty(\Omega) .$$

Proof of Theorems 8.4.2 and 8.4.3. An obvious modification of Lemmas 8.3.1 and 8.3.2 shows that it is sufficient to prove the two theorems locally when $P = P_m$. Thus, assuming that $0 \in \overline{\Omega}$ and writing $\Omega_\delta = \{x;\ x \in \Omega,\ |x| < \delta\}$, we shall prove (8.4.7) and (8.4.9) when $u \in C_0^\infty(\Omega_\delta)$ and δ is sufficiently small. Since grad $\varphi(0) \neq 0$ and $\varphi \in C^\infty$, we can again change coordinates in a neighbourhood of 0 so that φ becomes a linear function, $\varphi(x) = \langle x, N\rangle$.

With $v(x) = u(x)\, e^{\tau\langle x,N\rangle}$ where $u \in C_0^\infty(\Omega)$ we now have as in the proof of Theorem 8.3.1

$$\int |P_m(x,D)u|^2\, e^{2(\tau+A)\langle x,N\rangle}\, dx = \int |P_m(x,D+i\tau N)v|^2\, e^{2A\langle x,N\rangle}\, dx$$

$$\geq \int \left\{ |P_m(x,D+i\tau N)v|^2 - |P_m(x,D-i\tau N)v|^2 \right\} e^{2A\langle x,N\rangle} dx = \int F_\tau(x,D,\overline{D})v\overline{v}dx$$

where

$$F_{\tau,A}(x,\zeta,\overline{\zeta}) = e^{2A\langle x,N\rangle}(P_m(x,\zeta+i\tau N)\,\overline{P_m(x,\zeta+i\tau N)} - P_m(x,\zeta-i\tau N)\,\overline{P_m(x,\zeta-i\tau N)}) .$$

Here A is a constant which will be chosen later on in the proof in order to ensure the positivity of a certain quantity. Note that the coefficient of τ^j in the expansion of $F_{\tau,A}$ in powers of τ is of order $(2m - j;\ m)$ and is equal to 0 if $j = 0$. Since the coefficients of P_m are real, we have $F_{\tau,A}(x, \xi, \xi) = 0$ for real ξ. Hence it follows from Lemma 8.2.2 applied to $\tau^{-1} F_{\tau,A}$ that there exists a form

$$G_{\tau,A}(x, D, \overline{D}) = \sum_0^{2m-2} \tau^j\, G_A^{(j)}(x, D, \overline{D})$$

with continuous coefficients such that $G_A^{(j)}$ is of order $(2m - 2 - j;\ m-1)$ and

$$(8.4.10)\quad \tau\int G_{\tau,A}(x,D,\bar{D})\, v\bar{v}\, dx = \int F_{\tau,A}(x,D,\bar{D})v\bar{v}\, dx \le \int |P_m(x,D)u|^2 e^{2(\tau+A)\langle x,N\rangle} dx$$

Furthermore, (8.2.12) gives if $\tau \neq 0$ (cf. (8.3.5))

$$(8.4.11)\quad G_{\tau,A}(0,\xi,\xi) = 2\tau^{-1}\Big\{ \operatorname{Im} \sum_1^n P_{m,j}(0,\xi+i\tau N)\, \overline{P_m^{(j)}(0,\xi+i\tau N)} +$$

$$+ \operatorname{Im} P_m(0,\xi+i\tau N)\overline{\Big(\sum_1^n P_{m,j}^{(j)}(0,\xi+i\tau N)+2A\sum_1^n P_m^{(j)}(0,\xi+i\tau N)N_j\Big)}\Big\},\ \xi \in R_n .$$

In particular, we obtain when $\tau \to 0$

$$(8.4.12)\quad G_{o,A}(0,\xi,\xi) = 2\Big\{ \sum_{j,k=1}^n [P_{m,j}^{(k)}(x,\xi)\, P_m^{(j)}(x,\xi) - P_{m,j}(x,\xi)\, P_m^{(jk)}(x,\xi)]N_k +$$

$$+ \Big(\sum_1^n P_m^{(j)}(0,\xi)\, N_j\Big)\Big(\sum_1^n P_{m,j}^{(j)}(0,\xi) + 2A\sum_1^n P_m^{(j)}(0,\xi)\, N_j\Big)\Big\} \text{ if } \xi \in R_n,$$

$$P_m(0,\xi) = 0$$

We now claim that there are positive constants A, C_1, C_2, C_3 such that

$$(8.4.13)\quad |\xi+i\tau N|^{2(m-1)} \le C_1\, G_{\tau,A}(0,\xi,\xi) + C_2|P_m(0,\xi+i\tau N)|^2/|\xi+i\tau N|^2 +$$

$$+ C_3\, \tau^{2(m-1)},\ \xi \in R_n,\ \tau \in R_1;\ \xi + i\tau N \neq 0 ;$$

when the hypothesis of Theorem 8.4.3 are fulfilled we shall prove (8.4.13) with $C_3 = 0$. In view of the homogeneity it is sufficient to prove (8.4.13) when $|\xi + i\tau N| = 1$. First note that the hypothesis of Theorem 8.4.2 means that $G_{o,A}(0, \xi, \xi) > 0$ if $P_m(0, \xi) = 0$ and $\sum_1^n P_m^{(j)}(0, \xi) N_j = 0$, $\xi \in R_n$ and $|\xi| = 1$. Since the coefficient of A in (8.4.2) is $4(\sum_1^n P_m^{(j)}(0, \xi) N_j)^2$, it follows that we may choose A so large that

$$G_{o,A}(0, \xi, \xi) > 0 \text{ if } \xi \in R_n, P_m(0, \xi) = 0, |\xi| = 1 .$$

From now on we keep A fixed. For reasons of continuity we can now choose B so large that

$$G_{o,A}(0, \xi, \xi) + B|P_m(0, \xi)|^2 > 0 \text{ if } \xi \in R_n, \ |\xi| = 1 .$$

For a sufficiently large C we will have, again because of the continuity of the terms in the inequality,

$$G_{\tau,A}(0, \xi, \xi) + B|P_m(0,\xi+i\tau N)|^2 + C\tau^{2(m-1)} > 0 \text{ if } |\xi + i\tau N| = 1 .$$

If C_1 is chosen sufficiently large, the inequality (8.4.13) now follows with $C_2 = BC_1$ and $C_3 = CC_1$. - If the hypotheses of Theorem 8.4.3 are satisfied, and A is chosen as before, it follows that

$$G_{\tau,A}(0, \xi, \xi) > 0 \text{ if } P_m(0, \xi + i\tau N) = 0, |\xi + i\tau N| = 1$$

which we previously only knew when $\tau = 0$. Hence we can then choose B so large that

$$G_{\tau,A}(0, \xi, \xi) + B|P_m(0, \xi + i\tau N)|^2 > 0 \text{ if } |\xi + i\tau N| = 1 ,$$

which proves (8.4.13) with $C_3 = 0$.

We now multiply (8.4.13) by $|\hat{v}(\xi)|^2$ and integrate, which gives

$$V_{m-1}^2 = (2\pi)^{-n}\int|\xi+i\tau N|^{2(m-1)} |\hat{v}(\xi)|^2 d\xi \leq C_1\int G_{\tau,A}(0,D,\bar{D})v\bar{v}\, dx +$$

(8.4.14) $$+ C_2|||P_m(0,D + i\tau N)\, v|||_\tau^2 + C_3\, \tau^{2(m-1)}\int|v|^2\, dx ,$$

where the first equality is a definition and, for the sake of brevity, we have used the notation

(8.4.15) $$|||f|||_\tau^2 = (2\pi)^{-n}\int|\hat{f}(\xi)|^2 \; |\xi + i\tau N|^{-2}\, d\xi .$$

Since $G_{\tau,A}(x, D, \bar{D})$ has continuous coefficients and is of order $(2(m - 1);\ m - 1)$, we can conclude as in the proof of (8.3.8) that if ϵ is any given positive number we have

(8.4.16) $$\int G_{\tau,A}(0, D, \bar{D})\, v\bar{v}\, dx \leq \int G_{\tau,A}(x, D, \bar{D})\, v\bar{v}\, dx + \epsilon\, V_{m-1}^2, v \in C_0^\infty(\Omega_\delta)$$

if δ is sufficiently small. However, in order to handle the next term in (8.4.14) we first have to prove a lemma, which gives a sharper result than Theorem 2.2.5 for the norm (8.4.15).

Lemma 8.4.1. Let $a(x)$ be Lipschitz continuous with Lipschitz constant M when $|x| < \delta$, that is, $|a(x) - a(y)| \leq M|x-y|$ if $\max(|x|,|y|) < \delta$. If $a(0) = 0$ it then follows that

$$(8.4.17) \quad |||a(D_j + i\tau N_j)\, w|||_\tau \leq M(\delta + |\tau N|^{-1})||w||_2, \quad w \in C_0^\infty(\Omega_\delta) ,$$

where $||w||_2$ is the L^2 norm of w.

Proof. In view of the identity

$$a(D_j + i\tau N_j)\, w = (D_j + i\tau N_j)(aw) - (D_j a)w$$

and the trivial estimates

$$(8.4.18) \quad |||f|||_\tau^2 \leq |\tau N|^{-2}\, \|f\|_2^2, \quad |||(D_j + i\tau N_j)\, f|||_\tau^2 \leq \|f\|_2^2, \quad f \in L^2(R_n) ,$$

we have

$$|||a(D_j + i\tau N_j)\, w\,|||_\tau \leq \|aw\|_2 + |\tau N|^{-1}\|(D_j a)w\|_2 .$$

Since $|a| < \delta M$ in Ω_δ and $|D_j a| \leq M$, the inequality (8.4.17) follows.

End of the proof of Theorems 8.4.2 and 8.4.3. By hypothesis we have

$$P_m(x, D + i\tau N) = \sum_{|\alpha|=m} a_\alpha(x)\, (D + i\tau N)^\alpha$$

where $a_\alpha(x) \in C^1(\bar{\Omega})$. Since

$$\|(D + i\tau N)^\beta v\|_2 \leq V_{m-1}, \ |\beta| = m - 1 ,$$

it follows from Lemma 8.4.1 that with a constant C

$$(8.4.19) \quad |||P_m(0,D+i\tau N)v - P_m(x,D+i\tau N)v|||_\tau \leq C(\delta + |\tau N|^{-1}) V_{m-1}, \ v \in C_0^\infty(\Omega_\delta) .$$

Using (8.4.19), (8.4.18) and the triangle inequality we now obtain

$$|||P_m(0,D + i\tau N)v|||_\tau^2 \leq 2\, |||P_m(x,D+i\tau N)v|||_\tau^2 + 2\, |||P_m(0,D+i\tau N)v-P_m(x,D+i\tau N)v|||$$

$$\leq 2|\tau N|^{-2} \|P_m(x,D + i\tau N)v\|_2^2 + 2C^2(\delta + |\tau N|^{-1})^2 V_{m-1}^2 .$$

Combining this estimate with (8.4.14), (8.4.16) and (8.4.10) we have thus proved that when $u \in C_0^\infty(\Omega)$ and $u(x)e^{\tau\langle x,N\rangle} = v(x)$ we have

$$(8.4.20) \quad (1-\epsilon-2C_2(\delta+|\tau N|^{-1})^2)\, V_{m-1}^2 \leq C_1\tau^{-1}\int|P(x,D)u|^2 e^{2(\tau+A)\langle x,N\rangle}\, dx$$

$$+ 2C_2|\tau N|^{-2}\int|P(x,D)u|^2 e^{2\tau\langle x,N\rangle}\, dx + C_3\tau^{2(m-1)}\int|u|^2 e^{2\tau\langle x,N\rangle} dx .$$

When ϵ and δ are so small that $1 - \epsilon - 2C^2\delta^2 > \frac{1}{2}$, the inequality (8.4.7) follows from (8.4.20) for sufficiently large τ. If $C_3 = 0$, we obtain the inequality (8.4.9). This completes the proof.

<u>Remark</u>. A careful examination of the proof shows in fact that

if (8.4.2) is valid for all real characteristics satisfying (8.4.1) and if (8.1.11) is also fulfilled in case $\nu = 1$, then we have (8.1.ν) for large τ, $\nu = 1$ or 2, (for some constant K_3) if K_ν is replaced by $K_\nu + \epsilon$ where $\epsilon > 0$. The only place where a really different argument is needed is Lemma 8.3.1 where the functions φ_j should not be chosen as a partition of unity but so that $\sum \varphi_j^2 = 1$.

8.5 Estimates for principally normal operators. In Chapter VI we proved that existence of solutions in Ω of the differential equation $P(x, D)u = f$ (or, which is equivalent, some continuity of the inverse of the adjoint of P acting on $C_o^\infty(\Omega)$) requires that

$$(8.5.1) \quad C_{2m-1}(x,\xi) = i \sum_1^n (P_m^{(j)}(x,\xi)\, \overline{P_{m,j}(x,\xi)} - P_{m,j}(x,\xi)\, \overline{P_m^{(j)}(x,\xi)}) = 0$$

$$\text{if } P_m(x,\xi) = 0,\ \xi \in R_n,\ x \in \Omega .$$

It is also easy to see that (8.5.1) may be obtained as a limiting case of the inequalities in Theorem 8.1.1. In proving estimates for non-elliptic operators with non-real coefficients we need the following strengthened form of (8.5.1).

Definition 8.5.1. We shall say that $P(x, D)$ is principally normal in $\overline{\Omega}$ if the coefficients of P_m are in $C^1(\overline{\Omega})$ and there exists a differential operator $Q_{m-1}(x, D)$, homogeneous of degree $m-1$ in D, with coefficients in $C^1(\overline{\Omega})$, such that

$$(8.5.2) \quad C_{2m-1}(x,\xi) = P_m(x,\xi)\, \overline{Q}_{m-1}(x,\xi) + Q_{m-1}(x,\xi)\, \overline{P}_m(x,\xi) , \quad \xi \in R_n .$$

In particular, P is principally normal if $C_{2m-1}(x,\xi) = 0$ identically, that is, if the commutator of P and its adjoint is of order $\leq 2m-2$ (cf. Lemma 6.1.2). This is the reason for our terminology. Note that every operator with constant or real coefficients is principally normal.

It is clear that Q_{m-1} is uniquely determined by P_m unless P_m and $\overline{P}_m$ have a common factor, that is, P_m has a real factor. It is in fact the lack of uniqueness of Q_{m-1} when the coefficients of P_m are real which made it possible to obtain somewhat stronger results in section 8.4 than we are able to prove here.

We shall now study formally the limiting case of the conditions (8.1.11) and (8.1.12) when $\sigma \to 0$. To do so we note that if $\zeta = \xi + i\tau N$, with real τ and ξ, satisfies the equation $P_m(x, \zeta) = 0$, then the expression in brackets in the right hand side of (8.1.11) and (8.1.12) is equal to

$$\sum_{j,k=1}^{n} \partial^2\varphi/\partial x_j \partial x_k \, P_m^{(j)}(x,\zeta) \overline{P_m^{(k)}(x,\zeta)} + (2i\tau)^{-1} \Big\{ \sum_1^n \Big(P_{m,k}(x,\zeta) \overline{P_m^{(k)}(x,\zeta)} - P_m^{(k)}(x,\zeta) \overline{P_{m,k}(x,\zeta)} \Big) - i \Big(P_m(x,\zeta) \overline{Q_{m-1}(x,\zeta)} + Q_{m-1}(x,\zeta) P_m(x,\zeta) \Big) \Big\} .$$

In view of (8.5.2) this is now a polynomial in ξ and τ, hence has a meaning also when $\tau = 0$. In the following two theorems we shall require that it is positive at all real characteristics where in section 8.4 we only needed this hypothesis at characteristics satisfying (8.4.1).

This assumption is certainly not superfluous if P_m has no real factor and it will not affect the applications.

Theorem 8.5.1. Let Ω be a bounded open set, φ a real valued function in $C^\infty(\bar{\Omega})$ with $\operatorname{grad}\varphi(x) \neq 0$ when $x \in \bar{\Omega}$, and $P(x,D)$ a principally normal differential operator of order m such that the coefficients in the lower order terms are in L^∞ and those in $P_m(x,D)$ are in $C^2(\bar{\Omega})$. Assume further that with an operator Q_{m-1} satisfying the condition in Definition 8.5.1 we have

$$(8.5.3) \quad \sum_{j,k=1}^{n} \partial^2\varphi/\partial x_j \partial x_k \, P_m^{(j)}(x,\xi)\, \overline{P_m^{(k)}(x,\xi)} + \operatorname{Re}\sum_{j,k=1}^{n} (P_{m,k}^{(j)}(x,\xi)\, \bar{P}_m^{(k)}(x,\xi) - $$

$$- P_{m,k}(x,\xi)\, \bar{P}_m^{(kj)}(x,\xi)) \frac{\partial\varphi}{\partial x_j} + \operatorname{Im}\sum_1^n P_m^{(j)}(x,\xi) \frac{\partial\varphi}{\partial x_j} Q_{m-1}(x,\xi) > 0$$

if $x \in \bar{\Omega}$ and $0 \neq \xi \in R_n$ is a solution of the characteristic equation $P_m(x,\xi) = 0$. Then there is a constant K such that when τ is sufficiently large

$$(8.5.4) \quad \sum_{|\alpha| < m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau\varphi}\, dx \leq K\int (|P(x,D)u|^2 + \tau^{2m-1}|u|^2) e^{2\tau\varphi} dx,$$

$$u \in C_0^\infty(\Omega) .$$

Remark. When the coefficients are real, we may take for Q_{m-1} an arbitrary operator with purely imaginary coefficients. This gives back Theorem 8.4.2 apart from the stronger differentiability assumptions used here, for the inequality (8.5.3) may be satisfied where (8.4.1) is not fulfilled by choosing $Q_{m-1}(x,\xi) = iA \sum R_m^{(j)}(x,\xi)\, \partial\varphi/\partial x_j$ with a sufficiently large positive A. This in fact gives an alternative proof of Theorems 8.4.2 and 8.4.3.

<u>Theorem 8.5.2</u>. Assume that, in addition to the hypotheses of Theorem 8.5.1, we have

$$(8.5.5)\quad \sum_{j,k=1}^{n} \partial^2\varphi/\partial x_j \partial x_k \, P_m^{(j)}(x,\zeta)\, \overline{P_m^{(k)}(x,\zeta)} \; +$$

$$+ \, (2i\tau)^{-1} \sum_{1}^{n} (P_{m,k}(x,\zeta)\, \overline{P_m^{(k)}(x,\zeta)} - P_m^{(k)}(x,\zeta)\, \overline{P_{m,k}(x,\zeta)}) > 0$$

if $\zeta = \xi + i\tau \operatorname{grad} \varphi(x)$, with $x \in \bar{\Omega}$, $\xi \in R_n$ and $0 \neq \tau \in R_1$, satisfies the characteristic equation $P_m(x,\zeta) = 0$. Then there is a constant K such that for sufficiently large τ

$$(8.5.6)\quad \sum_{|\alpha| < m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau\varphi} dx \leq K \int |P(x,D)u|^2 e^{2\tau\varphi} dx, \quad u \in C_0^\infty(\Omega).$$

<u>Proof of Theorems 8.5.1 and 8.5.2</u>. It is again sufficient to prove these theorems locally for a convenient choice of lower order terms when φ is linear, $\varphi(x) = \langle x,N \rangle$. (See the proof of Theorems 8.4.2 and 8.4.3.) Thus we take $P(x,D) = P_m(x,D) + R_{m-1}(x,D)$ where R_{m-1} is an operator of order $m-1$ with C^1 coefficients which will be chosen later. With $u \in C_0^\infty(\Omega_\delta)$ and $v(x) = u(x)e^{\tau \langle x,N \rangle}$ we now form the trivial inequality

$$(8.5.7)\quad \int |P(x,D)u|^2 \, e^{2\tau \langle x,N \rangle} dx = \int |P(x,D + i\tau N)v|^2 \, dx \geq$$

$$\geq \int \{ |P_m(x,D+i\tau N)v|^2 - |\bar{P}_m(x,D-i\tau N)v|^2 + 2 \operatorname{Re} P_m(x,D+i\tau N)v \, \overline{R_{m-1}(x,D+i\tau N)v} \} \, dx \, .$$

To the difference $\int[|P_m(x,D+i\tau N)v|^2 - |\bar{P}_m(x,D-i\tau N)v|^2]dx$ we can apply Lemma 8.2.2 as in the proof of Theorem 8.3.1. Thus there exists a differential quadratic form

$$G_\tau(x,D;\bar{D}) = \sum_{o}^{2m-1} \tau^j \, G^{(j)}(x,D;\bar{D})$$

with C^1 coefficients and $G^{(j)}$ of order $(2m-1-j;\ m)$, such that the right-hand side of (8.5.7) is equal to $\int G_\tau(x,D,\bar{D})v\bar{v}\,dx$; thus

$$\int G_\tau(x,D,\bar{D})v\bar{v}\,dx \leq \int |P(x,D)u|^2 \, e^{2\tau < x,N >}\,dx \,, \tag{8.5.8}$$

and we have (cf. formula (8.3.5))

$$G_\tau(x,\xi,\xi) = 2\,\mathrm{Im}\sum_{1}^{n} P_{m,k}(x,\xi+i\tau N)\,\overline{P_m^{(k)}(x,\xi+i\tau N)} \; + \tag{8.5.9}$$

$$+ \; 2\,\mathrm{Re}\, P_m(x,\xi+i\tau N)\overline{(i\sum_{1}^{n} P_{m,k}^{(k)}(x,\xi+i\tau N) + R_{m-1}(x,\xi+i\tau N))}\;,\; \xi \in R_n,\; \tau \in R_1$$

In order to be able to use (8.5.2) we now choose

$$R_{m-1}(x,D) = -\,Q_{m-1}(x,D) - i\sum_{1}^{n} P_{m,k}^{(k)}(x,D)\;. \tag{8.5.10}$$

Since by assumption the coefficients of Q_{m-1} are in $C^1(\bar{\Omega})$ and those of P_m are in $C^2(\bar{\Omega})$, the coefficients of R_{m-1} are then in $C^1(\bar{\Omega})$. Furthermore, from (8.5.9), (8.5.1) and (8.5.2) it follows that $G_o(x,\xi,\xi) = 0$, $\xi \in R_n$, that is, $G^{(o)}(x,\xi;\xi) = 0$. We can therefore apply Lemma 8.2.2 to the differential quadratic form $G^{(o)}(x,D,\bar{D})$, which gives the estimate

$$(8.5.11) \qquad |\int G^{(o)}(x,D,\bar{D})v\bar{v}\,dx| \le C \sum_{|\alpha| \le m-1} \int |D^{\alpha}v|^2\,dx .$$

Now write

$$G'_{\tau}(x,D,\bar{D}) = \sum_{o}^{2m-2} \tau^j\, G^{(j+1)}(x,D,\bar{D}) .$$

We then have

$$G_{\tau}(x,D,\bar{D}) = G^{(o)}(x,D,\bar{D}) + \tau G'_{\tau}(x,D,\bar{D}) ,$$

and the coefficient of τ^j in G'_{τ} is of order $\le$ (2m-2-j; m). Furthermore, $G'_{\tau}(x,\xi,\xi) = \tau^{-1} G_{\tau}(x,\xi,\xi)$ so that (8.5.9) and (8.5.10) give

$$(8.5.12) \qquad G'_{\tau}(x,\xi,\xi) = 2\tau^{-1} \{\mathrm{Im} \sum_{1}^{n} P_{m,k}(x,\xi+i\tau N)\, P_m^{(k)}(x,\xi+i\tau N) -$$

$$- \mathrm{Re}\, P_m(x,\xi+i\tau N)\, \overline{Q_{m-1}(x,\xi+i\tau N)}\} , \quad \xi \in R_n .$$

In any term in $\int G'_{\tau}(x,D,\bar{D})\, v\bar{v}\, dx$ which involves a derivative of v of order m, we can make an integration by parts since the coefficients are in $C^1(\bar{\Omega})$. In view of Leibniz' formula this gives one term where the coefficient is differentiated and one where it is not; thus we obtain

$$\int G'_{\tau}(x,D,\bar{D})\, v\bar{v}\, dx = \int G''_{\tau}(x,D,\bar{D})\, v\bar{v}\, dx + \int G'''_{\tau}(x,D,\bar{D})\, v\bar{v}\, dx$$

where

$$(8.5.13) \qquad G''_{\tau}(x,\xi,\xi) = G'_{\tau}(x,\xi,\xi) ,$$

and the coefficient of τ^j in G''_τ is of order $(2m-2-j;\ m-1)$ and the coefficient of τ^j in G'''_τ is of order $(2m-3-j;\ m-1)$. Hence we have with a constant C

$$(8.5.14) \qquad \left| \int G'''_\tau(x,D,\bar{D})v\bar{v}\, dx \right| \leq C\tau^{-1} \sum_{|\alpha|<m} \tau^{2(m-1-|\alpha|)} \int |D^\alpha v|^2\, dx .$$

In view of (8.5.12) and (8.5.13), the condition (8.5.3) means that $G''_o(x,\xi,\xi) > 0$ if $0 \neq \xi \in R_n$ is a solution of the characteristic equation $P_m(x,\xi) = 0$; similarly (8.5.5) means that $G''_\tau(x,\xi,\xi) > 0$ if $\zeta = \xi + i\tau N$ with $\xi \in R_n$, $0 \neq \tau \in R_1$, and ζ satisfies the equation $P_m(x,\zeta) = 0$. We therefore obtain the estimate (8.4.13) with $G_{\tau,A}$ replaced by G''_τ, the constant C_3 being 0 if the hypotheses of Theorem 8.5.2 are fulfilled. The rest of the proof of Theorems 8.4.2 and 8.4.3 now applies without change except for the fact that instead of (8.4.10) we now have

$$(8.5.15) \qquad \tau \int G''_\tau(x,D,\bar{D})\, v\bar{v}\, dx = \int G_\tau(x,D,\bar{D})\, v\bar{v}\, dx - \int G^{(o)}(x,D,\bar{D})\, v\bar{v}\, dx -$$

$$- \tau \int G'''(x,D,\bar{D})\, v\bar{v}\, dx \leq \int |P(x,D)u|^2 e^{2\tau \langle x,N \rangle} dx + C\, V^2_{m-1}$$

in view of (8.5.8), (8.5.11) and (8.5.13). (We have used the notation V_{m-1} introduced in (8.4.14) and the obvious fact that the right-hand sides of (8.5.11) and (8.5.14) can be estimated by a constant times V^2_{m-1}.) The conclusion of the proof still proceeds as before, however, and may be left to the reader.

8.6 Pseudo-convexity. Let $\psi \in C^2$ in a neighbourhood of a point x^o and assume that $\text{grad}\, \psi(x^o) \neq 0$. Then the equation

$$\psi(x) = \psi(x^o) \tag{8.6.1}$$

defines a non singular level surface in a neighbourhood of x^o with orientation: we call the part of a neighbourhood of x^o where $\psi(x) > \psi(x^o)$ ($\psi(x) < \psi(x^o)$) the positive (negative) side of the surface. If ψ_1 defines the same surface with the same orientation, we have at the point x^o with a positive λ

$$\text{grad}\, \psi_1 = \lambda\, \text{grad}\, \psi\ ; \quad \sum_{j,k=1}^{n} (\partial^2\psi_1/\partial x_j \partial x_k) y_j z_k = \lambda \sum_{j,k=1}^{n} (\partial^2\psi/\partial x_j \partial x_k) y_j z_k \quad \text{if}$$

$$\sum_1^n z_j \partial\psi/\partial x_j = \sum_1^n y_j \partial\psi/\partial x_j = 0\ .$$

This shows that the following definition is independent of the function ψ defining the oriented surface (8.6.1).

Definition 8.6.1. Let P be either elliptic or principally normal. The oriented surface defined by (8.6.1) will be called pseudo-convex with respect to P at the point x if

$$\sum_{j,k=1}^{n} \partial^2\psi/\partial x_j \partial x_k\, P_m^{(j)}(x,\xi)\, \overline{P}_m^{(k)}(x,\xi) + \text{Re} \sum_{j,k=1}^{n} (P_{m,k}^{(j)}(x,\xi)\, \overline{P}_m^{(k)}(x,\xi) - P_{m,k}(x,\xi)\, \overline{P}_m^{(kj)}(x,\xi))\partial\psi/\partial x_j > 0 \tag{8.6.2}$$

if $0 \neq \xi \in R_n$ satisfies the equations

$$P_m(x,\xi) = 0, \qquad \sum_1^n P_m^{(j)}(x,\xi)\partial\psi/\partial x_j = 0\ . \tag{8.6.3}$$

The surface is called strongly pseudo-convex with respect to P at the point x if, in addition,

$$(8.6.4) \qquad \sum_{j,k=1}^{n} \partial^2\psi/\partial x_j \partial x_k P_m^{(j)}(x,\zeta)\overline{P_m^{(k)}(x,\zeta)} +$$

$$+ (2i\tau)^{-1}\sum_1^n P_{m,k}(x,\zeta)\ \overline{P_m^{(k)}(x,\zeta)} - P_m^{(k)}(x,\zeta)\ \overline{P_{m,k}(x,\zeta)}) > 0$$

if $\zeta = \xi + i\tau \operatorname{grad}\psi(x)$, with $\xi \in R_n$ and $0 \neq \tau \in R_1$ satisfies the equations

$$(8.6.5) \qquad P_m(x,\zeta) = 0, \ \sum_1^n P_m^{(j)}(x,\zeta)\partial\psi/\partial x_j = 0 .$$

Remark 1. When $m = 1$ there is no difference between pseudo-convexity and strong pseudo-convexity. In fact, (8.6.3) and (8.6.5) are then equivalent, and (8.6.4) reduces to (8.6.2) since $\operatorname{Im}\Sigma P_{m,k}(x,\xi)\overline{P_m^{(k)}(x,\xi)} = 0$ when $P_m(x,\xi) = 0$, in view of the definition of a principally normal operator.

Remark 2. If $P_m(x,N) = 0$, the condition (8.6.3) is fulfilled by $\xi = N$, and condition (8.6.2) with $\xi = N$ implies that (5.3.10) is fulfilled with P_m replaced by either $\operatorname{Re} P_m$ or $\operatorname{Im} P_m$.

Remark 3. Every surface with normal N at x is (strongly) pseudo-convex there if and only if there is no real ξ not proportional to N such that the equation $P_m(x,\xi + \tau N) = 0$ has a real (complex) double zero τ. (This is the condition required by Calderón [ ], which shows that Theorem 8.9.1 contains his results.)

We shall now prove the stability of the notions introduced in Definition 8.6.1.

Theorem 8.6.1. Let P be either elliptic or principally normal in a neighbourhood of x^o and assume that the coefficients of P_m are in C^1 there. Further, let ψ be a function with grad $\psi(x^o) \neq 0$ which is in C^2 in a neighbourhood of x^o. If the surface (8.6.1) is (strongly) pseudo-convex with respect to P at x^o, there then exists a neighbourhood Ω of x^o and a positive number ϵ such that every $\varphi \in C^2(\Omega)$ for which

$$|D^\alpha(\varphi - \psi)| < \epsilon \quad \text{in } \Omega, \quad |\alpha| \leq 2 ,$$

has (strongly) pseudo-convex level surfaces with respect to P everywhere in Ω.

Proof. Assume for example that P is principally normal and that the surface (8.6.1) is strongly pseudo-convex at x^o; the other three cases in the theorem are still simpler and may be left to the reader. Choose Q_{m-1} according to (8.5.2). The assumption in the theorem then means that the polynomial in ξ and τ given by

$$(8.6.6) \quad \sum_{j,k=1}^{n} \partial^2\psi/\partial x_j \partial x_k \, P_m^{(j)}(x,\zeta) \, \overline{P_m^{(k)}(x,\zeta)} +$$

$$+ \tau^{-1}\{\mathrm{Im} \sum_1^n P_{m,k}(x,\zeta) \, \overline{P_m^{(k)}(x,\zeta)} - \mathrm{Re}\, P_m(x,\zeta) \, \overline{Q_{m-1}(x,\zeta)}\} ,$$

where $\zeta = \xi + i\tau N$, is positive when $P_m(x,\zeta) = \sum_1^n P_m^{(j)}(x,\zeta)N_j = 0$, if $x = x^0$, $N = N^0 = \text{grad}\,\psi(x^0)$ and $\zeta \neq 0$, $\xi \in R_n$, $\tau \in R_1$. In fact, (8.6.6) reduces to (8.6.2) when $\tau = 0$ (cf. (8.5.3)), and if $\tau \neq 0$ it is obvious that (8.6.6) reduces to (8.6.4). Now let $\Sigma = \{(\xi,\tau);\ |\xi|^2 + \tau^2 = 1\}$ and let A be the closed subset of Σ where $P_m(x^0,\xi+i\tau N^0) = \sum_1^n P_m^{(j)}(\xi+i\tau N^0)N_j^0 = 0$. Then the polynomial (8.6.6) has a positive lower bound in A when $x = x^0$ and $N = N^0$; hence we can find a neighbourhood B of A in Σ where the polynomial still has a lower bound. Since B is a neighbourhood of A, it follows that

$$|P_m(x^0, \xi + i\tau N^0)| + |\sum_1^n P_m^{(j)}(\xi + i\tau N^0)\, N_j^0|$$

has a positive lower bound in the complement B' of B in Σ. If Ω and ϵ are sufficiently small, it thus follows that the equations

$$P_m(x,\xi + i\tau \text{ grad } \varphi(x)) = \sum_1^n P_m^{(j)}(x,\xi + i\tau \text{ grad } \varphi(x))\partial\varphi/\partial x_j = 0$$

have no solution $(\xi,\tau) \in B'$ and that the polynomial (8.6.6) with $\partial^2\psi/\partial x_j\partial x_k$ replaced by $\partial^2\varphi/\partial x_j\partial x_k$, with $x \in \Omega$ and $N = \text{grad}\,\varphi(x)$, is still positive in B. This proves the theorem.

We next study the relation of pseudo-convexity to the conditions under which we have obtained estimates in sections 8.3 and 8.5.

<u>Theorem 8.6.2</u>. Let P be principally normal and let $\psi \in C^2(\bar{\Omega})$ have pseudo-convex level surfaces in the compact set $\bar{\Omega}$. Then the hypothesis (8.5.3) of Theorem 8.5.1 is fulfilled by $\varphi = e^{\lambda\psi}$ provided that the constant λ is chosen large enough.

<u>Proof</u>. Replacing φ by $e^{\lambda\psi}$ in (8.5.3) and multiplying by $\lambda^{-1}e^{-\lambda\psi}$ afterwards, we find that we have to prove that

$$(8.6.7) \quad \lambda\Big|\sum_1^n P_m^{(j)}(x,\xi)\partial\psi/\partial x_j\Big|^2 + \sum_{j,k=1}^n \partial^2\psi/\partial x_j\partial x_k P_m^{(j)}(x,\xi)\,\bar{P}_m^{(k)}(x,\xi) +$$

$$+ \operatorname{Re}\sum_{j,k=1}^n (P_{m,k}^{(j)}(x,\xi)\,\bar{P}_m^{(k)}(x,\xi) - P_{m,k}(x,\xi)\,\bar{P}_m^{(kj)}(x,\xi))\partial\psi/\partial x_j +$$

$$+ \operatorname{Im}\sum_1^n P_m^{(j)}(x,\xi)\partial\psi/\partial x_j\,\bar{Q}_{m-1}(x,\xi) > 0$$

if $x \in \bar{\Omega}$, $0 \neq \xi \in R_n$, $P_m(x,\xi) = 0$ and λ is large enough. To prove this we note that the set of (x,ξ) with $x \in \bar{\Omega}$ and $\xi \in R_n$ such that $|\xi| = 1$, $P_m(x,\xi) = 0$ and the inequality opposite to (8.6.7) is valid, is a compact set, decreasing with λ. If it is not void for large λ, there hence exists an $x \in \bar{\Omega}$ and a $\xi \in R_n$ such that $|\xi| = 1$, $P_m(x,\xi) = 0$ and (8.6.7) fails to hold for every λ. But then it follows that $\sum_1^n P_m^{(j)}(x,\xi)\partial\psi/\partial x_j = 0$ so that (8.6.7) is valid for every λ by assumption. This contradiction proves the theorem.

Theorem 8.6.3. Let P be either elliptic, have real coefficients or be principally normal, and let $\psi \in C^2(\overline{\Omega})$ have strongly pseudo-convex level surfaces in $\overline{\Omega}$. Then the hypotheses of Theorems 8.3.1, 8.4.3 or 8.5.2, respectively, are satisfied by $\varphi = e^{\lambda\psi}$ if λ is sufficiently large.

The proof is essentially a repetition of that of Theorem 8.6.2 so it may be left to the reader.

We shall now discuss the local existence of functions with pseudo-convex level surfaces.

Theorem 8.6.4. If P is principally normal in a neighbourhood of x^0 and

$$\sum_1^n |P_m^{(j)}(x^0,\xi)|^2 \neq 0 \ , \quad 0 \neq \xi \in R_n \ , \tag{8.6.8}$$

it is possible to find a function with pseudo-convex level surfaces in a neighbourhood of x^0.

Proof. In view of Theorem 8.6.1 it is sufficient to prove that we can find $\psi \in C^2$ with grad $\psi(x^0) \neq 0$ and a pseudo-convex level surface at x^0. To do so we just have to choose

$$\psi(x) = A|x - x^0|^2 + \langle x - x^0, N \rangle$$

where $N \neq 0$ is fixed and A is chosen sufficiently large afterwards.

Remark. Even if (8.6.8) is fulfilled for every $x^o \in \bar{\Omega}$ it does not follow that there exists a function with pseudo-convex level surfaces in the whole of $\bar{\Omega}$. In fact, let P_m have real coefficients. Then there does not exist any function φ with pseudo-convex level surfaces in $\bar{\Omega}$ if there is a closed bicharacteristic contained in $\bar{\Omega}$. For on such a curve the function φ must attain its maximum at some point x. Then φ is stationary at x, that is, the bicharacteristic is a tangent to the level surface of φ there. Since φ must have a non positive second derivative along the bicharacteristic at the maximum point, it follows from the discussion at the beginning of section 8.5 that φ is not pseudo-convex. — An example where this remark applies is the first order differential operator

$$P(x,D) = x_1D_2 - x_2D_1$$

in the set $\Omega = \{x;\ 1 < x_1^2 + x_2^2 < 2\}$. Since the bicharacteristics are the circles $x_1^2 + x_2^2 = \text{constant}$, we cannot find any pseudo-convex function in the whole of $\bar{\Omega}$ although (8.6.8) is fulfilled for every $x \in \bar{\Omega}$. (A more sophisticated form of this example where Ω is simply connected can easily be given in the three-dimensional case; see Trèves [ ]).

It is also interesting to note that there may exist functions with pseudo-convex level surfaces even if (8.6.8) is not fulfilled for every $x \in \Omega$. For example, consider the Tricomi operator

$$P(x,D) = x_2D_1^2 + D_2^2 ,$$

If $\psi(x) = -x_2$, the left hand side of (8.6.2) reduces to $2\xi_1^2$, which is > 0 if $x_2\xi_1^2 + \xi_2^2 = 0$ and $\xi = (\xi_1,\xi_2) \neq 0$. More generally, we can improve Theorem 8.6.4 as follows.

Theorem 8.6.5. Assume that P is principally normal in a neighbourhood of x^o and that there is a vector $N \neq 0$ such that

$$(8.6.9) \qquad \mathrm{Re} \sum_1^n P_{m,k}(x^o,\xi)\, \overline{P}_m^{(kj)}(x^o,\xi)\, N_j < 0 \quad \text{if} \quad 0 \neq \xi \in R_n \quad \text{and}$$

$$P_m^{(j)}(x^o,\xi) = 0, \; j = 1, \ldots, n .$$

Then there exists a function ψ whose level surfaces are pseudo-convex with respect to P in a neighbourhood of x^o.

Proof. We only need to take

$$\psi(x) = A|x - x^o|^2 + < x - x^o, N >$$

with a sufficiently large A, and use Theorem 8.6.1.

Remark. When P_m has real coefficients, the condition (8.6.9) has a simple geometrical meaning. In fact, consider a solution ξ^o of the equations $P_m^{(j)}(x^o,\xi^o) = 0$, $j = 1, \ldots, n$, and the bicharacteristic

$$dx_j/dt = P_m^{(j)}(x,\xi) \; , \; d\xi_j/dt = -P_{m,j}(x,\xi)$$

with initial data $x = x^o$, $\xi = \xi^o$ for $t = 0$. Since (8.6.9) gives that $P_{m,j}(x^o,\xi^o) \neq 0$ for some j, this is a smooth curve in $R^n \times R^n$ with parameter t. Now we have $dx_j/dt = 0$ when $t = 0$ and since

$$d^2x_j/dt^2 = \sum_{k=1}^{n} P_{m,k}^{(j)}(x,\xi)\, dx_k/dt + \sum_{k=1}^{n} P_m^{(jk)}(x,\xi)\, d\xi_k/dt\ ,$$

it follows that for small t

$$x_j = -\sum_{k=1}^{n} P_m^{(jk)}(x^o,\xi^o)\, P_{m,k}(x^o,\xi^o)t^2 + o(t^2)\ .$$

It follows that the projection of the bicharacteristic into Ω, that is, the curve obtained by eliminating ξ, has a cusp at x^o whose tangent from x^o along the curve has the coordinates

$$-\sum_{k=1}^{n} P_m^{(jk)}(x^o,\xi^o)\, P_{m,k}(x^o,\xi^o)\ ,\ j = 1, \ldots , n\ .$$

The hypothesis of Theorem 8.6.5 thus means that all such cusps, if any, have to lie in the interior of a half space through x^o.

8.7 Estimates, existence and approximation theorems in $H_{(s)}$. In this section we shall prove that the L^2 estimates (8.1.2) lead to a priori estimates in the norm $\| \|_{(s)}$ for every real number s. (See section 2.4 for the definition.) By duality this will give an existence theory for the adjoint operator.

Theorem 8.7.1. Let P be a principally normal differential operator of order m with coefficients in $C^\infty(\Omega)$, and assume that there exists a function $\psi \in C^2(\Omega)$ with $\operatorname{grad} \psi \neq 0$ in Ω and with pseudo-convex level surfaces throughout Ω. If $u \in \mathcal{E}'(\Omega)$ and $P(x,D)u = f \in H_{(s)}$ it then follows that $u \in H_{(s+m-1)}$, and to every real number s and every compact set $K \subset \Omega$ there exists a constant $C_{s,K}$ such that

$$(8.7.1) \qquad \|u\|_{(s+m-1)} \leq C_{s,K}(\|P(x,D)u\|_{(s)} + \|u\|_{(s+m-2)}) , \quad \text{if } u \in \mathcal{E}'(K) .$$

Proof. Let Ω' be an open set with compact closure contained in Ω, chosen so that $K \subset \Omega'$. In view of Theorem 8.6.1 and Borel-Lebesgue's lemma we can approximate ψ in C^2 norm with a function $\psi' \in C^\infty(\Omega)$ so closely that $\operatorname{grad} \psi' \neq 0$ in $\overline{\Omega'}$ and the level surfaces of ψ' are pseudo-convex in $\overline{\Omega'}$. Using Theorem 8.6.2 we can then choose a constant λ so large that $\varphi = e^{\lambda\psi}$ satisfies the hypothesis of Theorem 8.5.1 in $\overline{\Omega'}$. Thus $\varphi \in C^\infty(\Omega)$ and we have

$$(8.7.2) \qquad \tau \sum_{|\alpha| = m-1} |D^\alpha u|^2 e^{2\tau\varphi}\, dx \leq C\{\int |P(x,D)u|^2 e^{2\tau\varphi}\, dx + $$

$$+ \sum_{|\alpha| \leq m-2} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau\varphi}\, dx\}, \quad u \in C_0^\infty(\Omega').$$

Now write $v(x) = u(x)e^{\tau\varphi(x)}$ and introduce the differential operator

$$P_\tau(x,D)v = e^{\tau\varphi}P(x,D)u = e^{\tau\varphi}P(x,D)(ve^{-\tau\varphi}) .$$

It is clear that the coefficients of P_τ are polynomials in τ, and that more precisely

$$(8.7.3) \qquad P_\tau(x,D) = \sum_{j+|\alpha| \le m} a_j^\alpha(x)\tau^j D^\alpha$$

where a_j is in $C^\infty(\Omega)$. It then follows from (8.7.2) that, with another constant C, we have

$$(8.7.4) \qquad \tau \sum_{|\alpha| = m-1} \|D^\alpha v\|_2^2 \le C\{\|P_\tau(x,D)v\|_2^2 + \sum_{|\alpha| \le m-2} \tau^{2(m-|\alpha|)-1}\|D^\alpha v\|_2^2\} ,$$

$$v \in C_o^\infty(\Omega') .$$

In fact, $D^\alpha u = e^{-\tau\varphi} D^\alpha v$ + terms of lower order.

We now assume that we have an element $u \in \mathcal{E}'(K)$ such that $P(x,D)u = f \in H_{(s)}$. Under the additional assumption that we already know that $u \in H_{(s+m-2)}$ we shall then prove that $u \in H_{(s+m-1)}$ and at the same time obtain an estimate (8.7.1). In doing so we note that our assumption means that $v = e^{\tau\varphi} u \in H_{(s+m-2)}$ and that $P_\tau(x,D)v \in H_{(s)}$ (see Theorem 2.2.5). We shall use the methods of section 2.4, choosing a function $\chi \in C_o^\infty(R_n)$ satisfying the hypotheses of Theorem 2.4.1 and such that

$$(8.7.5) \qquad K + \epsilon \operatorname{supp} \chi \subset \Omega', \ 0 \le \epsilon \le 1 .$$

Since $\operatorname{supp} v \subset K$ we then have $v * \chi_\epsilon \in C_0^\infty(\Omega')$ if $0 \le \epsilon \le 1$, hence (8.7.4) gives when $0 \le \epsilon \le 1$

$$(8.7.6) \qquad \tau \sum_{|\alpha|=m-1} \|(D^\alpha v)*\chi_\epsilon\|_2^2 \le C\{2\|(P_\tau(x,D)v)*\chi_\epsilon\|_2^2 +$$

$$+ 2\|P_\tau(x,D)(v*\chi_\epsilon)-(P_\tau(x,D)v)*\chi_\epsilon\|_2^2 + \sum_{|\alpha| \le m-2} \tau^{2(m-|\alpha|)-1}\|(D^\alpha v)*\chi_\epsilon\|_2^2.$$

We now multiply both sides by $\epsilon^{-2s}(1+\delta^2/\epsilon^2)^{-1}\, d\epsilon/\epsilon$ and integrate over ϵ from 0 to 1. In view of (2.4.9) this gives

$$(8.7.7) \qquad \tau \sum_{|\alpha| = m-1} (C_1\|D^\alpha v\|_{(s-1),\delta}^2 - \|D^\alpha v\|_{(s-1)}^2) \le C(2C_2\|P_\tau(x,D)v\|_{(s-1),\delta}^2 +$$

$$+ B + C_2 \sum_{|\alpha| \le m-2} \tau^{2(m-|\alpha|)-1}\|D^\alpha v\|_{(s-1),\delta}^2),$$

where

$$(8.7.8) \quad B = 2C_2 \int_0^1 \|P_\tau(x,D)(v*\chi_\epsilon) - (P_\tau(x,D)v)*\chi_\epsilon\|_2^2\ \epsilon^{-2s}(1+\delta^2/\epsilon^2)^{-1}\, d\epsilon/\epsilon .$$

In order to estimate the term B, which is the effect of the variation of the coefficients, we use the expression for P_τ given in (8.7.3). We may assume that $a_j^\alpha \in C_0^\infty(\Omega)$ since without changing these coefficients in Ω' we can multiply them by a function in $C_0^\infty(\Omega)$ which is equal to 1 in Ω'. Then it follows from Theorem 2.4.2 that

$$(8.7.9) \qquad \int_0^1 \|a_j^\alpha\tau^j(D^\alpha v)*\chi_\epsilon) - (a_j^\alpha\tau^j D^\alpha v)*\chi_\epsilon\|_2^2\ \epsilon^{-2s}(1+\delta^2/\epsilon^2)^{-1}\, d\epsilon/\epsilon$$

$$\le C_3\, \tau^{2j}\, \|D^\alpha v\|_{(s-2),\delta}^2 .$$

If $|\alpha| \neq 0$ we can write $D^\alpha = D_j D^\beta$ for some β with $|\beta| = |\alpha| - 1$ and obtain $\|D^\alpha v\|_{(s-2),\delta} \leq \|D^\beta v\|_{(s-1),\delta}$. From this remark and (8.7.9) it thus follows that the expression B defined by (8.7.8) has the estimate

$$(8.7.10) \qquad B \leq C_4 \sum_{|\alpha| \leq m-1} \tau^{2(m-|\alpha|-1)} \|D^\alpha v\|_{(s-1),\delta}^2 + \tau^{2m} \|v\|_{(s-2),\delta}^2$$

where C_4 is another constant, depending on s but independent of v and of δ. Using (8.7.10) in (8.7.7) we now obtain with another constant C

$$(8.7.11) \quad \tau \sum_{|\alpha|=m-1} \|D^\alpha v\|_{(s-1),\delta}^2 \leq C\{\|P_\tau(x,D)v\|_{(s-1),\delta}^2 + \sum_{|\alpha|=m-1} \|D^\alpha v\|_{(s-1),\delta}^2 + \sum_{|\alpha| \leq m-2} \tau^{2(m-|\alpha|)-1} \|D^\alpha v\|_{(s-1),\delta}^2 + \tau^{2m} \|v\|_{(s-2),\delta}^2\} .$$

Now choose $\tau > 2C$ and move the first sum in the right-hand side of (8.7.11) to the left. We then obtain, using the identity (2.4.2), with still another constant C

$$(8.7.12) \quad \tau \|v\|_{(s+m-2),\delta}^2 \leq C\{\|P_\tau(x,D)v\|_{(s-1),\delta}^2 + \sum_{j=0}^{m-2} \tau^{2(m-j)-1} \|v\|_{(s+j-1),\delta}^2 + \tau^{2m} \|v\|_{(s-2),\delta}^2\} .$$

Recalling that we have assumed that $v \in H_{(s+m-2)}$ and that $P_\tau(x,D)v \in H_{(s)}$ we now let $\delta \to 0$ in (8.7.12). In view of (2.4.4), this gives

$$(8.7.13) \quad \tau \|v\|_{(s+m-1)}^2 \leq C\{\|P_\tau(x,D)v\|_{(s)}^2 + \sum_{j=0}^{m-2} \tau^{2(m-j)-1} \|v\|_{s+j}^2 + \tau^{2m} \|v\|_{(s-1)}^2 ,$$

where C is still another constant. Since $s+m-2 \geq s-1$, this proves that $v \in H_{(s+m-1)}$ and introducing $v = e^{\tau\varphi}u$ we obtain (8.7.1) in view of Theorem 2.2.5.

Finally, assume that we only know that $u \in \mathcal{E}'(K)$ and $P(x,D)u \in H_{(s)}$. Choose a number σ such that $u \in H_{(\sigma+m-2)}$. Then it follows from what we have proved that $u \in H_{(\sigma'+m-1)}$ where $\sigma' = \inf(\sigma,s)$. Iterating this conclusion a finite number of times we obviously obtain $u \in H_{(s+m-1)}$. The proof is complete.

Corollary 8.7.1. Let the hypotheses of Theorem 8.7.1 be fulfilled and let $u \in \mathcal{E}'(K)$ satisfy the equation $P(x,D)u = 0$. Then $u \in C_0^\infty(K)$ and the set of all such functions u is finite dimensional for every compact set $K \subset \Omega$.

Proof. That $u \in C_0^\infty(K)$ follows immediately from Theorem 8.7.1. From (8.7.1) we obtain

$$\|u\|_{(1)} \leq C\|u\|_{(0)}$$

for all such u. Hence it follows from Theorem 2.2.3 that the functions u in the theorem form a locally compact Banach space with the norm $\|u\|_{(0)}$. Hence the space is finite dimensional, which completes the proof.

Theorem 8.7.2. Let the hypotheses of Theorem 8.7.1 be fulfilled and let Ω' be an open set with $\overline{\Omega}'$ compact $\subset \Omega$. Let $f \in H_{(s)}$ and assume that

$$f(u) = 0 \tag{8.7.14}$$

if $u \in C_0^\infty(\bar{\Omega}')$ and $P(x,D)u = 0$. Denote the adjoint of P by ${}^tP$, that is,

$$\int [P(x,D)u]v\,dx = \int u\,{}^tP(x,D)v\,dx;\ u,v \in C_0^\infty(\Omega)\ .$$

Then one can find $v \in H_{(s+m-1)}$ such that ${}^tP(x,D)v = f$ in Ω'.

Remark. It is obvious that P and ${}^tP$ satisfy the hypotheses of Theorem 8.7.1 at the same time, the principal part of ${}^tP$ being $(-1)^m P_m(x,D)$. Hence the theorem applies to P as well as to ${}^tP$.

Proof of the theorem. Let $u_1, \ldots, u_J$ be a basis in the finite dimensional space of solutions with support in $\bar{\Omega}'$ of the equation $P(x,D)u = 0$. (See Corollary 8.7.1.) Choose J functions $v_1, \ldots, v_J$ in $C_0^\infty(\Omega')$ so that

$$\int u_j v_k\,dx = 0 \text{ if } j \neq k, \text{ and } = 1 \text{ if } j = k;\ j,k = 1, \ldots, J\ .$$

We then claim that there is a constant C, depending on s, but not on u, so that

$$(8.7.15)\quad \|u\|_{(-s)}^2 \leq C^2 \|P(x,D)u\|_{(-s-m+1)}^2 \text{ if } u \in C_0^\infty(\bar{\Omega}') \text{ and}$$

$$\int u\,v_j\,dx = 0,\ j = 1, \ldots, J\ .$$

In fact, if this were not true, there would exist a sequence $u_\nu \in C_0^\infty(\bar{\Omega}')$ with $\|u_\nu\|_{(-s)} = 1$ such that $\int u_\nu\,v_j\,dx = 0$, $j = 1, \ldots, J$, and

(8.7.16)
$$\|P(x,D)u_\nu\|_{(-s-m+1)} \to 0,\ \nu \to \infty .$$

Since $\|u_\nu\|_{(-s)} = 1$ we may assume that u_ν converges strongly in $H_{(-s-1)}$ for the sequence is pre-compact there in view of Theorem 2.2.3. The limit u cannot be equal to 0, for if $\|u_\nu\|_{(s-1)} \to 0$ it follows from (8.7.16) and (8.7.1) that $\|u_\nu\|_{(-s)} \to 0$, which contradicts our assumption that $\|u_\nu\|_{(-s)} = 1$ for every ν. But the limit u is in $\mathcal{E}'(\overline{\Omega}')$ and satisfies the equation $P(x,D)u = 0$, hence is in C_0^∞ and we have $\int uv_j\, dx = 0$, $j = 1, \dots, J$. Thus u must be 0 after all and this contradiction proves (8.7.15).

Now consider the linear form

(8.7.17)
$$P(x,D)u \to f(u) \ ,\ u \in C_0^\infty(\overline{\Omega}') .$$

When $u \in C_0^\infty(\overline{\Omega}')$ and $\int uv_j\, dx = 0$, $j = 1, \dots, J$, we have in view of (8.7.15)

(8.7.18)
$$|f(u)| \leq C\|f\|_{(s)}\|P(x,D)u\|_{(-s-m+1)}$$

since $|f(u)| \leq \|f\|_{(s)}\|u\|_{(-s)}$. To see that (8.7.18) is valid for an arbitrary $u \in C_0^\infty(\overline{\Omega}')$ we now only have to apply (8.7.18) to $u - \sum_1^J u_j \int uv_j\, dx$. From Hahn-Banach's theorem and (8.7.18) it follows that the linear form (8.7.17) can be extended to a continuous linear form on $H_{(-s-m+1)}$, that is, there exists an element $v \in H_{(s+m-1)}$ with $\|v\|_{(s+m-1)} \leq C\|f\|_{(s)}$ such that

$$f(u) = v[P(x,D)u],\ u \in C_0^\infty(\Omega') .$$

But this means precisely that ${}^tP(x,D)v = f$ in Ω', which completes the proof.

We shall now prove an approximation theorem for the operator $P(x,D)$ ("The identity of weak and strong extensions").

<u>Theorem 8.7.3</u>. Let the hypotheses of Theorem 8.7.1 be fulfilled, let $u \in \mathcal{E}'(\Omega)$ and assume that for two real numbers s and t we have

$$u \in H_{(s)},\ P(x,D)u \in H_{(t)} . \tag{8.7.19}$$

If $\chi \in C_0^\infty(R_n)$, $\int \chi\, dx = 1$ and we set $\chi_\epsilon(x) = \epsilon^{-n}\chi(x/\epsilon)$, it then follows when $\epsilon \to 0$ that

$$u*\chi_\epsilon \to u \text{ in } H_{(s)},\quad P(x,D)(u*\chi_\epsilon) \to P(x,D)u \text{ in } H_{(t)} . \tag{8.7.20}$$

<u>Remark</u>. Since $P(x,D)u \in H_{(t)}$, it follows from Theorem 8.7.1 that $u \in H_{(t+m-1)}$ and since $u \in H_{(s)}$ we have $P(x,D)u \in H_{(s-m)}$. The most interesting case is therefore when $t+m-1 \le s$ and $s-m \le t$, that is,

$$t+m-1 \le s \le t+m ,$$

for otherwise one hypothesis (8.7.19) implies a strengthened form of the other. However, the general formulation will be useful below.

<u>Proof of Theorem 8.7.3</u>. That $u * \chi_\epsilon \to u$ in $H_{(s)}$ follows from Theorem 2.2.9. Next note that $P(x,D)u \in H_{(t)}$ implies that $u \in H_{(t+m-1)}$ in view of Theorem 8.7.1; hence $D^\alpha u \in H_{(t-1)}$ when $|\alpha| \le m$. Writing

$P(x,D)u = f$ and $P(x,D) = \sum_{|\alpha| \leq m} a^{\alpha}(x)D^{\alpha}$, where the coefficients may be changed outside a neighbourhood of the support of u so that they are in $C_o^{\infty}(\Omega)$, it thus follows from Theorem 2.4.3 that

$$P(x,D)(u*\chi_{\epsilon}) - f*\chi_{\epsilon} = \sum_{|\alpha| \leq m} [a^{\alpha}((D^{\alpha}u) * \chi_{\epsilon}) - (a^{\alpha}D^{\alpha}u) * \chi_{\epsilon}] \to 0$$

in $H_{(t)}$. Since $f * \chi_{\epsilon} \to f$ in $H_{(t)}$ when $\epsilon \to 0$, in view of Theorem 2.2.9 again, the proof is complete.

By duality the following theorem results from Theorem 8.7.3.

Theorem 8.7.4. Let the hypotheses of Theorem 8.7.1 be fulfilled and let

$$v \in H_{(s)}^{loc}(\Omega), \quad {}^tP(x,D)v \in H_{(t)}^{loc}(\Omega) . \tag{8.7.21}$$

Then there exists a sequence $v_{\nu} \in C_o^{\infty}(\Omega)$ such that when $\nu \to \infty$

$$v_{\nu} \to v \text{ in } H_{(s)}^{loc}(\Omega), \quad {}^tP(x,D)v_{\nu} \to {}^tP(x,D)v \text{ in } H_{(t)}^{loc}(\Omega) . \tag{8.7.22}$$

Remark. The most interesting case is again that where $s \leq t+m$. (Cf remark to Theorem 8.7.3.)

Proof. In the direct product space

$$H = H_{(s)}^{loc}(\Omega) \times H_{(t)}^{loc}(\Omega)$$

we consider the linear subspace G of all pairs $(v, {}^tP(x,D)v)$ belonging to the space and denote by G_∞ the set of such pairs with $v \in C_o^\infty(\Omega)$. Since H is metrizable we only have to show that the closure of G_∞ is equal to G. Thus consider a continuous linear form L on H. It is obvious that it can be written in the form

$$H \ni (v_1,v_2) \to < f_1,v_1 > - < f_2,v_2 >$$

where $f_1 \in H^c_{(-s)}(\Omega)$ and $f_2 \in H^c_{(-t)}(\Omega)$, and $< f_j,v_j >$ is the continuous extension of the bilinear form $\int f_j\, v_j\, dx$. (Cf. Theorem 2.2.8.) If L vanishes on G_∞ we then have

$$< f_1,v > = < f_2, {}^tP(x,D)v > , \quad v \in C_o^\infty(\Omega) ,$$

that is, $P(x,D)f_2 = f_1$. If we take X as in Theorem 8.7.3 and set $f_2^\epsilon = f_2 * X_\epsilon$ and $f_1^\epsilon = P(x,D)f_2^\epsilon$, it now follows that $f_2^\epsilon \in C_o^\infty(\Omega)$ and that

$$f_2^\epsilon \to f_2 \text{ in } H_{(-t)}, \quad f_1^\epsilon \to f_1 \text{ in } H_{(-s)} \text{ when } \epsilon \to 0 .$$

The support of f_2 obviously remains in a fixed compact subset of Ω. If $v \in H^{loc}_{(s)}(\Omega)$ and ${}^tP(x,D)v \in H^{loc}_{(t)}(\Omega)$ we have by definition of ${}^tP$ that

$$< f_1^\epsilon,v > - < f_2^\epsilon, {}^tP(x,D)v > = < f_1^\epsilon - P(x,D)f_2^\epsilon,v > = 0 ,$$

which in the limit when $\epsilon \to 0$ proves that the linear form L vanishes at $(v, {}^tP(x,D)v)$, hence in the whole of G. Hahn-Banach's theorem thus gives that G_∞ is dense in G, and the proof is complete.

Remark. Similar results with convergence in a topology which is more restrictive at the boundary may be obtained by a slight change of the methods when the boundary is smooth. See Hörmander [ ].

Also for the homogeneous equation ${}^tP(x,D)v = 0$ we can obtain a similar approximation theorem. (Note that it is much weaker than those proved in section 3.4, but that on the other hand no restriction on the domain is required here.

Theorem 8.7.5. Let the hypotheses of Theorem 8.7.1 be fulfilled and let Ω_s, Ω_∞ be open subsets of Ω such that $\overline{\Omega}_\infty$ is compact and contained in Ω_s. Set

$$N_s = \{v;\ v \in H_{(s)},\ {}^tP(x,D)v = 0 \ \text{ in } \ \Omega_s\} ,$$

which is a closed subspace of $H_{(s)}$, and set

$$N_\infty = \{v;\ v \in H_{(\infty)},\ {}^tP(x,D)v = 0 \ \text{ in } \ \Omega_\infty\}$$

where $H_{(\infty)} \in \bigcap_1^\infty H_{(k)} \subset C^\infty$. Then the closure of N_∞ in $H_{(s)}$ contains N_s.

Proof. a) Choose open sets $\Omega_{s+1}, \Omega_{s+2}, \ldots$ with compact closures such that

$$\Omega_s \supset \overline{\Omega}_{s+1} \supset \Omega_{s+1} \supset \overline{\Omega}_{s+2} \supset \cdots \supset \overline{\Omega}_\infty$$

and define N_k in the same way as N_s for $k \geq s$. Then it is sufficient to prove that the closure of N_{k+1} in H_k contains N_k for every $k \geq s$. For given $v \in N_s$ we can then recursively find $v_k \in N_k$, $k \geq s$, so that $v_s = v$ and

$$\|v_{k+1} - v_k\|_{(k)} < \epsilon\, 2^{-k}$$

where ϵ is any given positive number. It follows that

$$v_\infty = \lim_{j \to \infty} v_j = v_k + \sum_k^\infty (v_{j+1} - v_j)$$

exists in every $H_{(k)}$, hence $v_\infty \in N_\infty$, and

$$\|v_s - v_\infty\|_{(s)} = \|\sum_s^\infty (v_{j+1} - v_j\|_{(s)} \leq \epsilon \sum_s^\infty 2^{-j} ,$$

which proves that the closure of N_∞ contains N_s.

b) To prove that the closure of N_{k+1} contains N_k we let f be any element in $H_{(-k)}$ which is orthogonal to N_{k+1}; in view of Hahn-Banach's theorem we only have to prove that f is then orthogonal to N_k. Since N_{k+1} is the set of all $v \in H_{(k+1)}$ such that $v[P(x,D)u] = 0$ for every $u \in C_0^\infty(\Omega_{k+1})$, the orthogonal complement of N_{k+1} in $H_{(-k-1)}$ is the closure in $H_{(-k-1)}$ of $\{P(x,D)u;\ u \in C_0^\infty(\Omega_{k+1})\}$. Hence there exists a sequence $u_\nu \in C_0^\infty(\Omega_{k+1})$ such that

(8.7.23) $$P(x,D)u_\nu \to f \text{ in } H_{(-k-1)} .$$

at $(v, {}^tP(x,D)v)$, hence in the whole of G. Hahn-Banach's theorem thus gives that G_∞ is dense in G, and the proof is complete.

Remark. Similar results with convergence in a topology which is more restrictive at the boundary may be obtained by a slight change of the methods when the boundary is smooth. See Hörmander [ ].

Also for the homogeneous equation ${}^tP(x,D)v = 0$ we can obtain a similar approximation theorem. (Note that it is much weaker than those proved in section 3.4, but that on the other hand no restriction on the domain is required here.

Theorem 8.7.5. Let the hypotheses of Theorem 8.7.1 be fulfilled and let Ω_s, Ω_∞ be open subsets of Ω such that $\overline{\Omega}_\infty$ is compact and contained in Ω_s. Set

$$N_s = \{v;\ v \in H_{(s)},\ {}^tP(x,D)v = 0 \text{ in } \Omega_s\},$$

which is a closed subspace of $H_{(s)}$, and set

$$N_\infty = \{v;\ v \in H_{(\infty)},\ {}^tP(x,D)v = 0 \text{ in } \Omega_\infty\}$$

where $H_{(\infty)} \in \bigcap_1^\infty H_{(k)} \subset C^\infty$. Then the closure of N_∞ in $H_{(s)}$ contains N_s.

Proof. a) Choose open sets $\Omega_{s+1}, \Omega_{s+2}, \dots$ with compact closures such that

$$\Omega_s \supset \overline{\Omega}_{s+1} \supset \Omega_{s+1} \supset \overline{\Omega}_{s+2} \supset \cdots \supset \overline{\Omega}_\infty$$

and define N_k in the same way as N_s for $k \geq s$. Then it is sufficient to prove that the closure of N_{k+1} in H_k contains N_k for every $k \geq s$. For given $v \in N_s$ we can then recursively find $v_k \in N_k$, $k \geq s$, so that $v_s = v$ and

$$\|v_{k+1} - v_k\|_{(k)} < \epsilon\, 2^{-k}$$

where ϵ is any given positive number. It follows that

$$v_\infty = \lim_{j \to \infty} v_j = v_k + \sum_k^\infty (v_{j+1} - v_j)$$

exists in every $H_{(k)}$, hence $v_\infty \in N_\infty$, and

$$\|v_s - v_\infty\|_{(s)} = \|\sum_s^\infty (v_{j+1} - v_j\|_{(s)} \leq \epsilon \sum_s^\infty 2^{-j} ,$$

which proves that the closure of N_∞ contains N_s.

b) To prove that the closure of N_{k+1} contains N_k we let f be any element in $H_{(-k)}$ which is orthogonal to N_{k+1}; in view of Hahn-Banach's theorem we only have to prove that f is then orthogonal to N_k. Since N_{k+1} is the set of all $v \in H_{(k+1)}$ such that $v[P(x,D)u] = 0$ for every $u \in C_o^\infty(\Omega_{k+1})$, the orthogonal complement of N_{k+1} in $H_{(-k-1)}$ is the closure in $H_{(-k-1)}$ of $\{P(x,D)u;\ u \in C_o^\infty(\Omega_{k+1})\}$. Hence there exists a sequence $u_\nu \in C_o^\infty(\Omega_{k+1})$ such that

(8.7.23) $$P(x,D)u_\nu \to f \text{ in } H_{(-k-1)} .$$

Let $u^1, \ldots, u^J$ be a basis for the solutions of the differential equation $P(x,D)u = 0$ with support in $\overline{\Omega}_{k+1}$ (cf. Corollary 8.7.1) and choose $v^1, \ldots, v^J \in C_o^\infty(\Omega_{k+1})$ so that $\int u^j v^k \, dx = \delta_{jk}$ (Kronecker's delta). Since an inequality of the form (8.7.15) is valid for every s, it then follows that $u_\nu - \sum_1^J u^j \int u_\nu v^j \, dx$ has a limit $u \in H_{(m-k-2)}$ with support in $\overline{\Omega}_{k+1}$, and from (8.7.23) it follows that $P(x,D)u = f$. Since $f \in H_{(-k)}$ by assumption, it follows from Theorem 8.7.1 that $u \in H_{(m-k-1)}$. Now choose χ as in Theorem 8.7.3. For sufficiently small ϵ we then have $u^\epsilon = u * \chi_\epsilon \in C_o^\infty(\Omega_k)$, and $P(x,D)u^\epsilon \to f$ in $H_{(-k)}$ when $\epsilon \to 0$. If $v \in N_k$ we thus have

$$< f,v > = \lim_{\epsilon \to 0} < P(x,D)u^\epsilon, v > = \lim_{\epsilon \to 0} < u^\epsilon, {}^tP(x,D)v > = 0 \,.$$

The proof is complete.

As an application of this approximation theorem we can now give a supplement to Theorem 8.7.2.

<u>Theorem 8.7.6</u>. Let the hypotheses of Theorem 8.7.1 be fulfilled and let Ω' be an open set with $\overline{\Omega}'$ compact and contained in Ω. Let $f \in C^\infty(\Omega)$ and assume that

$$\int f \, u \, dx = 0$$

if $u \in C_o^\infty(\overline{\Omega}')$ and $P(x,D)u = 0$. Then there exists a function $v \in C^\infty(\Omega)$ such that ${}^tP(x,D)v = f$ in Ω'.

<u>Proof</u>. Since we may multiply f by a function in $C_o^\infty(\Omega)$ which is equal to 1 in Ω', it is no restriction to assume that $f \in C_o^\infty(\Omega) \subset H_{(\infty)}$.

Let Ω'' be an open set with compact closure $\subset \Omega$ such that $\Omega'' \supset \overline{\Omega}'$ and every solution of the differential equation $P(x,D)u = 0$ with support in $\overline{\Omega}''$ has its support in $\overline{\Omega}'$. (This choice is possible since there are only a finite number of linearly independent solutions.) In view of Theorem 8.7.2 we can then find $v_k \in H_{(k)}$ such that

$$ {}^tP(x,D)v_k = f \quad \text{in} \quad \Omega'', \; k = 1, 2, \ldots $$

Since ${}^tP(x,D)(v_{k+1}-v_k) = 0$ in Ω'', it follows from Theorem 8.7.5 that there is a function $w_k \in H_{(\infty)}$ such that ${}^tP(x,D)w_k = 0$ in Ω' and

$$ \|v_{k+1} - v_k - w_k\|_{(k)} < 2^{-k} . \tag{8.7.24} $$

It then follows that

$$ v = \lim_{k\to\infty}(v_k - \sum_1^{k-1} w_j) \tag{8.7.25} $$

exists in every $H_{(s)}$. In fact, if s is a positive integer, we have

$$ \lim_{k\to\infty}(v_k - \sum_1^{k-1} w_j) = v_s - \sum_1^{s-1} w_j + \sum_s^{\infty}(v_{k+1} - v_k - w_k) $$

where the series is convergent in $H_{(s)}$ since it follows from (8.7.24) that $\|v_{k+1} - v_k - w_k\|_{(s)} < 2^{-k}$, when $s \leq k$. Hence the function v defined by (8.7.25) is in $H_{(\infty)}$ and since it is obvious that ${}^tP(x,D)v = f$ in Ω', this completes the proof.

We finally give a result on the existence of fundamental solutions which is similar to Theorem 7.2.1. (We do not know how to prove a stronger result analogous to Theorem 7.3.3.)

Theorem 8.7.7. Let the hypotheses of Theorem 8.7.1 be fulfilled and let Ω' be an open set with compact closure contained in Ω. Assume that neither the differential equation $P(x,D)u = 0$ nor the equation ${}^tP(x,D)v=0$ has a solution $\neq 0$ with support in $\overline{\Omega}'$. Then there exists a linear mapping of $L^2(\Omega')$ into itself such that

$$P(x,D)\, Ef = f \quad \text{in } \Omega', \quad \text{if } f \in L^2(\Omega') ; \tag{8.7.26}$$

$$E\, P(x,D)u = u \quad \text{in } \Omega', \quad \text{if } u \in C_0^\infty(\Omega') ; \tag{8.7.27}$$

$$D^\alpha E \quad \text{is a bounded operator in } L^2(\Omega') \text{ if } |\alpha| < m . \tag{8.7.28}$$

For the proof we need an elementary lemma on linear transformations in Hilbert spaces.

Lemma 8.7.1. Let H' and H'' be two Hilbert spaces, and let T_0, T_1 be two linear mappings of H' into H'' such that

i) T_1 is a closed extension of T_0;

ii) The range of T_1 is equal to H'';

iii) T_0 has a continuous inverse, that is,

$$\|f\| \leq C\|T_0 f\|, \; f \in \mathcal{D}_{T_0} ,$$

where $\mathcal{D}_{T_0}$ is the domain of T_0.

Then there exists a bounded linear mapping E of H'' into H' such that

(8.7.29) $$T_1Ef = f, \; f \in H'' \;;$$

(8.7.30) $$ET_o f = f; \; f \in \mathcal{D}_{T_o} \; .$$

Proof. a) Using only i) and ii), we first prove that there exists a bounded linear mapping F of H'' into H' such that (8.7.29) is valid. In fact, let

$$N = \{f; \; f \in H', \; T_1 f = 0\}$$

which is a closed subspace of H' since T_1 is closed. Let N^o be the orthogonal complement of N in H'. Every $f \in H'$ then has a unique decomposition $f = g+h$ with $g \in N$ and $h \in N^o$, and since $T_1 g = 0$ it follows that $f \in \mathcal{D}_{T_1}$ if and only if $h \in \mathcal{D}_{T_1}$ and then we have $T_1 f = T_1 h$. Hence the restriction of T_1 to $\mathcal{D}_{T_1} \cap N^o$ is closed, one to one, and its range is equal to H''. Let F be the inverse of this mapping. Since it is also closed and is defined in the whole of H'', it follows from the closed graph theorem that F is bounded, and (8.7.29) follows from the construction.

b) It is no restriction to assume that T_o is closed, and then it follows from iii) that the range $\mathcal{R}_{T_o}$ of T_o is a closed subspace of H''. Let π be the orthogonal projection of H'' onto $\mathcal{R}_{T_o}$ and set

$$E = T_o^{-1} \pi + F(I-\pi)$$

where F is the mapping constructed in a) and I is the identity mapping. It is then obvious that E is bounded. If $f \in \mathcal{D}_{T_o}$ we have $\pi T_o f = T_o f$, hence $ET_o f = T_o^{-1} T_o f = f$, which proves (8.7.30). Since

$$T_1 E = T_1 T_o^{-1} \pi + T_1 F(I-\pi) = \pi + I-\pi = I$$

in view of i) and the fact that F satisfies (8.7.29), the proof is complete.

Proof of Theorem 8.7.7. Let $H' = \{u;\ D^\alpha u \in L^2(\Omega'),\ |\alpha| \leq m-1\}$ with the norm $\sum_{|\alpha| < m} \|D^\alpha u\|_2^2$, and let $H'' = L^2(\Omega')$. We define the domain of T_1 as the set of all $u \in H'$ such that $P(x,D)u \in H''$ and set $T_1 u = P(x,D)u$ for such u. It is then clear that T_1 is closed, and if we let T_o be the restriction of T_1 with the domain $C_o^\infty(\Omega')$, the condition i) in the lemma is fulfilled. That the range of T_1 is equal to H'' follows from Theorem 8.7.2 with $s = 0$ and condition iii) follows from (8.7.15) with $s = 1-m$. Hence the lemma applies, which proves the theorem.

Remark. A mapping E with the properties of Theorem 8.7.7 may be regarded as the Green's function of a boundary problem. See Visik [ ], Hörmander [ ].

8.8 The unique continuation of singularities. The main result of this section is the following

Theorem 8.8.1. Let P be a principally normal differential operator of order m with C^∞ coefficients, defined in a neighbourhood Ω of a point x^0, and let ψ be a function in $C^2(\Omega)$ such that $\operatorname{grad}\psi(x^0) \neq 0$ and the level surface $\psi(x) = \psi(x^0)$ is pseudo-convex at x^0. If $u \in \mathcal{D}'(\Omega)$ and

(i) $u \in C^\infty(\Omega^+)$, where $\Omega^+ = \{x;\ x \in \Omega,\ \psi(x) > \psi(x^0)\}$

(ii) $P(x,D)u = f \in C^\infty(\Omega)$,

it then follows that $u \in C^\infty(\Omega')$ for some neighbourhood Ω' of x^0.

The proof of Theorem 8.8.1 will require several steps. The most important one is the following lemma.

Lemma 8.8.1. Let P be a differential operator with coefficients in $C_0^\infty(R_n)$ and let φ be a continuous bounded function on R_1 such that

$$(8.8.1)\quad \tau \sum_{|\alpha|=m-1} \int |D^\alpha u|^2 e^{2\tau\varphi(x_n)} dx \leq C\left\{\int |P(x,D)u|^2 e^{2\tau\varphi(x_n)} dx + \sum_{|\alpha|\leq m-2} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau\varphi(x_n)} dx\right\},\ u \in C_0^\infty(\Omega),\ \tau \geq 1 .$$

Also assume that the planes x_n = constant are non-characteristic in Ω. If

$$(8.8.2)\qquad v \in \mathcal{E}'(\Omega) \text{ and } D^\alpha v \in H'_{(\varphi-(1/2))},\ |\alpha| < m ,$$

(8.8.3) $$P(x,D)v = g \in H^{\ell}_{(\varphi)} ,$$

it then follows that

(8.8.4) $$D^\alpha v \in H^{\ell}_{(\varphi)}, \ |\alpha| < m .$$

Proof. That the planes x_n = constant are non-characteristic means that the coefficient $c(x)$ of D_n^m in $P(x,D)$ is $\neq 0$ in Ω. After dividing $P(x,D)$ by $c(x)$ we may thus assume that

(8.8.5) $$P(x,D) = D_n^m + \sum_{|\alpha| \leq m, \alpha_n < m} a^\alpha(x) D^\alpha .$$

In view of (8.8.2) we have, in particular, $D^\alpha v \in H^{\ell}_{(\varphi-1)}$, $|\alpha| < m$. Using (2.4.23) we shall prove that the norms $\|D^\alpha v\|^{\ell}_{(\varphi-1),\delta}$ remain bounded when $\delta \to 0$ and $|\alpha| < m$, which will prove (8.8.4) in virtue of (2.4.22). Thus choose a function χ satisfying the hypotheses of Theorem 2.4.5 and with so small support that

$$\operatorname{supp} v + \epsilon \operatorname{supp} \chi \, dx' \subset \Omega, \ 0 \leq \epsilon \leq 1$$

where $\chi\, dx'$ is again the measure with density χ with respect to the Lebesgue measure in the plane $x_n = 0$. We wish to apply (8.8.1) to $v_\epsilon = v *' \chi$. To see that this is legitimate, we first note that (8.8.1) is valid for every $u \in \mathcal{E}'(\Omega)$ such that $D^\alpha u \in L^2$ when $|\alpha| \leq m$, for such a function can be approximated in $H_{(m)}$ by functions in $C_0^\infty(\Omega)$ with supports in a fixed compact subset of Ω (cf. Theorem 2.2.9). Now we have $D^\alpha v \in H'_{(\varphi-2)}$ if $|\alpha| \leq m$. In fact, this is implied by (8.8.2) if $|\alpha| \leq m$ and $\alpha_n < m$ and hence (8.8.3) gives that $D_n^m u \in H'_{(\varphi-2)}$ in view of (8.8.5). Thus it follows from Theorem 2.4.5 that $D^\alpha(v*'\chi_\epsilon) \in L^2$ when $|\alpha| \leq m$, so we may in fact apply (8.8.1) to $u = v_\epsilon = v*'\chi_\epsilon$. In doing so we choose $\tau = \log(e/\epsilon)$, noting that we then have $\tau \geq 1$ if $0 < \epsilon \leq 1$. Integration of (8.8.1) with respect to ϵ from 0 to 1 after multiplication by $(1 + \delta^2/\epsilon^2)^{-1}\epsilon^{-1}$ now gives

$$(8.8.6) \qquad \sum_{|\alpha|=m-1} \iint_0^1 |D^\alpha v*'\chi_\epsilon|^2 (\log \tfrac{e}{\epsilon})\, \epsilon^{-2\varphi(x_n)} (1 + \delta^2/\epsilon^2)^{-1} dx\, d\epsilon/\epsilon \leq C(M_\delta' + M'')$$

where

$$(8.8.7) \qquad M_\delta' = \iint_0^1 |P(x,D)v_\epsilon|^2 \epsilon^{-2\varphi(x_n)} (1 + \delta^2/\epsilon^2)^{-1}\, dx\, d\epsilon/\epsilon ,$$

$$(8.8.8) \quad M'' = \sum_{|\alpha|\leq m-2} \iint_0^1 |D^\alpha v*'\chi_\epsilon|^2 \epsilon^{-2\varphi(x_n)} (\log \tfrac{e}{\epsilon})^{2(m-|\alpha|)-1}\, dx\, d\epsilon/\epsilon .$$

The integral (8.8.8) is convergent. In fact, we have $D^\alpha v \in H'_{(\varphi+(1/2))}$ when $|\alpha| \leq m-2$ in view of (8.8.2) and the identity

$$(8.8.9) \qquad \|w\|'^2_{(\varphi+(1/2))} = \|w\|'^2_{(\varphi-(1/2))} + \sum_{j=1}^{n-1} \|D_j w\|'^2_{(\varphi-(1/2))}$$

valid for all $w \in \mathcal{E}'$ (with the usual convention that the norm in a complete space is extended to be $+\infty$ outside the space). Hence the convergence of the integral (8.8.8) follows from (2.4.23).

To estimate M_δ' we use the fact that according to (8.8.5)

$$P(x,D)v_\epsilon = [P(x,D)v]*'\chi_\epsilon + \sum_{|\alpha|\le m,\alpha_n<m} [a^\alpha((D^\alpha v)*'\chi_\epsilon) - (a^\alpha D^\alpha v)*'\chi_\epsilon] .$$

Estimating the square of the left-hand side by a constant times the sum of the squares of the terms in the right-hand side we now obtain in virtue of (2.4.23) and (2.4.24)

$$(8.8.10) \qquad M'_\delta \le C(\|P(x,D)v\|'^2_{(\varphi-1),\delta} + \sum_{|\alpha|\le m,\alpha_m<m} \|D^\alpha v\|'^2_{(\varphi-2),\delta} .$$

Introducing $g = P(x,D)v$ and using an analogue of (8.8.9), we thus have with another constant C

$$(8.8.11) \qquad M'_\delta \le C(\|g\|'^2_{(\varphi)} + \sum_{|\alpha|<m} \|D^\alpha v\|'^2_{(\varphi-1),\delta}) .$$

To estimate the left-hand side of (8.8.6) from below we note that it follows from (2.4.23) if $0<\theta<1$ that

$$(8.8.12) \quad C_1\|w\|'^2_{(\varphi-1),\delta} \le (\log \tfrac{e}{\theta})^{-1}\int_0^1\!\!\int |w*'\chi_\epsilon|^2(\log \tfrac{e}{\epsilon})\epsilon^{-2\varphi(x_n)}(1+\delta^2/\epsilon^2)^{-1}dx\, d\epsilon/\epsilon$$

$$+ \theta^{-1}\int_0^1\!\!\int |w*'\chi_\epsilon|^2\, \epsilon^{-2\varphi(x_n)+1}\, dx\, d\epsilon/\epsilon + \|w\|'^2_{(\varphi-1)} ,$$

$$w \in H'_{(\varphi-(1/2))} .$$

In fact, this follows immediately by considering separately the integral (2.4.23) from 0 to θ and from θ to 1. Using Theorem 2.4.5 again we obtain from (8.8.12) with another constant C

$$(8.8.13)\quad \|w\|^{\prime\,2}_{(\varphi-1),\delta} \le C\Big\{(\log \tfrac{e}{\theta})^{-1}\iint |w*'\chi_\epsilon|^2(\log \tfrac{e}{\epsilon})\epsilon^{-2\varphi(x_n)}(1+\tfrac{\delta^2}{\epsilon^2})^{-1}\,dx\,\frac{d\epsilon}{\epsilon}$$

$$+\ \theta^{-1}\|w\|^{\prime\,2}_{(\varphi-(1/2))}\Big\},\quad w \in H'_{(\varphi-(1/2))}\ .$$

Summing up the inequalities (8.8.6), (8.8.11) and (8.8.13), applied to $w = D^\alpha v$, we have now proved that there is a constant C such that for $0 < \delta < 1$ and $0 < \theta < 1$ we have

$$(8.8.14)\quad \sum_{|\alpha|=m-1}\|D^\alpha v\|^{\prime\,2}_{(\varphi-1),\delta} \le C\Big\{(\log \tfrac{e}{\theta})^{-1}[\|g\|^{\prime\,2}_{(\varphi)} + \sum_{|\alpha|<m}\|D^\alpha v\|^{\prime\,2}_{(\varphi-1),\delta} + M''] +$$

$$+\ \theta^{-1}\sum_{|\alpha|<m}\|D^\alpha v\|^{\prime\,2}_{(\varphi-(1/2))}\Big\}\ .$$

Now choose θ so small that $C < \frac{1}{2}\log(e/\theta)$. With a constant C_θ depending on θ it then follows from (8.8.14) that

$$(8.8.15)\quad \sum_{|\alpha|=m-1}\|D^\alpha v\|^{\prime\,2}_{(\varphi-1),\delta} \le \|g\|^{\prime\,2}_{(\varphi)} + \sum_{|\alpha|\le m-2}\|D^\alpha v\|^{\prime\,2}_{(\varphi)} + M'' + C_\theta \sum_{|\alpha|<m}\|D^\alpha v\|^{\prime\,2}_{(\varphi-(1/2))}\ .$$

As already noted it follows from (8.8.2) and (8.8.9) that $D^\alpha v \in H'_{(\varphi+(1/2))} \subset H'_{(\varphi)}$ when $|\alpha| \le m-2$. Hence the right-hand side of (8.8.15) is finite, and when $\delta \to 0$ this proves that $D^\alpha v \in H'_{(\varphi)}$ when $|\alpha| = m-1$ also. The proof of the lemma is complete.

<u>Lemma 8.8.2</u>. If the assumption (8.8.2) is weakened to

$$(8.8.2)'\qquad v \in \mathcal{E}'(\Omega)\ ,\quad D^\alpha v \in H'_{(s)}\ ,\quad |\alpha| < m\ ,$$

for some real number s, while the other hypotheses of Lemma 8.8.1 are left unchanged, it still follows that (8.8.4) holds.

Proof. We can choose a positive integer k such that $\varphi - k/2 < s$ everywhere. Then we have

$$(8.8.16) \qquad D^{\alpha}v \in H'_{(\varphi-k/2)}, \ |\alpha| < m ,$$

so that all assumptions of Lemma 8.8.1 are fulfilled with φ replaced by $\varphi - (k-1)/2$ if k is positive. Hence it follows from Lemma 8.8.1 that (8.8.16) can be improved to

$$(8.8.17) \qquad D^{\alpha}v \in H'_{(\varphi-(k-1)/2)}, \ |\alpha| < m .$$

Repeating this argument k times, we obtain (8.8.4), which proves the lemma.

We next eliminate the condition $v \in \mathcal{E}'(\Omega)$ in (8.8.2)'.

Lemma 8.8.3. Let P and φ satisfy the same assumptions as in Lemma 8.8.1. Let x^0 be a point in Ω and u an element in $\mathcal{D}'(\Omega)$ such that $P(x,D)u = f \in C^{\infty}(\Omega)$ and $u \in C^{\infty}(\omega)$ where ω is an open subset of Ω such that

$$\omega \supset \{x;\ x \in \Omega,\ \varphi(x_n) \geq \varphi(x_n^0),\ x \neq x^0\} .$$

Then it follows that $u \in C^{\infty}$ in a neighbourhood of x^0.

Proof. Let Ω' be an open neighbourhood of x^o such that $\overline{\Omega}' \subset \Omega$. By assumption the planes x_n = constant are non-characteristic, $P(x,D)u = f \in C^\infty(\Omega')$ and u is of finite order in Ω'. Hence it follows from Theorem 4.3.1 that there is a number s such that

(8.8.18) $$D^\alpha u \in H'^{loc}_{(s)}(\Omega'), \ |\alpha| \leq m .$$

Now let $\psi \in C_o^\infty(\Omega')$ be chosen so that $\psi = 1$ in a neighbourhood Ω'' of x^o, and set $v = \psi u$, $g = P(x,D)v$. We then have in view of (8.8.18)

(8.8.19) $$D^\alpha v \in H'_{(s)}, \ |\alpha| \leq m; \quad \text{hence} \quad g \in H'_{(s)} .$$

Furthermore, it is obvious that $g \in C^\infty(\omega)$ and since $g = f$ in Ω'' we have even $g \in C^\infty(\Omega'' \cup \omega)$. Hence we can find an $\epsilon > 0$ such that $g \in C^\infty(\Omega_{4\epsilon})$ where we have used the notation

$$\Omega_t = \{x;\ x \in \Omega,\ \varphi(x_n) > \varphi(x_n^o)-t\} .$$

Assuming as we may that $\varphi(x_n^o) = 0$, we now claim that

(8.8.20) $$g \in H'_{(\lambda(\varphi+2\epsilon))} \quad \text{if} \quad -\lambda\epsilon < s .$$

In fact, we can write $g = g_1 + g_2$ where $g_1 \in C_o^\infty$ and $\varphi(x) \leq -3\epsilon$ in the support of g_2. Then we have $\lambda(\varphi+2\epsilon) \leq -\lambda\epsilon < s$ in the support of g_2, hence $g_2 \in H'_{(\lambda(\varphi+2\epsilon))}$ if $-\lambda\epsilon < s$, and since $g_1 \in C_o^\infty$ this proves (8.8.20). In view of (8.8.19) we can now apply Lemma 8.8.2 and conclude that

(8.8.21) $$D^\alpha v \in H'_{(\lambda(\varphi+2\epsilon))} \quad \text{if} \quad |\alpha| < m \quad \text{and} \quad \lambda > -s/\epsilon .$$

Since $u = v$ in Ω'' it follows in particular that

$$D^\alpha u \in H'^{loc}_{(t)}(\Omega'' \cap \Omega_\epsilon), \; |\alpha| < m ,$$

for an arbitrary t, for we can choose λ so large that $\lambda\epsilon > t$ also. In view of Corollary 4.3.1 (or rather its proof), it now follows that $u \in C^\infty(\Omega'' \cap \Omega_\epsilon)$, which completes the proof of the lemma.

Proof of Theorem 8.8.1. a) We first assume that the surface $\psi(x) = \psi(x^0)$ is non-characteristic at x^0. Let ψ_2 be the Taylor expansion of second order of ψ at x^0 and set with a positive ϵ

$$\psi'(x) = \psi_2(x) - \epsilon|x-x^0|^2 .$$

We then have $\psi'(x) < \psi(x)$ for all $x \neq x^0$ in a neighbourhood of x^0, and for sufficiently small ϵ it follows from Theorem 8.6.1 that there is a neighbourhood Ω' of x^0 such that the level surfaces of ψ' are pseudo-convex and non-characteristic with respect to P in Ω'. Hence Theorem 8.6.2 shows that we may choose λ so large that the estimate (8.1.2) holds with $\varphi = e^{\lambda\psi'}$ and Ω replaced by Ω'. Replacing Ω' by a smaller neighbourhood of x^0, if necessary, we can make a C^∞ change of coordinates so that $\varphi(x) = x_n$ in the new coordinates. But then the hypotheses of Lemma 8.8.3 are all fulfilled so it follows that $u \in C^\infty$ in a neighbourhood of x^0, which proves Theorem 8.8.1 in this case.

b) Now let the surface $\psi(x) = \psi(x^o)$ be characteristic at x^o. In view of Remark 2 following Definition 8.6.1 and the proof of Corollary 5.3.2 (applied to either $\text{Re } P_m$ or $\text{Im } P_m$) the assertion of Theorem 8.8.1 follows from the case already proved in a). The proof is complete.

Example. A solution of the Tricomi equation

$$(x_2 D_1^2 + D_2^2)u = 0$$

in an open set Ω belongs to $C^\infty(\Omega)$ if it is in $C^\infty(\Omega_-)$ where $\Omega_- = \{x;\ x \in \Omega,\ x_2 < 0\}$. For it follows from Theorem 7.4.1 that $u \in C^\infty(\Omega_+)$ where $\Omega_+ = \{x;\ x \in \Omega,\ x_2 > 0\}$ and using Theorem 8.8.1 with $\varphi(x) = -x_2 - (x_1 - t)^2$ we obtain that $u \in C^\infty$ in a neighbourhood of any point $(t,0) \in \Omega$, and the equation is elliptic where $x_2 > 0$.

We shall now construct examples which show that one cannot relax very much the condition of pseudo-convexity in Theorem 8.8.1.

Theorem 8.8.2. Let P be a differential operator with constant coefficients and principal part P_m and let N be a real vector with $P_m(N) = 0$ but

$$P_m'(N) = (P_m^{(1)}(N), \ldots , P_m^{(n)}(N)) \neq 0 .$$

Denote the real projection of the bicharacteristic with direction $P_m'(N)$ by $\sum$, that is, (8.8.22) $$\Sigma = \{\mathrm{Re}(zP'_m(N));\ z \in C\}\ ,$$

which is a linear subspace of R_n of dimension 1 or 2. Then there exists a solution $u \in C^m(R_n)$ of the equation $P(D)u = 0$ which is infinitely differentiable in $\complement\Sigma$ but is not in C^{m+1} in the neighbourhood of any point in Σ.

Proof. The set U of all $u \in C^m(R_n)$ which satisfy the equation $P(D)u = 0$ and are infinitely differentiable outside Σ is a Fréchet space with the obvious topology. (That is, the least fine topology for which the mappings $U \to C^m(R_n)$ and $U \to C^\infty(\complement\Sigma)$ are continuous.) Now let K_j be the countable set of all closed spheres with positive radius and center in a dense countable subset of Σ. If we can prove that the set M_j of all $u \in U$ which are in C^{m+1} in K_j is of the first category for every j, the set $M = \bigcup M_j$ will also be of the first category and every $u \notin M$ will have the desired property.

Since any translation of R_n along Σ maps U onto itself, it is thus sufficient to prove that the set of all $u \in U$ which are in C^{m+1} in the sphere $\{x;\ |x| \le R\}$ is of the first category for every $R > 0$. Assume therefore that this were not true for a certain $R > 0$. In view of the closed graph theorem (see Bourbaki [1], ex. 3, Chap. I, p. 39) we can then find a compact set $K \subset R_n$, a compact set $K' \subset \complement\Sigma$ and constants C and N such that

(8.8.23) $$\sum_{|\alpha| \le m+1} \sup_{|x| \le R} |D^\alpha u(x)| \le C \left\{ \sum_{|\alpha| \le m} \sup_K |D^\alpha u| + \sum_{|\alpha| \le N} |D^\alpha u| \right\},$$

$u \in U$.

By constructing suitable solutions of the equation $P(D)u = 0$ we shall see that this leads to a contradiction.

Let Σ^o be the annihilator of Σ in R_n, that is,

$$\Sigma^o = \{\xi;\ \xi \in R_n,\ < P_m'(N),\ \xi > = 0\} .$$

Σ^o is of dimension $n-1$ or $n-2$ and Σ^o is the real tangent plane of the characteristic surface at N. A variable point in Σ^o will be denoted by σ and we shall write $d\sigma$ for the Lebesgue measure in Σ^o. Let η be a real vector which is not contained in Σ^o.

<u>Lemma</u> 8.8.4. The algebraic equation in τ

$$P[(\tau\eta + N + \sigma)/\delta^2] = 0,\ \sigma \in \Sigma^o , \tag{8.8.24}$$

has a solution $\tau(\sigma,\delta)$ which is analytic in a neighbourhood of $(0,0)$, and when $(\sigma,\delta) \to (0,0)$ we have $\tau(\sigma,\delta) = 0(|\sigma|^2 + |\delta|^2)$.

<u>Proof</u>. If we write $P = P_m + P_{m-1} + \cdots$ where P_k is homogeneous of degree k, the equation (7.6) becomes

$$P_m(\tau\eta + N + \sigma) + \delta^2\ P_{m-1}(\tau\eta + N + \sigma) + \cdots = 0 .$$

When $\tau = \sigma = \delta = 0$ the derivative with respect to τ is $< P_m'(N),\ \eta >$ which is $\neq 0$ by assumption. Hence it follows by the implicit function theorem that (8.8.24) has an analytic solution τ vanishing when $\delta = \sigma = 0$, and it is obviously even in δ. Since $P_m(N+\sigma) = 0(|\sigma|^2)$ when $\Sigma^o \ni \sigma \to 0$, it follows that the Taylor expansion of $\tau(\sigma,\delta)$ cannot have any first order terms at all, which completes the proof of the lemma.

<u>End of the proof of Theorem 8.8.2</u>. Take a function $\varphi \in C_o^\infty(\sum^o)$ with $\int \varphi(\sigma)d\sigma = 1$ and set

$$(8.8.25) \qquad u_\delta(x) = \int \exp(i\delta^{-2} < x, \tau\eta + N + \sigma >) \varphi(\sigma/\delta)\, d(\sigma/\delta)$$

where τ is the function discussed in Lemma 7.3. For sufficiently small δ the definition (8.8.25) makes sense, $u_\delta \in C^\infty(R_n)$ and $P(D)u_\delta = 0$ in view of (8.8.24). We shall prove that (8.8.23) cannot hold for any C and N when $u = u_\delta$ and $\delta \to 0$.

First note that differentiation of (8.8.25) under the integral sign gives

$$(8.8.26) \qquad \sup_K |D^\alpha u| = O(\delta^{-2|\alpha|})$$

since $|\tau\delta^{-2}|$ and $|\sigma|$ are bounded when $\sigma/\delta \in \operatorname{supp} \varphi$ and $\delta \to 0$. On the other hand, let y be a vector with $< y,N > = 1$. (The theorem is trivial if $m = 1$ for the differential operator $P(D)$ then acts only along $\sum$. Hence we may assume that $m > 1$ and then it follows that $N \neq 0$ since $P'_m(N) \neq 0$.) We now have for integer $j > 0$

$$< y,D >^j u_\delta(0) = \delta^{-2j} \int (1 + < y, \tau\eta + \sigma >)^j \varphi(\sigma/\delta) d(\sigma/\delta) =$$

$$= \delta^{-2j} \int (1 + < y, \tau(\delta\sigma,\delta)\eta + \delta\sigma >)^j \varphi(\sigma) d\sigma .$$

When $\delta \to 0$, the last integral converges to 1. Hence the left-hand side of (8.8.23) with $u = u_\delta$ tends to ∞ at least as fast as $\delta^{-2(m+1)}$ when $\delta \to 0$.

To complete the proof we shall now show that the last sum in the right-hand side of (8.8.23) tends to 0 when $u = u_\delta$ and $\delta \to 0$. With the notation

$$\tau_\delta^*(\sigma) = \delta^{-2} \tau(\delta\sigma,\delta)$$

we have after a substitution of variables

$$(8.8.27) \qquad u_\delta(x)e^{-i\langle x,N\rangle/\delta^2} = v_\delta(x) = \int \exp(i\langle x,\tau_\delta^*\eta + \sigma/\delta\rangle)\, \varphi(\sigma)d\sigma .$$

It will clearly be sufficient for us to prove that all derivatives of v_δ tend to 0 in K' faster than any power of δ. Differentiation of (8.8.27) gives

$$(8.8.28) \qquad D^\alpha v_\delta(x) = \int e^{i\langle x/\delta,\sigma\rangle} e^{i\langle x,\eta\rangle\tau_\delta^*}(\tau_\delta^*\eta + \sigma/\delta)^\alpha\, \varphi(\sigma)d\sigma .$$

Now recall that for functions $\psi \in C_o^\infty(R_k)$ with supports in a fixed compact set and any integer $j \geq 0$, one can estimate $|\xi|^j|\hat{\psi}(\xi)|$ by a constant (independent of ψ) times an upper bound for the derivatives of ψ of order $\leq j$ (see the proof of Lemma 1.7.1). Denoting the distance from x to $\sum$ by $|x|_\Sigma$ we thus obtain from (8.8.28) that $|x/\delta|_\epsilon^j\, |D^\alpha v_\delta(x)|$ can be estimated by means of an upper bound for $e^{i\langle x,\eta\rangle\tau_\delta^*}(\tau_\delta^*\eta+\sigma/\delta)^\alpha\varphi(\sigma)$ and the derivatives with respect to σ of order $\leq j$. When $x \in K'$, the distance $|x|_\Sigma$ is bounded from below and since all derivatives of τ_δ^* are uniformly bounded in $\operatorname{supp} \varphi$ we obtain that

$$\delta^{-j}|D^\alpha v_\delta(x)| = O(\delta^{-|\alpha|}) \quad \text{when} \quad x \in K' .$$

Since j is arbitrary, this proves the assertion that $D^\alpha v_\delta(x)$ tends to 0 in K' faster than any power of δ. The proof is complete.

8.9 The uniqueness of the Cauchy problem. In analogy to Theorem 8.8.1 we shall prove

Theorem 8.9.1. Let $P(x,D)$ be a differential operator of order m with bounded measurable coefficients in a neighbourhood Ω of a point x^0. Also assume either that P is principally normal and that the coefficients in the principal part are in $C^2(\Omega)$, or else that P_m has real C^1 coefficients or that P_m is elliptic with C^1 coefficients. Let ψ be a function in $C^2(\Omega)$ such that $\operatorname{grad} \psi(x^0) \neq 0$ and the level surface $\psi(x) = \psi(x^0)$ is strongly pseudo-convex at x^0. If $u \in H_{(m)}^{loc}(\Omega)$ satisfies the equation $P(x,D)u = 0$ and $u = 0$ in $\{x;\ x \in \Omega,\ \psi(x) > \psi(x^0)\}$, it then follows that $u = 0$ in a neighbourhood of x^0.

Proof. As in the proof of Theorem 8.8.1 we can, using Theorems 8.6.1 and 8.6.3, find a neighbourhood Ω' of x^0, a function $\varphi \in C^\infty(\Omega')$ and an open set $\omega \subset \Omega'$ with $\omega \supset \{x;\ x \in \Omega',\ \varphi(x) \geq \varphi(x^0),\ x \neq x^0\}$ such that $u = 0$ in ω and

$$(8.9.1) \quad \tau \sum_{|\alpha| \leq m-1} \int |D^\alpha v|^2 e^{2\tau\varphi} dx \leq C \int |P(x,D)v|^2 e^{2\tau\varphi} dx, \quad v \in C_0^\infty(\Omega') .$$

It is clear that (8.9.1) must also be valid for all $v \in \mathcal{E}'(\Omega) \cap H_{(m)}$ since such functions v can be approximated in $H_{(m)}$ by functions in $C_0^\infty(\Omega)$ with supports in a fixed bounded set. (In fact, (8.9.1) is valid for all $v \in \mathcal{E}'(\Omega) \cap H_{(m-1)}$ such that $P(x,D)v \in L^2$. This follows by using Theorem

2.4.3 with $s = -1$. However, we have not proved that Theorem 2.4.3 is then valid for arbitrary $a \in C_0^1$, so we refrain from the weakening of the assumption $u \in H_{(m)}^{loc}$ to $u \in H_{(m-1)}^{loc}$ which this could give. See, however, Friedrichs [ ], Hörmander [ ]).

Now let $\psi \in C_0^\infty(\Omega')$ be equal to 1 in a neighbourhood Ω'' of x^0, and set $v = \psi u$. We then have $v \in H_{(m)}$ and $P(x,D)v = 0$ in $\omega \cup \Omega''$ because $v = u = 0$ in ω and $v = u$ in Ω''. This proves that there is an $\epsilon > 0$ such that $\varphi(x) \leq \varphi(x^0) - \epsilon$ when $x \in \operatorname{supp} P(x,D)v$. With $\Omega_\epsilon = \{x;\ x \in \Omega,\ \varphi(x) > \varphi(x^0) - \epsilon\}$, we thus obtain using (8.8.1) that

$$\tau \sum_{|\alpha| \leq m-1} \int_{\Omega_\epsilon} |D^\alpha v|^2 e^{2\tau(\varphi(x^0) - \epsilon)} dx \leq C \int |P(x,D)v|^2 e^{2\tau(\varphi(x^0) - \epsilon)} dx ,$$

hence

$$\sum_{|\alpha| \leq m-1} \int_{\Omega_\epsilon} |D^\alpha v|^2 dx \leq C\tau^{-1} \int |P(x,D)v|^2 dx .$$

When $\tau \to \infty$ the right hand side $\to 0$ and it follows that $v = 0$ in Ω_ϵ. Recalling that $v = u$ in Ω'', we now have $u = 0$ in $\Omega'' \cap \Omega_\epsilon$ which completes the proof.

<u>Remark</u>. Theorem 8.9.1 may also be regarded as a theorem on unique continuation for solutions of a differential inequality

$$|P_m(x,D)u| \leq K \sum_{|\alpha| < m} |D^\alpha u| .$$

In fact, if u satisfies this condition and we set

$$a_\alpha = -(P_m(x,D)u)\overline{(D_\alpha u)}\Big(\sum_{|\beta|<m} |D_\beta u|^2\Big)^{-1}, \quad |\alpha| < m,$$

the operator

$$P(x,D) = P_m(x,D) + \sum_{|\alpha|<m} a_\alpha(x)D_\alpha$$

has bounded coefficients and $P(x,D)u = 0$, so that Theorem 8.9.1 can be applied.

Corollary 8.9.1. Let P have C^∞ coefficients in a neighbourhood Ω of x^0 and be either elliptic or principally normal. Further, let ψ be a function in $C^2(\Omega)$ such that $\operatorname{grad}\psi(x^0) \neq 0$ and the level surface $\psi(x) = \psi(x^0)$ is strongly pseudo-convex at x^0. If $u \in \mathcal{D}'(\Omega)$ satisfies the equation $P(x,D)u = 0$ and $u = 0$ in $\{x;\ x \in \Omega,\ \psi(x) > \psi(x^0)\}$, it then follows that $u = 0$ in a neighbourhood of x^0.

Proof. From Theorem 8.8.1 it follows that $u \in C^\infty$ in a neighbourhood of x^0 and hence the corollary follows from Theorem 8.9.1.

2.4 <u>The spaces</u> $H_{(s)}$. In Chapters VIII, IX and X, where we deal with classes of differential equations with variable coefficients defined by conditions on the principal part only, the following special spaces $H_{p,k}$ will be important.

<u>Definition 2.4.1</u>. We shall denote by $H_{(s)}$ the space H_{2,k_s} where

$$k_s(\xi) = (1+|\xi|^2)^{s/2}$$

and we shall write $\|u\|_{(s)} = \|u\|_{2,k_s}$, thus

(2.4.1) $$\|u\|_{(s)}^2 = (2\pi)^{-n} \int |\hat{u}(\xi)|^2 (1+|\xi|^2)^s \, d\xi \, .$$

When s is a non-negative integer, the space $H_{(s)}$ obviously consists of all $u \in L^2$ such that $D^\alpha u \in L^2$ when $|\alpha| \leq s$. Also note that the obvious identity

(2.4.2) $$\|u\|_{(s+1)}^2 = \|u\|_{(s)}^2 + \sum_1^n \|D_j u\|_{(s)}^2$$

shows that for arbitrary s the space $H_{(s+1)}$ consists of those $u \in H_{(s)}$ such that $D_j u \in H_{(s)}$ for $j = 1, \ldots, n$.

We shall now show that for arbitrary s one can express the norm (2.4.1) simply in terms of the L^2 norms of regularizations of u. (These results are only needed in sections 8.7 and 8.8.) In order to overcome some technical difficulties in sections 8.7 and 8.8 we shall also consider the following two parameter family of norms

$$(2.4.3) \qquad \|u\|_{(s-1),\delta}^2 = (2\pi)^{-n} \int |\hat{u}(\xi)|^2 (1+|\xi|^2)^s (1+|\delta\xi|^2)^{-1} \, d\xi \ .$$

This norm is clearly equivalent to $\|u\|_{(s-1)}$ when $u \in H_{(s-1)}$ and $\delta \neq 0$. It is clear that

$$(2.4.4) \qquad \|u\|_{(s-1),\delta} \nearrow \|u\|_{(s)} \quad \text{when} \quad \delta \searrow 0, \ u \in H_{(s-1)} \ .$$

(We interpret $\|u\|_{(s)}$ as $+\infty$ if $u \notin H_{(s)}$.)

Let $\chi \in C_0^\infty(R_n)$ and assume that

$$(2.4.5) \qquad \hat{\chi}(\xi) = O(|\xi|^k), \ \xi \to 0 \ ,$$

but that

$$(2.4.6) \qquad \hat{\chi}(t\xi) = 0 \ \text{ for all real } \ t \Rightarrow \xi = 0 \ .$$

It is clear that every χ with $\hat{\chi}(0) = \int \chi \, dx \neq 0$ has these two properties for $k = 0$, and when $k > 0$ we only have to apply a power of the Laplacean to such a function in order to fulfill (2.4.5) and (2.4.6). If $\epsilon > 0$ we shall write

$$(2.4.7) \qquad \chi_\epsilon(x) = \epsilon^{-n} \chi(x/\epsilon) \ ,$$

thus

$$(2.4.8) \qquad \hat{\chi}_\epsilon(\xi) = \hat{\chi}(\epsilon\xi) \ .$$

Theorem 2.4.1. If (2.4.5) and (2.4.6) are valid and $s < k$ it follows that there exist positive constants C_1 and C_2, independent of δ and u but depending on s and χ, such that when $0 < \delta \leq 1$

$$(2.4.9)\qquad C_1\|u\|_{(s-1),\delta}^2 \leq \int_0^1 \|u * \chi_\epsilon\|_2^2\, \epsilon^{-2s}(1+\delta^2/\epsilon^2)^{-1}\, d\epsilon/\epsilon + \|u\|_{(s-1)}^2$$

$$\leq C_2\|u\|_{(s-1),\delta}^2,\ u \in H_{(s-1)}\ .$$

The second inequality is still true if (2.4.6) fails to hold.

Proof. It is sufficient to assume that $u \in S$ in the proof, for S is dense in $H_{(s-1)}$ (Lemma 2.2.1) and both sides of (2.4.9) are continuous in $H_{(s-1)}$. Using Parseval's formula we obtain

$$(2.4.10)\qquad \int_0^1 \|u * \chi_\epsilon\|_2^2\, \epsilon^{-2s}(1+\delta^2/\epsilon^2)^{-1}\, d\epsilon/\epsilon = \int |\hat{u}(\xi)|^2\, F(\xi)\, d\xi$$

where

$$(2.4.11)\qquad F(\xi) = \int_0^1 |\hat{\chi}(\epsilon\xi)|^2\, \epsilon^{-2s}(1+\delta^2/\epsilon^2)^{-1}\, d\epsilon/\epsilon\ .$$

Writing $\eta = \xi/|\xi|$, which is thus a unit vector, and introducing $t = \epsilon|\xi|$ as a new variable, we obtain

$$F(\xi) \leq |\xi|^{2s} \int_0^\infty |\hat{\chi}(t\eta)|^2\, t^{-2s}(1+|\delta\xi|^2/t^2)^{-1}\, dt/t$$

$$\leq |\xi|^{2s}(1+|\delta\xi|^2)^{-1}\left\{ \int_0^1 |\hat{\chi}(t\eta)|^2\, t^{-2s}\, dt/t + \int_1^\infty |\hat{\chi}(t\eta)|^2 t^{2-2s} dt/t \right\}\ .$$

Since it follows from (2.4.5) that $|\hat{\chi}(t\eta)| \leq M\, t^k$ for a constant M and since $\hat{\chi} \in S$, the two integrals in the right-hand side of this estimate are bounded functions of η when $|\eta| = 1$, which proves the latter part of (2.4.9). To prove the other part, we note that when $|\xi| \geq 1$ the same argument gives

$$F(\xi) \geq |\xi|^{2s}(1+|2\delta\xi|^2)^{-1} \int_{1/2}^{1} |\hat{\chi}(t\eta)|^2\, t^{-2s}\, dt/t ,$$

and the integral in the right-hand side is a continuous function of η which is always different from 0 in view of (2.4.6) and the analyticity of t. Hence for some positive C_1 we have

$$F(\xi) \geq C_1(1+|\xi|^2)^s(1+|\delta\xi|^2)^{-1} \text{ when } |\xi| \geq 1 . \tag{2.4.12}$$

Since

$$\tfrac{1}{2}(1+|\xi|^2)^s(1+|\delta\xi|^2)^{-1} \leq (1+|\xi|^2)^{s-1} \text{ when } |\xi| \leq 1,\quad 0 < \delta \leq 1 ,$$

the first part of (2.4.9) follows from (2.4.12) if $C_1 \leq 1/2$. The proof is complete.

<u>Corollary 2.4.1</u>. From the same assumptions as in Theorem 2.4.1 it follows that

$$C_1\|u\|_{(s)}^2 \leq \int_0^1 \|u * \chi_\epsilon\|^2\, \epsilon^{-2s}\, d\epsilon/\epsilon + \|u\|_{(s-1)}^2 \leq C_2\|u\|_{(s)}^2,\quad u \in H_s . \tag{2.4.13}$$

Proof. If we let $\delta \to 0$ in (2.4.9), the estimate (2.4.13) follows in view of (2.4.4).

We next give an estimate for the commutator of regularization and multiplication with smooth functions, which is important in studying operators with variable coefficients.

Theorem 2.4.2. Let $a \in S$ and let χ satisfy (2.4.5). If $s < k$, there exists a constant C_3, independent of δ and u, such that when $0 < \delta \leq 1$ we have

$$(2.4.14) \quad \int_0^1 \|a(u*\chi_\epsilon)-(au)*\chi_\epsilon\|^2 \epsilon^{-2s}(1+\delta^2/\epsilon^2)^{-1} d\epsilon/\epsilon \leq C_3\|u\|_{(s-2),\delta}^{\;2}, \quad u \in H_{(s-2)}.$$

Proof. We may assume in the proof that $u \in S$ for S is dense in $H_{(s-2)}$ and the left-hand side of (2.4.14) is obviously semi-convex from below in $\mathcal{O}'$ even. The commutator to estimate is

$$U(x,\epsilon) = \int (a(x)-a(x-y))\, u(x-y)\, \chi_\epsilon(y)\, dy .$$

We shall use Taylor's formula with remainder term,

$$a(x)-a(x-y) = -\sum_{0<|\alpha|<N} (-y)^\alpha a^{(\alpha)}(x) + R_N(x,y) ,$$

where N will be chosen later and we have used the notation

$$a^{(\alpha)} = i^{|\alpha|} D^\alpha a/\alpha! .$$

If we write $\chi^{\alpha}(y) = -(-y)^{\alpha}\chi(y)$, we thus have

$$(2.4.15)\quad U(x,\epsilon) = \sum_{0<|\alpha|<N} a^{(\alpha)}(x)\epsilon^{|\alpha|} u * \chi^{\alpha}_{\epsilon}(x) + \int R_N(x,y)u(x-y)\chi_{\epsilon}(y)dy \ .$$

It follows from 2.4.5 that $\hat{\chi}^{\alpha}(\xi) = O(|\xi|^{k-|\alpha|})$, $\xi \to 0$, hence Theorem 2.4.1 gives since $a^{(\alpha)}$ is bounded

$$\int_0^1 \int |a^{(\alpha)}|^2 |u * \chi^{\alpha}_{\epsilon}|^2 \epsilon^{-2(s-|\alpha|)}(1+\delta^2/\epsilon^2)^{-1}\, dx\, d\epsilon/\epsilon \leq C\|u\|^2_{(s-|\alpha|-1),\delta}$$

$$\leq C\|u\|^2_{(s-2),\delta} \quad \text{if} \quad |\alpha| \neq 0 \ .$$

It thus only remains to estimate the last term in (2.4.15), for which we introduce the notation

$$U_N(x,\epsilon) = \int R_N(x,y)\, u(x-y)\, \chi_{\epsilon}(y)\, dy \ .$$

First note that for fixed x we have

$$|\epsilon^{-s-(1/2)}\, U_N(x,\epsilon)| \leq \|u\|_{(s-2)} \|\epsilon^{-s-(1/2)}\, R_N(x,\cdot)\, \chi_{\epsilon}(\cdot)\|_{(2-s)} \ .$$

Let the support of χ be contained in the sphere $|x| \leq R$ and set

$$A(x) = \sup_{|\alpha|=N;\,|x-y|\leq R} |D^{\alpha}a(y)| \ .$$

Since $a \in S$, we have $A \in L^2$. We can estimate $D_y^\alpha R_N(x,y)$ when $|x-y| \leq R$ by a constant times $A(x)|x-y|^{N-|\alpha|}$ if $|\alpha| \leq N$. Hence we obtain if $s \geq 2$ that

$$\|\epsilon^{-s-(1/2)} R_N(x,.)\chi_\epsilon(.)\|_{(2-s)} \leq \|\epsilon^{-s-(1/2)} R_N(x,.)\chi_\epsilon(.)\|_{(o)} = O(A(x)\epsilon^{N-s-(n+1)/2},$$

which is $O(A(x))$ when $0 < \epsilon < 1$ if we choose $N \geq s+(n+1)/2$. On the other hand, if $s < 2$ and σ is an integer $\geq 2-s$, we have if $N \geq \sigma+s+(n+1)/2$ that

$$\|\epsilon^{-s-(1/2)} R_N(x,.)\chi_\epsilon(.)\|_{(2-s)} \leq \|\epsilon^{-s-(1/2)} R_N(x,.)\chi_\epsilon(.)\|_{(\sigma)} = O(A(x))$$

when $0 < \epsilon < 1$. In both cases it follows that

$$\int_0^1 \int |U_N(x,\epsilon)|^2 \epsilon^{-2s}(1+\delta^2/\epsilon^2)^{-1} dx \frac{d\epsilon}{\epsilon} \leq C\|u\|_{(s-2)}^2 \leq C\|u\|_{(s-2),\delta}^2 .$$

The proof is complete.

Corollary 2.4.2. From the same assumptions as in Theorem 2.4.1 it follows that

$$(2.4.16) \quad \int_0^1 \|a(u * \chi_\epsilon)-(au) * \chi_\epsilon\|^2 \epsilon^{-2s} d\epsilon/\epsilon \leq C_3\|u\|_{(s-1)}^2, \quad u \in H_{(s-1)} .$$

We shall also need the following result which is very closely related to Theorem 2.4.2.

Theorem 2.4.3. If $u \in H_{(s-1)}$ and $a \in S$, we have

$$(2.4.17) \qquad a(u * \chi_\epsilon) - (au) * \chi_\epsilon \to 0 \text{ in } H_{(s)} \text{ when } \epsilon \to 0 .$$

Proof. The statement is obvious if $u \in C_0^\infty(R_n)$ and since this is a dense subset of $H_{(s-1)}$ it is thus sufficient to prove the estimate

$$(2.4.18) \quad \|a(u * \chi_\epsilon) - (au) * \chi_\epsilon\|_{(s)} \leq C_s\|u\|_{(s-1)}, \quad u \in S, \quad 0 < \epsilon < 1 .$$

(Cf. the proof of Theorem 1.2.1.) With the notations of the proof of Theorem 2.4.2 we thus have to prove that $\|U(x,\epsilon)\|_{(s)} \leq C_s\|u\|_{(s-1)}$. But this follows from (2.4.15) since

$$\|u * \epsilon\chi_\epsilon^\alpha\|_{(s)} \leq \|u\|_{(s-1)} \sup |\hat{\chi}^\alpha(\xi)|(\epsilon^2+|\xi|^2)^{1/2}$$

and the norm of the remainder term in (2.4.15) in any space $H_{(\sigma)}$ can be estimated in terms of $\|u\|_{(s-1)}$ if N is chosen large enough. In fact, this follows by an obvious change of the estimates in the proof of Theorem 2.4.2 which may be left to the reader.

In section 8.8 we shall also have to consider spaces of distributions which can be regarded as square integrable functions of x_n with values in $H_{(\varphi)}(R_{n-1})$ where φ is a function of x_n. More precisely, we introduce the following definition.

Definition 2.4.2. Let φ be a bounded continuous function in R_1. We then denote by $H'_{(\varphi)}$ the space of all $u \in S'(R_n)$ such that the partial Fourier transform $\hat{u}_n$ (defined by (1.7.21) and (1.7.23)) is locally square integrable and

(2.4.19) $\|u\|'^{\,2}_{(\varphi)} = (2\pi)^{1-n} \int |\hat{u}_n(\xi',x_n)|^2 (1+|\xi'|^2)^{\varphi(x_n)} \, d\xi' \, dx_n < \infty$.

Theorem 2.4.4. $H'_{(\varphi)}$ is a Hilbert space with the norm $\|u\|'_{(\varphi)}$, $C_0^\infty(R_n)$ is dense in $H'_{(\varphi)}$ and if $\psi \in S$, $u \in H'_{(\varphi)}$ we have $\psi u \in H'_{(\varphi)}$.

Proof. The completeness follows immediately as in the proof of Theorem 2.2.1 from the fact that every function $\hat{u}_n$ such that the integral (2.4.19) is finite must automatically be in S' since φ is bounded. To prove that $C_0^\infty(R_n)$ is dense we only have to note that if $v \in H'_{(\varphi)}$ and

$$(2.4.20) \qquad \int \hat{u}_n(\xi',x_n) \, \overline{\hat{v}_n(\xi',x_n)} (1+|\xi'|^2)^{\varphi(x_n)} \, d\xi' \, dx_n = 0$$

for every $u \in C_0^\infty$, it follows that $\hat{v}_n(\xi',x_n)(1+\xi^2)^{\varphi(x_n)} = 0$ for (2.4.20) means that the partial Fourier transform of this function is 0. Since the last statement follows immediately from Theorem 2.2.5, the proof is complete.

As above we shall also use the norms

$$(2.4.21) \quad \|u\|'_{(\varphi-1),\delta} = \{(2\pi)^{1-n} \int |\hat{u}_n(\xi',x_n)|^2 (1+|\xi'|^2)^{\varphi(x_n)} (1+|\delta\xi|^2)^{-1} d\xi' dx_n\}^{1/2}$$

which are equivalent to $\|u\|_{(\varphi-1)}$ when $u \in H_{(\varphi-1)}$ and $\delta > 0$. When $\delta \searrow 0$ we obviously have

$$(2.4.22) \qquad \|u\|'_{(\varphi-1),\delta} \nearrow \|u\|'_{(\varphi)}, \quad u \in H'_{(\varphi-1)}$$

with the usual convention that $\|u\|'_{(\varphi)}$ is defined as $+\infty$ if $u \notin H'_{(\varphi)}$.

Furthermore, Theorems 2.4.1 and 2.4.2 can immediately be extended to the spaces $H'_{(\varphi)}$. Thus let $\chi \in C_0^\infty(R_{n-1})$, denote by $\chi\, dx'$ the measure $\chi(x')dx_1 \cdots dx_{n-1}$ in the plane $x_n = 0$ and set for $u \in \mathcal{S}'(R_n)$

$$u *' \chi = u * (\chi\, dx') .$$

Theorem 2.4.5. Let (2.4.5) be valid (for $\xi \in R_{n-1}$) and assume that φ is bounded, $\sup \varphi < k$. Then there exist positive constants C_1 and C_2, independent of δ and of u but depending on χ and on φ such that when $0 < \delta \leq 1$

$$(2.4.23) \qquad C_1 \|u\|'^2_{(\varphi-1),\delta} \leq \iint_0^1 |u*'\chi_\epsilon|^2 \epsilon^{-2\varphi(x_n)} (1+\delta^2/\epsilon^2)^{-1}\, dx\, d\epsilon/\epsilon +$$

$$+ \|u\|'^2_{(\varphi-1)} \leq C_2 \|u\|'^2_{(\varphi-1),\delta} , \quad u \in H'_{(\varphi-1)} .$$

In particular, this means that $u *' \chi_\epsilon \in L_2$ when $\epsilon > 0$.

Proof. The theorem follows from (2.4.9) since we may assume that $u \in C_0^\infty(R_n)$ in view of Theorem 2.4.4 and since the proof of Theorem 2.4.1 shows that the constants C_1 and C_2 of that theorem remain bounded when s belongs to a compact subset of the interval $-\infty < s < k$.

Similarly, Theorem 2.4.2 gives

Theorem 2.4.6. Let $a \in S$ and let χ satisfy (2.4.5). If φ is bounded, $\sup \varphi < k$, there then exists a constant C_3, independent of δ and u, such that when $0 < \delta \leq 1$ we have

$$(2.4.24)\quad \iint_0^1 |a(u*'X)-(au)*'X|^2 e^{-2\varphi(x_n)}(1+\delta^2/\epsilon^2)^{-1}dx d\epsilon/\epsilon \le C_3\|u\|'^{\,2}_{(\varphi-2),\delta},$$

$$u \in H'_{(\varphi-2)} \quad .$$

PSEUDO-PERIODIC FUNCTIONS

by

J. P. Kahane

These lectures are devoted to the subject of pseudo-periodic functions, which were first introduced by Paley and Wiener in their book (1934) but have not received much attention since then. First, we attempt to give a general idea of the material to be covered:

If $s(x) = \sum a_n e^{i\lambda_n x}$, then $s(x)$ is almost periodic, with spectrum $\{\lambda_n\}$; $s(x)$ does not look like a periodic function except (as was observed by Bochner in 1926) when $\{\lambda_n\}$ is regular ($\equiv$ uniformly discrete, in Beurling's terminology), i.e., $\lambda_{n+1}-\lambda_n \geq \delta > 0$. We call $\delta=\inf|\lambda_{n+1}-\lambda_n|$ the step of the sequence $\{\lambda_n\}$.

The following definition was given by Paley and Wiener (1934): f is pseudo-periodic is f is locally square integrable and if for any two sufficiently large intervals I and J the norms in $L^2(I)$ and $L^2(J)$ of all linear combinations of translates of f are equivalent, that is, the ratio of thes norms is bounded from above and below by positive constants. Paley and Wiener proved the following:

1) f <u>is pseudo-periodic</u> $\Longleftrightarrow$ f <u>is a.p</u> (<u>in the sense of Stepanoff</u>) <u>and</u> $\{\lambda_n\}$ <u>is regular</u>

2) <u>ps. period of</u> $\{\lambda_n\} < 8\pi/\delta$

where ps. period $\{\lambda_n\} = \inf |I|$, the inf being taken over all intervals I which satisfy the condition above.

In 1936, Ingham proved the following:

$$\text{ps.period } \{\lambda_n\} \leq 2\pi/\delta \ .$$

This is easily seen to be the best possible such estimate (consider an ordinary periodic function).

The following is also known (1957):

$$\underline{\text{ps.period}} = 2\pi\, \Delta(\Lambda) \qquad (\Lambda = \{\lambda_n\})$$

where $\Delta(\Lambda)$ is the upper uniform density of Λ, defined in any one of the following three equivalent ways:

a) $\{\mu_n\}$ is of uniform density $D \iff \mu_n - nD = O(1)$; then $\Delta(\Lambda) =$ inf of the densities of the $\{\mu_n\} \supset \Lambda$ having a uniform density.

b) Let $n(r,r+t)$ be the number of points of Λ in $[r,r+t[$, r real, $t > 0$. Then

$$\Delta(\Lambda) = \lim_{t\to\infty} \sup_{-\infty < r < +\infty} \frac{n(r,r+t)}{t}$$

c) $$\Delta(\Lambda) = \lim_{t\to\infty} \overline{\lim_{|r|\to\infty}} \frac{n(r,r+t)}{t}$$

Also to be discussed is the following, which connects the theory of pseudoperiodic functions to the theory of interpolation by entire functions of exponential type:

Let $T = \{\tau > 0 \mid \forall \{b_n\} \in \ell^2 \ \exists\ \Phi \in L^2$, Φ the restriction of an entire function of exponential type $\leq \tau$ s.t. $\Phi(\lambda_n) = b_n\}$.

Then $\inf_T \tau = \frac{1}{2}$ ps.per.$\{\lambda_n\}$.

(We remark that Beurling, in his seminar in 1959, obtained stronger results in the L^∞ case.)

Finally, we shall discuss the following problem of Mandelbrojt: given a class of functions C, defined on a domain D, a set Λ, a set $G \subset D$, and a property P. Let (Λ, G, P) be the following proposition: if $f \in C$ has spectrum in Λ and satisfies P in G, then f satisfies P everywhere in D.

As an example, we take for P any one of the following properties:

1) $f \in C^\infty$
2) f is analytic
3) f is locally representable as an absolutely convergent Fourier series
4) f has derivatives (in the sense of Schwartz) which are locally in L^2.

We assume that Λ is regular, and that $G = I$ is an interval. Then:

if $|I| > 2\pi\, \Delta(\Lambda)$, then (Λ, G, P) is true

if $|I| < 2\pi\, \Delta(\Lambda)$, then (Λ, G, P) is not true, at least for 3) and 4).

We now begin to give the details; first, some definitions.

R^P is p-dimensional Euclidean space. We say that $G \subset R^P$ is a domain if it is the closure of an open bounded set. We write

$$\int_G f = \int_G f(x)dx = \int \cdots_G \int f(x_1, \ldots, x_p)dx_1 \cdots dx_p$$

$$E^2 = E^2(R^P) = \{f \in \text{complex valued functions defined on } R^P \mid \forall \text{ domains } G \int_G |f|^2 dx < \infty\}$$

E^2 is a Frechet space with the seminorms

$$\|f\|_G = (\int_G |f|^2)^{1/2}.$$

We define, as usual, the translates f_t of f, $t \in R^P$, by

$$f_t(x) = f(x-t) .$$

Then, if $\mathcal{U}$ is a fixed domain (say, the unit cube about the origin),

$$\|f_t\|_{\mathcal{U}} \text{ is a family of seminorms for } E^2.$$

We say that f is <u>E^2 bounded</u> if $\sup_{t \in R^P} \|f_t\|_{\mathcal{U}} < \infty$. Then BE^2 (the space of E^2 bounded functions) is a Banach space with the norm $\|f\| = \sup_{t \in R^P} \|f_t\|_{\mathcal{U}}$.

Next, the definition of <u>E^2 almost-periodic functions</u> (Stepanoff):

1) $f \in BE^2$ and $\{f_t\}_{t \in R^P}$ is relatively compact in BE^2 or, equivalently,

2) f is in the space, in BE^2, of the trigonometric polynomials with real frequencies.

If, as in the Bohr case, we define

$$a(\lambda) = \lim_{\xi \to \infty} \frac{1}{\xi^p} \int_{\xi\mathcal{U}} f(x)e^{-i\lambda \cdot x}dx \qquad \lambda \in R^p$$

$$\lambda \cdot x = \lambda_1 x_1 + \cdots + \lambda_p x_p ;$$

then the a.p. spectrum of $f = \Lambda$ is given by

$$\Lambda = \{\lambda | a(\lambda) \neq 0\} .$$

Equivalently, $\Lambda = \{\lambda | e^{i\lambda x}$ is in the space of translates of f in $BE^2\}$. We shall also need the following important fact:

$$f = \lim_{\substack{BE^2 \\ j \to \infty}} \sum_{\substack{\lambda \in \Lambda \\ \text{finite}}} a_j(\lambda) e^{i\lambda \cdot x}$$

We recall the definition of E^2 mean periodicity (Schwartz):

1) Define $\tau(f)$ = span of $\{f_t\}_{t \in R^p}$ in E^2. Then f is E^2 mean-periodic $\Longleftrightarrow \tau(f) \neq E^2$ or, as is easily seen to be equivalent,

1') f is E^2 mean-periodic $\Longleftrightarrow \exists\, \varphi \in L^2$ with compact support s.t. $f * \varphi = 0$.

Then, the problem of harmonic analysis for E^2 mean-periodic functions f is as follows: to find the $P(x)e^{i\lambda \cdot x} \in \tau(f)$ where $P(x)$ is a polynomial, and λ is a complex p-tuple.

We define $\text{spectrum}_{\text{m.p.}} f = \{\lambda | e^{i\lambda \cdot x} \in \tau(f)\}$. The $\text{spectrum}_{\text{m.p.}}$ of f is said to be simple if no $P(x)e^{i\lambda \cdot x} \in \tau(f)$ with $\deg P \geq 1$.

The problem of harmonic synthesis, then, is:

Does the set of $P\, e^{i\lambda \cdot x} \in \tau(f)$ span $\tau(f)$?

In the case $p = 1$, this problem was solved by Schwartz in 1949. It is not solved for $p \geq 2$.

Finally, we define <u>E^2 pseudoperiodicity</u>:

$$f \text{ is } E^2 \text{ ps.per.} \iff \tau(f) \subset BE^2 .$$

We shall prove the following

<u>Theorem</u>: Let us consider the following statements

1) <u>f is E^2 ps.per.</u>

2) <u>f is E^2 a.p with regular spectrum</u>

3) f is E^2 <u>m.p with spectrum</u> $\begin{cases}\text{real}\\ \text{simple}\\ \text{regular}\end{cases}$ <u>and harmonic synthesis holds.</u>

<u>Then</u> 1) $\iff$ 2) $\iff$ 3) <u>and</u>

$$\underline{\text{m.p spectrum}}\ f = \underline{\text{a.p spectrum}}\ f .$$

We need the following

<u>Lemma I</u>: Let F be a closed subspace of E^2, $F \subset BE^2$. Then $\exists$ G s.t. on F $\| \ \|$ is equivalent to $\| \ \|_G$.

<u>Proof</u>: If G is <u>any</u> domain, then, for any $f \in BE^2$, we have $\|f\|_G \leq \sqrt{k}\, \|f\|$, where k is such that G is covered by k translates of $\mathcal{U}$. On the other hand, F, being closed in E^2 and contained in BE^2, is a closed subspace of BE^2 (since the topology of BE^2 is stronger than that of E^2). Thus, F is a Fréchet space as a subspace of E^2, and a Banach space as a subspace of BE^2, with a stronger topology. By a theorem of Banach, the two topologies are equivalent; if one defines the topology in E^2 by an increasing sequence of semi-norms $\|f\|_{G_i}$ $(G_{i+1} \supset G_i)$, then there exists an i s.t. $\| \ \|$ and $\| \ \|_{G_i}$ are equivalent.

It is easily seen that if $\tau(f) \subset BE^2$, $\text{spectrum}_{\text{m.p.}}\ f$ is real and simple. Furthermore, it follows from Lemma I that this spectrum is also regular; if not, then we can find $\lambda_n, \lambda'_n \in \text{spectrum}_{\text{m.p.}}\ f$ s.t. $|\lambda_n - \lambda'_n| \to 0$ as $n \to \infty$, and $\lambda_n \neq \lambda'_n$. Then $f_n = e^{i\lambda_n \cdot x} - e^{i\lambda'_n \cdot x} \in \tau(f)$, and

$$\frac{\|f_n\|}{\|f_n\|_G} \to \infty \qquad \text{as } n \to \infty, \quad \text{for any domain } G,$$

contradicting Lemma I. We have thus proved:

<u>Proposition 1</u>: 1) $\Longrightarrow$ $\text{spectrum}_{\text{m.p.}}\ f$ is real, simple, and regular.

We shall prove now

<u>Proposition 2</u>: $f\ E^2$ ps.per. $\Rightarrow f\ E^2$ a.p.

<u>Lemma 2</u>: If $g \in C^\infty$ and $\forall\ \alpha = (\alpha_1, \ldots, \alpha_p)$,

$$D^\alpha g = \frac{\partial^{\alpha_1 + \cdots + \alpha_p}}{\partial x_1^{\alpha_1} \cdots \partial x_p^{\alpha_p}} g \in B\,E^2,$$

then

$$\sup_{x \in R^p} |g(x)| < \infty .$$

<u>Proof</u>: Choose $h \in C^\infty$, with compact support G and s.t. $h(0) = 1$; then

$$\sup_{x \in R^p} \|D^\alpha(g_t h)\|_G < \infty, \qquad \text{for all } \alpha .$$

Taking Fourier transforms, we obtain, in particular,

$$\sup_{t \in R^P} \|(1 + r^2)^P \mathcal{F}(g_t h)\|_{L^2(R^P)} < \infty .$$

Therefore, by Schwarz's inequality,

$$\sup_{t \in R^P} \int_{R^P} |\mathcal{F}(g_t h)| < \infty$$

$\therefore \sup_{t \in R^P} |g_t h| < \infty$; in particular,

$$\sup_{t \in R^P} |g_t h(0)| = \sup_{t \in R^P} |g(-t)| < \infty ,$$

and Lemma 2 is proved.

Now, choose $\eta_\epsilon \in C^\infty$, $\eta_\epsilon \geq 0$, $\int_{R^P} \eta_\epsilon = 1$, η_ϵ has support in $\{x| \; |x| \leq \epsilon\}$. Then $D^\alpha(f * \eta_\epsilon)$ is uniformly bounded, by Lemma 2. Thus $\{(f * \eta_\epsilon)_t\}$ is rel. compact in the uniform topology. Therefore, it is rel. compact in $L^2(G)$, for any G. Therefore, by Lemma 1, it is rel. compact in BE^2. Thus $f * \eta_\epsilon$ is an E^2 a.p. function. Letting $\epsilon \to 0$, $f * \eta_\epsilon \to f$ in BE^2. Thus, f is E^2 a.p., and Proposition 2 is proved.

<u>Proposition 3</u>: $\text{Spectrum}_{\text{m.p.}}\ f = \text{spectrum}_{\text{a.p.}}\ f$ and harmonic synthesis holds, for f satisfying 1.

<u>Proof</u>: This follows from Lemma 1, Proposition 2, and the fact that we have

$$f = \lim_{\substack{BE_2 \\ j \to \infty}} \sum_{\substack{\lambda \in \Lambda \\ \text{finite}}} a_j(\lambda)\, e^{i\lambda x} .$$

The proof of our Theorem will be complete once we have proven

<u>Proposition 4</u>: If $\Lambda \subset R^P$ is a regular sequence, and we consider finite sums $s(x) = \sum_{\lambda\in\Lambda} a(\lambda)e^{i\lambda\cdot x}$, then $\|s\|_G$ is equivalent to $\|s\|$ for some G.

SOME APPLICATIONS OF ENTIRE FUNCTIONS OF EXPONENTIAL TYPE TO HARMONIC ANALYSIS (I)

The Multiplier Problem for Fourier Transforms of Measures with Compact Support

by

P. Malliavin*

$\mathcal{M}$ will denote the algebra of measures defined on the real line and with compact support, $\mathcal{M}_a$ will denote the part of $\mathcal{M}$ consisting of measures having their support contained in an interval of length a. The main problem that we shall consider here can be formulated as follows: "how fast" can the Fourier transform $\hat{\mu}(x)$ of a measure $\mu \in \mathcal{M}_a$ $\mu \not\equiv 0$, tend to zero as x tends to $\pm\infty$ through real values? A well known result is that if we define $I(\hat{\mu})$ by

$$I(\hat{\mu}) = \int_{-\infty}^{+\infty} \left|\log|\hat{\mu}(x)|\right| \frac{dx}{1+x^2} ,$$

then for all $\mu \in \mathcal{M}$, $\mu \not\equiv 0$ we have

(1) $$I(\hat{\mu}) < \infty$$

The condition (1) describes completely how fast the Fourier transform of a measure having its support in the half line must vanish at ∞. (This is an easy result; a proof can be found for instance in Paley-Wiener).

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*The results which are summarized here come from a joint work with A. Beurling and the complete proofs will appear elsewhere.

However, it is obvious that more than the condition (1) is needed to give a complete description in the case of measures with compact support. We now define the notion of a multiplier. Let $S(x)$ be a given continuous function on the real line. We shall call a measure $\mu \in \mathcal{M}_\epsilon$ a <u>multiplier of type</u> ϵ <u>for</u> S, provided that

$$\hat{\mu}(x)\, S(x) \quad \text{is bounded for } x \text{ real.}$$

We obtain using (1) an obvious necessary condition for the existence of a multiplier, namely

$$\int \log^+ |S(x)| \, \frac{dx}{1+x^2} < \infty$$

or supposing, as we can do without any loss of generality $|S(x)| > 1$,

(2) $$I(S) < \infty \quad .$$

Conversely we can state

<u>Theorem 1</u>: <u>Suppose that</u> $\log|S(x)|$ <u>is uniformly continuous, and that</u> (2) <u>is satisfied then</u> S <u>has multipliers of arbitrary small type</u>.

<u>Theorem 2</u>: <u>Suppose that</u> S <u>is the restriction to the real axis of an entire function of exponential type and that</u> (2) <u>is satisfied</u>. <u>Then</u> S <u>has multipliers of arbitrary small type</u>.

The theorems 1 and 2 will be derived, independently of each other, a more general statement will be given later in Theorem 4, when the technical background has been sufficiently developed.

It is easy to deduce some consequences of Theorem 2 for the algebra of measures. Let us denote by Ω the class of entire functions which can be written as the quotient of the Fourier transforms of two measures with compact support.

$$\Omega = \{F \mid F \text{ entire}, \quad F = \frac{\hat{\mu}}{\hat{\nu}} \text{ where } \mu, \nu \in \mathcal{M}\}$$

Then we have the following characterization of Ω.

<u>Proposition 1</u>. $F \in \Omega$ <u>if and only if</u> F <u>is an entire function of exponential type such that</u>

$$I(F) < \infty .$$

This statement can be compared to the theorem of Nevanlinna which says that a holomorphic (or, indeed, meromorphic) function of bounded characteristic in the upper half plane $\operatorname{Im} z > 0$ can be written as the quotient of two functions holomorphic and bounded in $\operatorname{Im} z > 0$. However this analogy cannot be carried further to meromorphic functions: it is easy to construct, using for instance the result of the next lecture, a meromorphic function, G, of exponential type, such that $I(G) < \infty$, and such that G cannot be written as the quotient of two elements of $\hat{\mathcal{M}}$. Now using the fact that Theorem 2 allows us to choose multipliers of arbitrary small type, we have

<u>Proposition 2</u>. <u>Given measures</u> $\mu, \nu \in \mathcal{M}$ <u>such that</u> $\frac{\hat{\mu}}{\hat{\nu}}$ <u>is entire, then for every</u> $\epsilon > 0$, <u>it is possible to find</u> $\nu_1 \in \mathcal{M}_\epsilon$ <u>such that</u> $\mu * \nu_1$ <u>belongs to the ideal generated by</u> ν; <u>that is, we have</u> $\mu * \nu_1 = \nu * \omega$ <u>for some</u> $\omega \in \mathcal{M}$.

The propositions 1 and 2 are immediate consequences of Theorem 2, and we leave their proof to the reader. Another application of Theorem 2 will be given in the next lecture.

We can also give several other formulations of Theorem 1. If S is given, we shall associate with S the class $\mathcal{A}_S$ of functions f such that

$$\int_{-\infty}^{+\infty} |\hat{f}(x)\ S(x)|\, dx < \infty$$

Then theorem (1) answers the following question: <u>does there exist in $\mathcal{A}_S$ a function with compact support?</u> Denote by $\mathcal{A}'_S$ the dual of $\mathcal{A}_S$; $\mathcal{A}'_S$ is a space of generalized distributions and we can now read Theorem 1 as the following theorem of regularization in this space: <u>suppose the hypothesis of Theorem 1 fulfilled, then for every</u> $T \in \mathcal{A}'_S$ <u>and for every</u> $\epsilon > 0$, <u>it is possible to find</u> $\varphi \in C_0^\infty$, <u>with support contained in an interval of length</u> ϵ, <u>such that</u> $T * \varphi \in C^\infty$.

We will give now some indications of the proof of Theorems 1 and 2.

The technique used can be described as follows: First we shall reduce the problem of constructing multipliers to a problem of potential theory on the real line, for a certain kernel satisfying the principle of balayage. This will allow us to apply then the full strength of potential theory and to solve our problem by using an extremal method.

I Reduction to a potential problem in one dimension

We shall denote by G the function

$$G(u) = \log\left|\frac{1+u}{1-u}\right| \qquad u > 0 .$$

G is the restriction to the positive real axis R^* of the Green function for the half plane $\operatorname{Re} z > 0$. If φ is a measure with support in R^* we shall denote by U^φ its potential defined by

$$U^\varphi = G * \varphi$$

where $*$ denotes the convolution on the multiplicative group R^*, that is

$$U^\varphi(x) = \int_0^\infty G\left(\frac{x}{t}\right) d\varphi(t) .$$

Now we can state the following representation formula.

<u>Proposition 3</u>: <u>Let</u> f <u>be an even entire function of exponential type</u> a <u>such that</u>

$$I(f) < \infty ;$$

<u>then there exists a measure</u> φ <u>such that</u>

$$\log|f(x)| = -xU^\varphi(x) \qquad x > 0 ;$$

furthermore φ satisfies ,

(4) $$\lim_{t \to \infty} \inf \frac{td\varphi}{dt} \geq -a$$

(5) $$\lim_{R = \infty} \int_0^R d\varphi$$

exists and is finite.

Conversely, let φ_1 be a measure satisfying (4) and (5); then for every $\eta > 0$, it is possible to find a positive measure ρ such that

(6) $$h(z) = \int_0^\infty \log|1 - z^2 t^{-2}| d\rho(t)$$

will satisfy

$$h(z) < (a + \eta)|z| + O(1)$$

(7) $$h(x) = -xU^{\varphi_1}(x) + O(\log x) ,$$ for x real large enough .

Remark: It is easy to go from the function h given by a positive continuous measure ρ to an entire function of exponential type. We shall call a function h of the kind given by (6) and satisfying $\log|S(x)| + h(x) < 0$ a multiplier in the wide sense.

Proof of Proposition 3: Let us first suppose that all the zeros of f are real; let $n(t)$ = numbers of zeros of $f \in (0,t)$. Then

$$\log|f(x)| = \int_0^\infty \log|1 - x^2 t^{-2}| dn(t)$$

or making an integration by part

$$\frac{\log|f(x)|}{x} = \text{P. V.} \int_0^\infty \frac{2x}{x^2 - t^2} \frac{n(t)}{t}\, dt$$

or defining the measure φ by

$$\int_0^R d\varphi = \frac{n(R)}{R} ,$$

we get by another integration by part

$$\log|f(x)| = -x \int_0^\infty G(\tfrac{x}{t}) d\varphi(t) = -xU^\varphi(x) .$$

Now the fact that $I(f) < \infty$ implies (cf., Boas, Entire Functions) that $\lim \frac{n(R)}{R}$ exists and is equal to the type of f. We have

$$d\varphi(t) = \frac{dn(t)}{t} - \frac{n(t)}{t}\frac{dt}{t} > -(a + \epsilon)\frac{dt}{t}$$

if t is sufficiently large, and this proves (4).

Let us now consider the case where the zeros of f are no more real. We shall reduce it in an obvious way to the case where f has only real zeros if we prove the following lemma.

Lemma 1: *Let f be an even entire function of exponential type such that*

$$I(f) < \infty ;$$

then there exists a measure $d\mu$, positive, with support on R^*, such that

$$\log|f(x)| = \int_0^\infty \log|1 - x^2 t^{-2}| d\mu(t) , \quad \text{for } x \in R^* .$$

Furthermore, denoting by $N(r)$ the numbers of zeros of f in $|z| < R$ we have

$$(8) \qquad N(R) = 2\int_0^R d\mu(t) + o(R) .$$

Proof: Let denote by

$$W_\theta(x) = \log|1 - x^2 e^{-2i\theta}|$$

by Λ the sequence of zeros of f contained in the angle

$$\frac{-\pi}{2} < \arg z \leq \frac{\pi}{2}$$

and let

θ_λ = argument of λ ,

$\delta_{|\lambda|}$ = Dirac Mass put at the point $|\lambda|$.

With these notations the Weierstrass factorization of f can be written, $*$ denoting always the convolution on the multiplicative group of positive reals,

$$\log|f(x)| = \sum_{\lambda\in\Lambda} (W_{\theta_\lambda} * \delta_{|\lambda|})(x) .$$

Now we shall use the factorization

$$W_\theta = W_o * K_\theta$$

where

$$K_\theta(x) = \frac{2}{\pi} \frac{x(x^2 + 1)|\sin \theta|}{x^4 - 2x^2 \cos 2\theta + 1} ;$$

factorization which can be proved looking to the Mellin transform of both members. Then we will get

$$\log|f(x)| = W_o * \sum_{\lambda \in \Lambda} K_{\theta_\lambda} * \delta_{|\lambda|}$$

the interversion of integration used to obtain this formula being justified by the fact that $I(f) < \infty$ implies

(9) $$\Sigma\, |\theta_\lambda|\,|\lambda|^{-1} < \infty$$

which with the fact

$$K_\theta(x) = O(\theta x) \qquad 0 < x < \tfrac{1}{2}$$

implies the absolute convergence of

$$\Sigma\, K_{\theta_\lambda} * \delta_{|\lambda|} .$$

Let $d\mu$ the measure equal to the sum of this series, $d\mu$ is positive. Finally (8) results from (9), as an elementary computation shows.

Now with Lemma 1 the first part of the proposition 1 is proved in the case of complex zeros as in the case of real zeros.

Proof of the constructive part of Proposition 3: We have now to show how, given a potential U^{φ_1}, we can construct a function h, of the form given in (6), such that (7) holds. Denote by

$$\varphi_1(R) = \int_0^R d\varphi_1 , \quad \beta = \lim_{R=\infty} \varphi(R)$$

and let ω be the measure defined by its differential $d\omega$:

$$d\omega = d(x[\varphi_1(x) - \beta + a + \eta]) .$$

We have then

$$d\left(\frac{\omega(x)}{x}\right) = d\varphi_1(x) ;$$

hence

$$\int_0^\infty \log|1 - x^2t^{-2}|d\omega(t) = -xU^{\varphi_1}(x) .$$

Furthermore for x sufficiently large, $x > M$, ω is a positive measure. Let ρ = positive part of ω. Then

$$\int_0^\infty \log|1 - x^2t^{-2}|[d\rho(t) - d\omega(t)] = \int_0^M \log|1 - x^2t^{-2}|[d\rho(t) - d\omega(t)] ,$$

and this last integral is $O(\log x)$ for x large, which proves (7).

II Solution of an extremal problem

We shall reduce the multiplier problem to a problem on the potentials U^φ. Let denote by

$$\sigma(x) = \frac{\log|S(x)|}{|x|}$$

(We can suppose without loss of generality that $\sigma > 0$, that $\sigma(x) = O(I)$ near zero, and that $\sigma(x)$ is even – if it were not the case we shall introduce $S_1(x) = S(x)S(-x)$). Now a being a positive number given, let us consider the convex set $\mathcal{A} = \mathcal{A}(\sigma, a)$ of measures φ defined by

$$\mathcal{A}(\sigma,a) = \{\varphi | U^\varphi \geq \sigma \text{ and } d\varphi \geq -a \frac{dt}{t}\} . \tag{10}$$

Now reading the Proposition 3 we can state: S <u>has multipliers in the wide sense of type arbitrary small if and only if for every</u> $b > 0$ <u>there exists</u> $\varphi \in \mathcal{A}(\sigma, b)$ <u>such that</u>

$$\lim_{R = \infty} \int_0^R d\varphi \tag{11}$$

<u>exists and is finite</u>.

The next step consists in replacing the condition (11) by a condition on U^φ. As $\sigma > 0$ we have, for all $\varphi \in \mathcal{A}(\sigma, a)$,

$$U^\varphi > 0 .$$

Let denote

$$\ell(\varphi) = \int_0^\infty U^\varphi(x)\frac{dx}{x} . \tag{12}$$

Then $\ell(\varphi)$ is a number finite or infinite well defined. We have

<u>Lemma 2</u>: <u>The condition</u> (11) <u>holds if and only if</u>

$$\ell(\varphi) < \infty .$$

<u>Proof</u>: We have

$$\int_0^R U^\varphi(x)\frac{dx}{x} = \int_0^\infty G(\tfrac{R}{t})\varphi(t)\frac{dt}{t} .$$

If $\varphi(t)$ tends to a finite limit when $t \to \infty$ it will be the same for the integral.

To prove the converse we shall proceed as follows. Let α denote a given number $\alpha > 1$, and denote by h_α a four times differentiable function such that

$$h_\alpha(t) = 1 , \qquad \text{for } t > \alpha$$

$$h_\alpha(t) = 0 \qquad \text{for } t < 1 .$$

Then let k_α be the bounded function defined by the convolution equation

$$U^{\omega_\alpha} = h_\alpha \qquad \text{where} \quad d\omega_\alpha = k_\alpha(t)\frac{dt}{t} .$$

(This equation is easily solved by the Mellin transform).

We have

$$k_\alpha(t) \to \frac{2}{\pi} \qquad \text{when} \quad t \to \infty .$$

Hence

$$\lim_{x \to \infty} \int h_\alpha\left(\frac{x}{t}\right)d\varphi(t) = \frac{2}{\pi}\int_0^\infty U^\varphi(t)\frac{dt}{t} .$$

This is true for all $\alpha > 1$. Furthermore

$$d\varphi \geq -b\,\frac{dt}{t} .$$

These two facts imply that (11) is fulfilled and the lemma is proved.

Now we can state our extremal problem. Minimize the integral $\ell(\varphi)$ when $\varphi \in \mathcal{A}(\sigma,b)$.

Let us denote by σ_b the function defined by

(13) $$\sigma_b(x) = \inf U^\varphi(x) \qquad \text{for all} \quad \varphi \in \mathcal{A}(\sigma,b).$$

Now using the theorem of the infimum envelope of a family of potentials, we see that

$$\sigma_b = U^{\theta_b} \tag{14}$$

where θ_b is a measure satisfying

$$d\theta_b \geq -b\,\frac{dt}{t}\ .$$

Hence $\theta_b \in \mathcal{A}(\sigma, b)$ and as obviously we have

$$\ell(\theta_b) \leq \ell(\varphi) \qquad \text{for all} \quad \varphi \in \mathcal{A}(\sigma, b)$$

θ_b will give us the solution of our extremal problem. We have then

__Theorem 3.__ S __has multipliers in the wide sense of arbitrary small type if and only if for all__ $b > 0$

$$\int_1^{+\infty} \sigma_b(x)\frac{dx}{x} < \infty \tag{15}$$

(where σ_b is defined by (10) and (13)).

Now the problem is to evaluate (15). We shall do that using the properties of the extremal measure θ_b. Let us denote by $\| \|_2$ the energy norm on the measures defined by

$$\|\varphi\|_2 = (\int U^{\varphi} d\varphi)^{\frac{1}{2}}$$

If σ can be written as a potential

$$\sigma = U^{\rho} ,$$

we shall denote by $\mathcal{D}(\sigma)$ the Dirichlet integral of σ defined by

$$(16) \qquad \mathcal{D}(\sigma) = \|\rho\|_2^2 .$$

We shall denote by $(u|v)_{\mathcal{D}}$ the scalar product associated to the Dirichlet integral.

Let us define

$$(17) \qquad \mathcal{D}_1(\sigma) = \int_0^{\infty} |\sigma(x)|^2 \frac{dx}{x} + \int_{-\infty}^{+\infty} \frac{du}{u^2} \int_0^{\infty} |\sigma(xe^u) - \sigma(x)|^2 \frac{dx}{x} .$$

Then looking to the Mellin transform of G it is easy to show that $\mathcal{D}(\sigma)$ and $\mathcal{D}_1(\sigma)$ define equivalent norms.

We can now state our main theorem.

Theorem 4: Suppose that σ has a finite Dirichlet integral and that (2) holds; then S has multipliers in the wide sense of arbitrary small type.

Proof: We shall study an extremal problem which has in fact the same extremum function that the function σ_b introduced in Theorem 3 has. We will not use this fact in the proof, and for the sake of brevity, we will not write the proof of this fact which we mention only to explain the success of the method.

Recall that a function f, such that $\mathcal{D}(f) < \infty$ is called a *pure potential* if there exists a measure positive ω, such that

$$f = U^{\omega} .$$

Then it is a well known and elementary fact that: *f is a pure potential if and only if*

$$(f|g)_{\mathcal{D}} > 0 \qquad \text{for all } g > 0 .$$

We have the following use of this characterization.

Lemma 3: Let f be a given function such that

$$\mathcal{D}(f) < \infty .$$

Let

$$B_f = \{g | \mathcal{D}(g) < \infty \text{ and } g \geq f \text{ almost everywhere for the Lebesgue measure}\} .$$

Denote by f^* the projection of the origin on the closed convex set B_f. Then

f^* is a pure potential $f^* = U^{\omega}$.

$f^* = f$ almost everywhere for the measure ω.

For every pure potential h we have

$$\mathcal{D}(f^* - h) \leq \mathcal{D}(f-h) .$$

Proof: If $g > 0$ we have

$$f^* + tg \in B_f \qquad \text{for all} \quad t > 0 .$$

Hence

$$\mathcal{D}(f^* + tg) \geq \mathcal{D}(f^*) \qquad t > 0$$

which implies

$$(f^*|g)_{\mathcal{D}} > 0 \qquad \text{for all} \quad g > 0 ;$$

hence f^* has to be a pure potential.

Now if $t > -1$ we have

$$(1 + t)f^* \geq (1 + t)f .$$

Hence

$$f^* + t(f^* - f) \in B_f \qquad \text{for} \quad t > -1 ,$$

which implies as before

$$(f^*|f^* - f)_{\mathcal{D}} = 0$$

or

$$\int (f^* - f)d\omega = 0 .$$

As $\omega > 0$, $f^* > f$, this implies $f^* = f$ a. e. for the measure ω. Finally

$$\mathcal{D}(f^* - h) - \mathcal{D}(f - h) = \mathcal{D}(f^*) - \mathcal{D}(f) + (h|f - f^*)_{\mathcal{D}}'.$$

Now if h is a pure potential the last integral is negative as $f - f^* \leqslant 0$. From the definition of f^* we have

$$\mathcal{D}(f^*) \leq \mathcal{D}(f)$$

and this proves the lemma.

<u>Lemma 4</u>: <u>Let</u> σ <u>be a given function such that</u>

$$\mathcal{D}(\sigma) < \infty ,$$

<u>and let</u> ψ <u>be an absolutely continuous measure given, such that the restriction of</u> ψ <u>to every compact set</u> K <u>is of finite energy</u>.

<u>Then there exists a measure</u> θ, <u>such that</u>

$$U^\theta \geq \sigma$$

<u>almost everywhere for the Lebesgue measure</u>, <u>and such that</u>

$$\int (U^\theta - \sigma) d\psi < \infty .$$

Proof: Let K_N be an increasing sequence of compact sets, $\bigcup K_N$ being the positive real line. Let ψ_N be the restriction of ψ to K_N and let us apply the lemma 3 to the function

$$f_N = \sigma + U^{\psi_N} ;$$

denote by f_N^* the projection of the origin to the convex set B_{f_N}. Then

$$f_N^* = U^{\omega_N} \qquad \omega_N > 0 .$$

Let

$$\theta_N = \omega_N - \psi_N .$$

We shall have by Lemma 3

(18) $$\|\theta_N\|_2^2 = \mathcal{D}(f_N^* - U^{\psi_N}) < \mathcal{D}(\sigma) ,$$

$$U^{\theta_N} \geq \sigma \quad \text{a. e. for the Lebesgue measure}$$

$$\int (U^{\theta_N} - \sigma) d(\theta_N + \psi_N) = 0 .$$

Let us now select a sequence θ_{N_j} which converges weakly to a measure θ. We will have still

$$U^{\theta} \geq \sigma \qquad \text{a. e.}$$

From another hand

$$\int (U^{\theta_N} - \sigma) d\theta_N = -\int (U^{\theta_N} - \sigma) d\psi_N .$$

Denote by h_N the characteristic function of K_N; this equality gives us using (18)

$$\int (U^{\theta_N} - \sigma) h_N d\psi < 2\mathcal{D}(\sigma) .$$

As we integrate positive functions this means that

$$\int (U^{\theta} - \sigma) d\psi < 2\mathcal{D}(\sigma) < \infty .$$

<u>Proof of Theorem 4</u>: Let b be a positive number given; denote by ψ the measure

$$d\psi = b \frac{dx}{x}$$

and apply Lemma 4 to (σ, ψ). Then we construct in this way a measure θ such that

$$d\theta \geq - b \frac{dx}{x} .$$

From another hand

$$\int U^{\theta} \frac{dx}{x} = \int (U^{\theta} - \sigma) \frac{dx}{x} + \int \sigma \frac{dx}{x}$$

By (2) the last integral of the second member is convergent. Using the lemma 4 we get that the first integral is convergent so

$$\int U^{\theta} \frac{dx}{x} < \infty$$

Now we have only to finish to remark that the fact that $d\theta > -b\frac{dx}{x}$ implies that U^{θ} is lower semi continuous, with the hypothesis that $S(x)$ is continuous and $|S(x)| > 1$, we get that

$$U^{\theta} - \sigma$$

is a lower semi continuous function, positive almost everywhere, hence everywhere positive and this proves the theorem.

III. Demonstration of the Theorems 1 and 2

We shall first reduce each of these theorems to the Theorem 4, that is we shall construct a majorant σ_1 of the given σ such that

$$\mathcal{D}(\sigma_1) < \infty .$$

Let us first remark that if $\log|S(x)|$ is uniformly continuous we can find a function $k(x)$, satisfying a uniform Lipschitz condition

$$|k(x) - k(x')| < A|x - x'|$$

and such that

$$k(x) = \log |S(x)| + O(1) .$$

Let

$$\sigma_1(x) = \frac{k(x)}{x} .$$

Then (2) implies with the uniform continuity that $\sigma_1(x) = o(1)$, and therefore

(19) $$|\sigma_1(e^u x) - \sigma_1(x)| < Au \qquad |u| < 1 .$$

Let us denote by

$$\Phi(\xi) = \sigma(e^\xi) .$$

We have just remarked in (19) that Φ is uniformly Lipschitz.

We have

<u>Lemma</u> 5. <u>Let</u> Φ <u>a function which is uniformly Lipschitz on the real line and such that</u>

$$\int_{-\infty}^{+\infty} |\Phi(\xi)|d\xi < \infty .$$

<u>Then</u>

$$\mathcal{D}_2(\Phi) = \int_{-\infty}^{+\infty} \frac{du}{u^2} \int_{-\infty}^{+\infty} |\Phi(\xi + u) - \Phi(\xi)|^2 d\xi < \infty$$

<u>Proof</u>: Denote by Φ^* the symetrically decreasing function which is equimeasurable with Φ. We have

$$\mathcal{D}_2(\Phi) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{|\Phi(\xi) - \Phi(\eta)|^2}{(\xi - \eta)^2} d\xi \ d\eta$$

Let

$$K_n(\xi) = \min\left(n , \frac{1}{\xi^2}\right) .$$

Then

$$\mathcal{D}_2(\Phi) = \lim_n \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[\Phi(\xi) - \Phi(\eta)\right]^2 K_n(\xi - \eta)d\xi \ d\eta$$

We have, using a theorem of Hardy, Littlewood and Polya

$$\int \int \left[\Phi(\xi) - \Phi(\eta)\right]^2 K_n(\xi - \eta) \ d\xi d\eta < \int \int \left[\Phi^*(\xi) - \Phi^*(\eta)\right]^2 K_n(\xi - \eta) \ d\xi d\eta$$

and passing to the limit we get

$$\mathcal{D}_2(\Phi) < \mathcal{D}_2(\Phi^*)$$

Let now $u > o$. We have by the Lipschitz condition

$$\left(\Phi^*(\xi+u) - \Phi^*(\xi)\right)^2 < Au \, |\Phi^*(\xi+u) - \Phi^*(\xi)|$$

Now if we remark that

$$\int_{-\infty}^{+\infty} |\Phi^*(\xi+u) - \Phi^*(\xi)| \, d\xi = \int_{-\infty}^{-\frac{u}{2}} \left[\Phi^*(\xi+u) - \Phi^*(\xi)\right] d\xi$$

$$+ \int_{-u/2}^{+\infty} \left[\Phi^*(\xi) - \Phi^*(\xi+u)\right] d\xi$$

$$= 2 \int_{-u/2}^{u/2} \Phi^*(\xi) d\xi < A_1 u$$

We obtain finally

$$\int (\Phi^*(\xi+u) - \Phi^*(\xi))^2 \, d\xi \leq A_2 \, u^2$$

which proves the lemma.

Now we can remark that with the hypothesis of the Theorem 1, we have σ_1 bounded and

$$\int |\sigma_1(x)| \, \frac{dx}{x} < \infty$$

which implies $\int |\sigma_1(x)|^2 \frac{dx}{x} < \infty$, and so the convergence of $\mathcal{D}_1(\sigma_1)$ is a consequence of the convergence of the integral $\mathcal{D}_2$ defined in the lemma 5.

Then the hypothesis of the Theorem 1 implies the existence of multipliers in the wide sense arbitrary small type.

Now consider the hypothesis of Theorem 2, that is

$$S(x) = f(x)$$

where f is an entire function of exponential type. Introducing

$$f_1(z) = f(z)\overline{f}(\overline{z}) + f(-z)\overline{f}(-\overline{z}) + 1$$

we can, without loss of generality, suppose that f is even, and greater than 1 on the real axis. Then using Proposition 3 we can write

$$(20) \qquad \begin{cases} \log|f(x)| = -xU^{\varphi}(x) > 0 \quad \text{where} \\ d\varphi \geq -a\,\dfrac{dt}{t} \\ \int U^{\varphi}(x)\,\dfrac{dx}{x} > -\infty\,. \end{cases}$$

We have

<u>Lemma 6</u>: <u>The conditions</u> (20) <u>imply that</u> φ <u>is a measure of finite energy</u>.

<u>Proof</u>: Let $d\chi$ denote the measure $\dfrac{dx}{x}$ restricted to the interval $(1, +\infty)$. Let denote by $d\chi^*$ the swept measure of the measure $d\chi$ on the interval $(0, 1)$. Then we have

$$d\chi^* = h(t)\,\frac{dt}{t}$$

where $h > 0$,

$$h(t) < M \qquad \text{if} \quad 0 < t < \frac{1}{2} ,$$

$$\int_{\frac{1}{2}}^{1} h(t) \frac{dt}{t} < M ,$$

M being a suitable absolute constant.

Let us denote by $d\mu_N$ the swept measure of $d\mu$ on the interval $(0, N)$. We have

$$d\mu_N \geq -b \frac{dt}{t} \quad -b\, h(\frac{t}{N}) \frac{dt}{t} .$$

As U^{μ} is continuous, we have the fact that μ_N is of finite energy and as $U^{\mu_N} < 0$ on the support of μ_N

$$\int U^{\mu_N} d\mu_N < -\int U^{\mu_N} d\mu_N^- ,$$

Therefore

$$< -\int_{0}^{\frac{N}{2}} U^{\mu}(b + M) \frac{dt}{t} + M \max_{\frac{N}{2} < t < N} |U^{\mu}(t)|$$

expression which is bounded uniformly in N. Therefore the energy norms of the sequence μ_N are uniformly bounded, μ is of finite energy. This proves the lemma and shows also the hypothesis of Theorem 2 implies $\mathcal{J}(\sigma) < \infty$ and we can therefore apply Theorem 4 and get a multiplier in the wide sense for S.

Obtaining an entire function from a multiplier in the wide sense involves only routine computations that we shall not develop here.

SOME APPLICATIONS OF ENTIRE FUNCTIONS OF EXPONENTIAL TYPE TO HARMONIC ANALYSIS (II)

On the Closure of a Sequence of Exponential on a Segment

by

P. Malliavin*

Λ being a given sequence of complex numbers, the following problem has been raised by Polya: When is the set $\{e^{i\lambda x}\}_{\lambda\in\Lambda}$ complete in the space $L_2(I)$ of square integrable functions on the segment I? This problem has been studied extensively — in particular, by Paley-Wiener and by Levinson.

It is obvious that the property of completeness is conserved if we translate the interval; that is, it depends only on the length $\ell(I)$ of the interval.

Let us recall some classical results. We introduce the counting function $n(x)$ of the sequence Λ defined, for $x > 0$, by $n(x) =$ number of $\lambda \in \Lambda$ such that $\operatorname{Re}\lambda \geq 0$ and $|\lambda| < x$ and defined, for $x < 0$, by $-n(x) =$ number of $\lambda \in \Lambda$ such that $\operatorname{Re}\lambda < 0$ and $|\lambda| < -x$. Then we have (Paley-Wiener)

$$\ell(I) < 2\pi \limsup \frac{n(x)}{x}$$

implies completeness, and a more elaborate result (Cartwright, Levinson)

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*The results which are summarized here are coming from a joint work with A. Beurling and these complete proofs will appear elsewhere.

$$(1) \qquad \ell(I) < 2\pi \lim_{\epsilon \downarrow 0} \limsup_{x} \frac{n(x+\epsilon x)-n(x)}{\epsilon x}$$

implies completeness. However, this last result does not settle the question; it does not give a necessary and sufficient condition for completeness. A result which could be considered as satisfactory would be the determination of the number

$$R(\Lambda) = \sup \text{ of } \ell(I)$$

for the intervals I for which $\{e^{i\lambda x}\}$ is complete in $L_2(I)$. We shall call $R(\Lambda)$ the <u>radius of totality</u> of Λ, and our purpose is to give an explicit determination of $R(\Lambda)$.

For this purpose, we shall introduce a density called the <u>effective density</u> of the sequence Λ as follows.

Let us denote by J_a the class of differentiable functions k for which $0 \leq k' \leq a$, and let

$$n_a(x) = \inf k(x) \quad \text{for all } k \in J_a \text{ satisfying}$$

$$k(x) \geq n(x) \quad .$$

n_a can be called the <u>shadow function</u> of n for the slope a: the reason for this terminology is the following. Let us suppose that the region of the (x,y) plane defined by $\{(x,y) | y < n(x)\}$ is opaque. Consider now an illumination of the plane by parallel rays of light in the direction a. Then $y = n_a(x)$ is the curve separating the dark region of the plane from the illuminated region.

If $a > a'$ we have $n_a < n_{a'}$; this is a consequence of the obvious inclusion $J_a \supset J_{a'}$.

This implies that if we consider the integral

$$\Omega(\Lambda,a) = \int_{-\infty}^{+\infty} [n_a(x)-n(x)] \frac{dx}{1+x^2}$$

then there will be a critical slope a_o such that for values of $a > a_o$ the integral Ω will converge, while for values of $a < a_o$ the integral will diverge. We call this critical slope a_o the <u>effective density</u> of Λ, and denote it by $D_e(\Lambda)$:

$$D_e(\Lambda) = \inf \{a|\Omega(\Lambda,a) < \infty\} .$$

Before we state our main theorem, we make a preliminary remark. It is well known (cf. for instance Boas: Entire functions) that if $\sum|\operatorname{Im} \frac{1}{\lambda}| = \infty$ then $R(\Lambda) = \infty$. We dispose this trivial case by assuming from now on that

(3) $$\sum|\operatorname{Im} \frac{1}{\lambda}| < \infty .$$

Now we can state

<u>Theorem 1</u>. $$R(\Lambda) = 2\pi D_e(\Lambda) .$$

To prove the theorem requires, on one hand, a uniqueness theorem, and on the other, an actual construction. Let us start with the uniqueness theorem which is the easier. We have to prove:

Proposition 1. Let $f(z)$ *be an entire function such that*

$$|f(z)| < e^{b|y|} \qquad (z = x + iy) .$$

Let Λ *be a sequence such that*

$$D_e(\Lambda) > \frac{b}{\pi}$$

and such that

$$f(\Lambda) = 0 .$$

Then f *vanishes identically.*

If $f \not\equiv 0$ it is well known that

$$(4) \qquad \int_{-\infty}^{+\infty} \log|f(x)| \frac{dx}{1+x^2} > -\infty .$$

The idea of the proof of Proposition 1 is to evaluate the contribution to this integral from the intervals near which there are many points of Λ. For this evaluation the following lemma is useful.

Lemma 1. ϵ *and* h *being two given numbers,* $\epsilon \geq 0$, $h > 0$, *we shall denote by* $A(\epsilon,h)$ *the family of functions* $u(z)$, *subharmonic in the complex plane, and satisfying*

$$u(z) < \pi|y| \qquad z = x + iy$$

$$\int_{V_\epsilon} d\mu \geq 2 + 2h$$

where $2\pi d\mu$ *is the positive measure equal to the Laplacian of* u (in the sense of distributions) *and where* V_ϵ *is the neighborhood of* $(-1,1)$ *defined by*

$$V_\epsilon = \{z \,|\, [\text{distance from } z \text{ to the segment } (-1,1)] \le \epsilon\} .$$

Let

$$\varphi(h,\epsilon) = \sup \int_{-1}^{+1} u(x)dx \qquad \underline{\text{for}} \quad u \in A(h,\epsilon)$$

Then for fixed positive h, it is possible to find an $\epsilon > 0$ such that

$$\varphi(h,\epsilon) < 0 .$$

Let us postpone the proof of this lemma and show at this point how it implies Proposition 1. We choose a, a' such that

$$\frac{b}{\pi} < a' < a < D_e(\Lambda) .$$

Let $\{O_m\}$ be the family of intervals defined by

$$\{x \,|\, n_a(x) > n(x)\} = \bigcup O_m \qquad O_m = (\alpha_m, \alpha_m + \ell_m)$$

We have

(5) $$\int_{O_m} dn = a\ell_m$$

$$\int_{O_m} [n_a(x)-n(x)] \frac{dx}{x^2} < a\ell_m^2 \alpha_m^{-2} .$$

Therefore the hypothesis $a < D_e(\Lambda)$ implies

(6) $$\sum \left(\frac{\ell_m}{\alpha_m}\right)^2 = \infty .$$

Furthermore we can suppose $\ell_m = o(\alpha_m)$. (Otherwise, the result (1) already implies that $f \equiv 0$.) Then

$$\int_{O_m} \log|f(x)| \frac{dx}{x^2} \sim \alpha_m^{-2} \int_{O_m} \log|f(x)|\,dx \ .$$

Let denote by $\mathcal{H}_m$ an affine transformation which maps the interval O_m onto $(-1,+1)$. Let

$$h = \frac{\pi a'}{b} - 1$$

and let ϵ be the number determined in Lemma 1 such that

$$\varphi(h,\epsilon) = -\eta < 0 \ .$$

Now let Γ be the set if integers defined by

> $m \in \Gamma$ if there are more than $a'\ell_m$ points of $\mathcal{H}_m(\Lambda)$ contained in V_ϵ.

Using (5), we find that there exists a constant $\eta_1 > 0$ such that

$$\sum_{|\lambda| \in O_m} |\mathrm{Im}\,\frac{1}{\lambda}| > \eta_1\, \ell_m^2\, \alpha_m^{-2} \qquad \text{for } m \notin \Gamma \ .$$

Together with (3) this implies

$$\sum_{m \notin \Gamma} \left(\frac{\ell_m}{\alpha_m}\right)^2 < \infty \ .$$

On the other hand, Lemma 1 shows that

$$\int_{O_m} \log|f(x)|\,dx < -\eta\, \ell_m^2 \qquad \text{for } m \in \Gamma$$

which implies with (4) that

$$\sum_{m \in \Gamma} \left(\frac{\ell_m}{\alpha_m} \right)^2 < \infty .$$

This contradicts (6) and Proposition 1 is proved.

Proof of Lemma 1. Let us have fixed h. Then a weak compactness argument shows that $\varphi(h,\epsilon)$ is a continuous function of ϵ. It is therefore sufficient to show that

$$\varphi(h,0) < 0 .$$

By considering, if necessary, $\frac{1}{2}[u(z) + u(\bar{z})]$ we see that we can suppose that $u(z) = u(\bar{z})$. Let $d\mu_1$ be the restriction of $d\mu$ to the real axis. We have

$$\frac{\partial u}{\partial y} dx = d\mu_1 .$$

If u_1 is that harmonic function in $\operatorname{Im} z > 0$ having the same values on the real axis as u and if $u_2(z)$ is defined by

$$u_2(z) = u_1(z) - \int_0^\infty \log |1-z^2t^{-2}| dt$$

then we have

$$u_2 < 0 \qquad \text{for all } z$$

$$u_2(x) = u(x) \quad \text{for } x \text{ real}$$

$$\frac{\partial u_2}{\partial y} dx \geq d\mu_1 - dx .$$

Now suppose that there exist a function $\beta(z)$ harmonic in $\operatorname{Im} z > 0$, and a positive number δ such that

$$(7) \qquad \begin{cases} \beta \leq -\delta & \text{for } |x| < 1 \\ -\delta < \beta \leq 0 & \text{for } 1 < |x| < 1 + \frac{h}{2} \\ \beta = 0 & \text{for } |x| > 1 + \frac{h}{2} \end{cases}$$

$$(7') \qquad \begin{cases} \frac{\partial\beta}{\partial y} \text{ is bounded on the real axis} \\ \frac{\partial\beta}{\partial y} \geq 0 \quad \text{for } |x| < 1 \\ \frac{\partial\beta}{\partial y} \leq 0 \quad \text{for } |x| > 1 \ . \end{cases}$$

Now we have

$$\int_{-\infty}^{+\infty} \left(u_2 \frac{\partial\beta}{\partial y} - \beta \frac{\partial u_2}{\partial y}\right) dx = 0 \ .$$

From (7) we deduce that

$$\int_{-\infty}^{+\infty} \beta \frac{\partial u_2}{\partial y} < -\delta \int_{-1}^{+1} (d\mu_1 - dx) + \delta \int_{1 < |x| < 1 + \frac{h}{2}} dx < -\delta h \ .$$

Now (7') gives

$$\left(\int_{-1}^{1} u_2\right) \sup \frac{\partial\beta}{\partial y} < -\delta h \ .$$

which proves the lemma. We have only to prove the existence of a harmonic function satisfying (7) and (7'). To do this we construct a family

$\mathcal{C}_\delta (0 < \delta < 1/4)$ of convex, C^∞ curves, symmetric in the y-axis, as follows. We take the rectangle with vertices at $-1, +1, 1-i, -1-i$, and "round off" the corners with C^∞ curves such that the points $1-\delta$, $1-i\delta$ and $-i$ lie on the straight part of it, and call the resulting curve $\mathcal{C}_\delta$. Let F_δ be the function which maps the interior of $\mathcal{C}_\delta$ on the half-plane $\operatorname{Im} z > 0$, such that

$$F_\delta(0) = \infty \ , \quad F_\delta(1-i\delta) = 1 \ , \quad F_\delta(-i) = 0 \ .$$

Let ψ_δ be the inverse function of F_δ and denote by $\beta = \operatorname{Im} \psi_\delta$. Then β will satisfy (7) and (7') if

$$F_\delta(1-\delta) < 1 + \frac{h}{2} \ .$$

This inequality can be realized if we have chosen δ sufficiently small, because F_δ tends as $\delta \to 0$ to a mapping function of the rectangle onto the half-plane.

The Construction

To get estimates from above for $R(\Lambda)$ we need to construct entire function f of exponential type vanishing on Λ satisfying

$$\text{(8)} \qquad \int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty$$

and of sufficiently small type.

The condition (8) is not very easy to handle. We shall replace it by

$$\text{(8')} \qquad \int_{-\infty}^{+\infty} |\log|f(x)|| \ \frac{dx}{1 + x^2} < \infty \ .$$

The multiplier theorem of the preceding lecture shows that this replacement does not change the problem. That is, if we have a function f satisfying (8'), then we can multiply it by a function g of arbitrary small type and obtain a new function satisfying (8).

The construction is quite long and we cannot give its details here. But let us describe one aspect of the proof. For simplicity let us suppose that Λ is a real, even sequence and let

$$\theta(x) = \frac{n_a(x)-n(x)}{x} .$$

Then if $a > D_e(\Lambda)$ we have, essentially, that $\theta \in L_1(R^*)$ where $L_1(R^*)$ denotes the space of summable functions on the positive real axis for the measure dx/x. The assertion (8') can also be written

$$\frac{\log|f(x)|}{x} \in L_1(R^*) .$$

The function f is constructed in such a way that we have, essentially, on the real axis

$$\frac{\log|f(x)|}{x} = K * \varphi + \text{P.V.} \int_{1/e}^{e} \frac{2}{\log t} \theta(\tfrac{x}{t}) \frac{dt}{t} .$$

Here, K is the kernel

$$K(t) = \frac{2t}{1 - t^2} + \frac{2\omega(t)}{\log t}$$

where ω is the characteristic function of the interval $(1/e, e)$. Since $K \in L_1(R^*)$, it follows that $K * \varphi \in L_1(R^*)$. Thus the problem reduces to the evaluation of the L_1 norm of the above principal value. For that, writing $\xi = \log x$, we shall prove the following lemma on the truncated Hilbert transform.

Lemma 2. *Let $\Phi_1(\xi)$ be a function having its support in $\xi > 0$ satisfying a one-sided Lipschitz condition*

$$\Phi_1(\sigma+\xi) - \Phi_1(\xi) < a\sigma \qquad \text{for } \sigma > 0$$

and let ξ_m be an increasing sequence tending to $+\infty$ such that

$$\sum (\xi_{m+1}-\xi_m)^2 < \infty .$$

Suppose further that

$$\int_{\xi_m}^{\xi_{m+1}} \Phi_1(\sigma)d\sigma = 0 \tag{9}$$

then

$$\Phi_1 \text{ and } \tilde{\Phi}_1 \in L_1(R)$$

where

$$\tilde{\Phi}_1(\sigma) = \text{P.V.} \int_{-1}^{+1} \Phi(\sigma+\xi) \frac{d\xi}{\xi} .$$

The proof of this lemma is straightforward: let ψ_m be the restriction of Φ_1 to the interval (ξ_m,ξ_{m+1}), then $\int|\tilde{\psi}_m|$ is evaluated on the interval $(2\xi_m-\xi_{m+1}, 2\xi_{m+1}-\xi_m)$ by using the Schwarz inequality and on the complement of this interval by using the asymptotic expansion at infinity of $\tilde{\psi}_m$ of which the first term is zero by reason of (9). These arguments, suitably detailed, prove the lemma.

Now the function $\Phi(\xi) = \theta(e^{\xi})$ will not satisfy the hypothesis of Lemma 2, but it can be shown that it is possible to find a sequence $\Lambda_1 \supset \Lambda$, such that for Λ_1 the hypothesis of Lemma 2 will be satisfied, and such that $D_e(\Lambda_1) < D_e(\Lambda) + \epsilon$.