On the Bank–Laine conjecture

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Abstract

We resolve a question of Bank and Laine on the zeros of solutions of $w'' + Aw = 0$ where A is an entire function of finite order.

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1 Introduction and result

The asymptotic distribution of zeros of solutions of linear differential equations with polynomial coefficients is described quite precisely by asymptotic integration methods; cf. [10] and [11, Chapter 8]. While certain differential equations with transcendental coefficients such as the Mathieu equation were considered early on, the first general results concerning the frequency of the zeros of the solutions of

$$w'' + Aw = 0 \tag{1.1}$$

with a transcendental entire function A appear to be due to Bank and Laine [2, 3].

For an entire function f, denote by $\rho(f)$ the order and by $\lambda(f)$ the exponent of convergence of the zeros of f. If A is a polynomial of degree n, then $\rho(w) = 1 + n/2$ for every solution w of (1.1), while $\rho(w) = \infty$ for every solution w if A is transcendental.

Let w_1 and w_2 be linearly independent solutions of (1.1). Bank and Laine proved that if A is transcendental and $\rho(A) < \frac{1}{2}$, then

$$\max\{\lambda(w_1), \lambda(w_2)\} = \infty.$$

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It was shown independently by Rossi [19] and Shen [20] that this actually holds for $\rho(A) \leq \frac{1}{2}$. Bank and Laine also showed that in the case of non-integer $\rho(A)$ we always have

$$\max\{\lambda(w_1), \lambda(w_2)\} \geq \rho(A), \tag{1.2}$$

and they gave examples of functions A of integer order for which there are solutions w_1 and w_2 both without zeros.

A problem left open by their work – which later became known as the Bank–Laine conjecture – is whether $\max\{\lambda(w_1), \lambda(w_2)\} = \infty$ whenever $\rho(A)$ is not an integer. This question has attracted considerable interest; see [13] for a survey, as well as, e.g., [8], [9] and [12, Chapter 5].

We answer this question by showing that the estimate (1.2) is best possible for a dense set of orders in the interval $(1, \infty)$.

Theorem. *Let p and q be odd integers. Then there exists an entire function A of order*

$$\rho(A) = 1 + \frac{\log^2(p/q)}{4\pi^2}$$

for which the equation (1.1) *has two linearly independent solutions w_1 and w_2 such that $\lambda(w_1) = \rho(A)$ while w_2 has no zeros.*

By an extension of the method it should be possible to achieve any prescribed order $\rho(A) > 1$; see Remark 2 at the end.

If w_1 and w_2 are linearly independent solutions of (1.1), then the Wronskian $W(w_1, w_2) = w_1 w_2' - w_1' w_2$ is a non-zero constant. The solutions are called normalized if $W(w_1, w_2) = 1$.

It is well-known that the ratio $F = w_2/w_1$ satisfies the Schwarz differential equation (see, for example [11]):

$$S[F] := \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 = 2A.$$

These meromorphic functions F are completely characterized by a topological property: they are locally univalent. More precisely, consider the equivalence relation on meromorphic functions $F_1 \sim F_2$ if $F_1 = L \circ F_2$, where L is a fractional linear transformation. Then the map $F \mapsto S[F]$ is a bijection between the equivalence classes of locally univalent meromorphic functions and all entire functions.

Normalized solutions w_1, w_2 are recovered from F by the formulas

$$w_1^2 = \frac{1}{F'}, \quad w_2^2 = \frac{F^2}{F'}.$$

So zeros of F are zeros of w_2 and poles of F are zeros of w_1.

A meromorphic function F is locally univalent if and only if $E = F/F'$ is an entire function with the property that $E(z) = 0$ implies $E'(z) \in \{-1, 1\}$. Such entire functions E are called *Bank–Laine functions*. If w_1 and w_2 is a normalized system of solutions of (1.1) and $F = w_2/w_1$, then

$$E = \frac{F}{F'} = w_1 w_2.$$

The converse is also true: every Bank–Laine function is the product of two linearly independent solutions of (1.1).

It turns out that the Schwarzian derivative has the following factorization:

$$2S[F] = B[F/F'],$$

where

$$B[E] := -2\frac{E''}{E} + \left(\frac{E'}{E}\right)^2 - \frac{1}{E^2}. \tag{1.3}$$

Thus every Bank–Laine function E is a product of two linearly independent solutions of (1.1) with $4A = B[E]$, a fact discovered by Bank and Laine [2, 3].

A considerable part of the previous research related to the Bank–Laine conjecture has concentrated on the study of Bank–Laine functions. There are a number of papers where Bank–Laine functions of finite order with various other properties are constructed [1, 4, 6, 14, 15, 16, 18]. In all examples constructed so far, for which the order could be determined, it was an integer. In our construction we have $\rho(E) = \rho(A)$; see Remark 1. Thus our theorem also yields the first examples of Bank–Laine functions of finite non-integral order.

In the proof of our theorem we use the fact that the functions F have a topological characterization. Starting with two elementary locally univalent functions, we paste them together by a quasiconformal surgery. The resulting function is locally univalent, and the asymptotics of $\log |F/F'|$ can be explicitly computed. A different kind of quasiconformal surgery was used in [4, 13].

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2 Proof of the theorem

For every integer $m \geq 0$ we consider the polynomial

$$P_m(z) = \sum_{k=0}^{2m} (-1)^k \frac{z^k}{k!}.$$

Then the entire function

$$g_m(z) = P_m(e^z) \exp e^z$$

satisfies

$$g_m'(z) = (P_m'(e^z) + P_m(e^z))\, e^z \exp e^z = \frac{1}{(2m)!} \exp\left(e^z + (2m+1)z\right)$$

and thus it has the following properties:

a) $g_m'(z) \neq 0$ for all $z \in \mathbf{C}$,

b) g_m is increasing on $\mathbf{R}$, and satisfies $g_m(x) \to 1$ as $x \to -\infty$ as well as $g_m(x) \to +\infty$ as $x \to +\infty$.

From now on, we fix two distinct non-negative integers m and n, and will sometimes omit them from notation. Notice that g_m and g_n are locally univalent entire functions. We are going to restrict g_m to the upper half-plane H^+ and g_n to the lower half-plane H^-, and then paste them together, using a quasiconformal surgery, producing an entire function F. Then our Bank–Laine function will be $E = F/F'$ and thus $A = B[E]/4$ as in (1.3).

It follows from b) that there exists an increasing diffeomorphism $\phi\colon \mathbf{R} \to \mathbf{R}$ such that $g_m(x) = (g_n \circ \phi)(x)$ for $x \in \mathbf{R}$. Let

$$k = \frac{2m+1}{2n+1}.$$

We show that the asymptotic behavior of the diffeomorphism ϕ is the following:

$$\phi(x) = x + O(e^{-x/2}), \quad \phi'(x) \to 1, \quad x \to +\infty, \tag{2.1}$$

and

$$\phi(x) = kx + c + O(e^{-\delta|x|}), \quad \phi'(x) \to k, \quad x \to -\infty, \tag{2.2}$$

with

$$c = \frac{1}{2n+1} \log \frac{(2n+1)!}{(2m+1)!} \quad \text{and} \quad \delta = \frac{1}{2}\min\{1, k\}.$$

In order to prove (2.1), we note that

$$\log g_m(x) = e^x + O(x) = e^x \left(1 + O(xe^{-x})\right), \quad x \to +\infty.$$

The equation $g_m(x) = g_n(\phi(x))$ easily implies that $\frac{2}{3}x \le \phi(x) \le 2x$ for large x. Thus we also have

$$\begin{aligned} \log g_n(\phi(x)) &= e^{\phi(x)} \left(1 + O\big(\phi(x)e^{-\phi(x)}\big)\right) \\ &= e^{\phi(x)} \left(1 + O\big(xe^{-2x/3}\big)\right), \quad x \to +\infty. \end{aligned}$$

Combining the last two equations we obtain

$$e^{\phi(x)-x} = 1 + O(xe^{-2x/3}), \quad x \to +\infty,$$

from which the first statement in (2.1) easily follows. For the second statement in (2.1) we use

$$\phi' = \frac{g_m'}{g_m} \frac{g_n \circ \phi}{g_n' \circ \phi}, \tag{2.3}$$

so that

$$\begin{aligned} \phi'(x) &= \frac{(2n)!}{(2m)!} e^{(2m+1)x-(2n+1)\phi(x)} \frac{P_n(e^{\phi(x)})}{P_m(e^x)} \\ &\sim e^{(2m+1)x-(2n+1)\phi(x)+2n\phi(x)-2mx} \\ &= e^{x-\phi(x)} = 1 + o(1), \quad x \to +\infty. \end{aligned}$$

In order to prove (2.2) we note that

$$P_m(w) = e^{-w} + \frac{w^{2m+1}}{(2m+1)!} + O(w^{2m+2}), \quad w \to 0,$$

and thus

$$P_m(w)e^w = 1 + \frac{w^{2m+1}}{(2m+1)!} + O(w^{2m+2}), \quad w \to 0.$$

Hence

$$\begin{aligned} g_m(x) &= 1 + \frac{e^{(2m+1)x}}{(2m+1)!} + O(e^{(2m+2)x}) \\ &= 1 + \frac{e^{(2m+1)x}}{(2m+1)!}(1 + O(e^x)), \quad x \to -\infty. \end{aligned}$$

The equation $g_m(x) = g_n(\phi(x))$ now yields

$$\frac{(2m+1)!}{(2n+1)!} e^{(2n+1)\phi(x)-(2m+1)x} = 1 + O(e^x) + O(e^{\phi(x)}), \quad x \to -\infty$$

and hence

$$\phi(x) = \frac{2m+1}{2n+1}x + \frac{1}{2n+1}\log\frac{(2n+1)!}{(2m+1)!} + O(e^x) + O(e^{\phi(x)}), \quad x \to -\infty,$$

which gives the first statement in (2.2). For the second statement in (2.2) we use (2.3) and obtain

$$\begin{aligned} \phi'(x) &\sim \frac{(2n)!}{(2m)!} e^{(2m+1)x-(2n+1)\phi(x)} = \frac{(2n)!}{(2m)!} e^{(2m+1)x-(2n+1)(kx+c+o(1))} \\ &= \frac{(2n)!}{(2m)!} e^{-(2n+1)c+o(1)} = k + o(1). \end{aligned}$$

Let $D = \mathbf{C}\backslash\mathbf{R}_{\leq 0}$, and $p \colon D \to \mathbf{C}$, $p(z) = z^\mu$, the principal branch of the power. Here μ is a complex number to be determined so that p maps D onto the complement G of a logarithmic spiral Γ, with

$$p(x + i0) = p(kx - i0), \quad x < 0. \tag{2.4}$$

It will be convenient to consider also the map $z \to \mu z$ obtained from p by a logarithmic change of the variable: if $w = p(z)$ then $\log w = \mu \log z$, cf. Figure 1.

This shows (taking $x = 0$ in Figure 1) that with $a_- = \log k - i\pi$ and $a_+ = i\pi$ we have $\mathrm{Re}(\mu a_-) = \mathrm{Re}(\mu a_+)$; that is, $\mathrm{Re}(\mu(\log k - i\pi)) = \mathrm{Re}(\mu i\pi)$. Moreover, $\mathrm{Im}(i\pi/\mu) = \pi$. A simple computation now yields that

$$\mu = \frac{2\pi}{4\pi^2 + \log^2 k}(2\pi - i\log k).$$

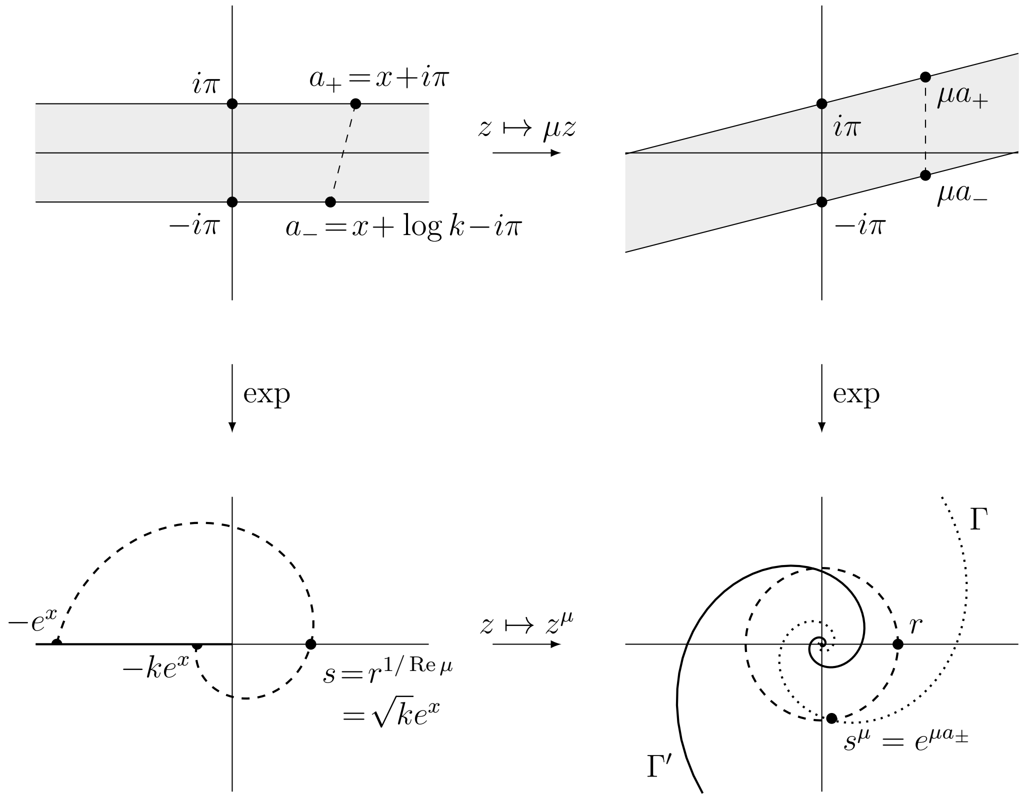

Figure 1: Sketch of the map p and the logarithmic change of variable, for $k = \frac{1}{5}$ and $\mu \approx 0.9384 + 0.2403i$. (The actual spirals Γ and Γ' wind much slower than drawn.)

The inverse map $h = p^{-1}$ is a conformal homeomorphism $h\colon G \to D$. Let $\Gamma' = p(\mathbf{R}_{\geq 0})$. The two logarithmic spirals Γ and Γ' divide the plane into two parts, G^+ and G^- which are images under p of the upper and lower half-planes, respectively.

The function V defined by

$$V(z) = \begin{cases} (g_m \circ h)(z), & z \in G^+, \\ (g_n \circ h)(z), & z \in G^-, \end{cases}$$

is analytic in $G^+ \cup G^-$ and has a jump discontinuity on Γ and Γ'. In view of (2.1), (2.2) and (2.4), this discontinuity can be removed by a small change in the independent variable. In order to do so, we consider the strip $\Pi =$

$\{z\colon |\mathrm{Im}\, z| < 1\}$ and define a quasiconformal homeomorphism $\tau\colon \mathbf{C} \to \mathbf{C}$, commuting with the complex conjugation, which is the identity outside of Π and satisfies

$$\tau(x) = \phi(x), \quad x > 0, \quad \text{and} \quad \tau(kx) = \phi(x), \quad x < 0. \tag{2.5}$$

Our homeomorphism can be given by an explicit formula: for $y = \mathrm{Im}\, z \in (-1, 1)$ we put

$$\tau(x + iy) = \begin{cases} \phi(x) + |y|(x - \phi(x)) + iy, & x \geq 0 \\ \phi(x/k) + |y|(x - \phi(x/k)) + iy, & x < 0. \end{cases}$$

The Jacobian matrix D_τ of τ is given for $x > 0$ and $0 < |y| < 1$ by

$$D_\tau(x + iy) = \begin{pmatrix} \phi'(x) + |y|(1 - \phi'(x)) & \pm(x - \phi(x)) \\ 0 & 1 \end{pmatrix},$$

and we see using (2.1) that

$$D_\tau(x + iy) \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 0 < |y| < 1,\ x \to \infty,$$

Similarly, using (2.2) we find that

$$D_\tau(x + iy) \to \begin{pmatrix} 1 & \mp c \\ 0 & 1 \end{pmatrix}, \quad 0 < |y| < 1,\ x \to -\infty.$$

We conclude that τ is quasiconformal in the plane.

Now we modify V to obtain a continuous function and define $U\colon \mathbf{C} \to \mathbf{C}$,

$$U(z) = \begin{cases} (g_m \circ h)(z), & z \in G^+ \cup \Gamma \cup \Gamma' \cup \{0\}, \\ (g_n \circ \tau \circ h)(z), & z \in G^-. \end{cases} \tag{2.6}$$

It follows from (2.4) and (2.5) that U is continuous and quasiregular in the plane. The existence theorem for solutions of the Beltrami equation [17, §V.1] yields that there exists a a quasiconformal homeomorphism $\psi\colon \mathbf{C} \to \mathbf{C}$ with the same Beltrami coefficient as U. The function $F = U \circ \psi^{-1}$ is then entire.

We note that U is regular in $\mathbf{C}\backslash X$, where $X = p(\Pi^-)$, and Π^- is the lower half of Π. Let $\Delta = \{z\colon |z| > 1\}$. It is easy to see that $X \cap \Delta$ has finite logarithmic area; that is,

$$\int_{X\cap\Delta} \frac{dx\,dy}{x^2 + y^2} = \int_{\Pi^-\cap\Delta} \frac{|p'(z)|^2}{|p(z)|^2} dx\,dy = |\mu|^2 \int_{\Pi^-\cap\Delta} \frac{dx\,dy}{x^2 + y^2} < \infty.$$

Thus the Beltrami coefficient of U (and hence of ψ) satisfies the hypotheses of the Teichmüller–Wittich–Belinskii theorem [17, §V.6]. This theorem shows that ψ is conformal at ∞ and may thus be normalized to satisfy

$$\psi(z) \sim z, \quad z \to \infty. \tag{2.7}$$

Now we want to differentiate the asymptotic relation (2.7). We write $\psi(z) = z + \psi_0(z)$ so that $\psi'(z) = 1 + \psi_0'(z)$. Then $|\psi_0(z)| \leq \alpha(z)$ for some function α satisfying $\alpha(z) = o(z)$ as $z \to \infty$. We may assume that $\alpha(z) \to \infty$ as $z \to \infty$. We use the Cauchy formula

$$\psi_0'(z) = \frac{1}{2\pi i} \int_{C_z} \frac{\psi_0(\zeta)}{(\zeta - z)^2} d\zeta$$

with a circle C_z centered at z. Choosing the radius $\beta(z)$ of this circle to satisfy

$$\alpha(z) = o(\beta(z)), \quad \beta(z) = o(z), \quad z \to \infty$$

and putting $Y = \{z \colon \operatorname{dist}(z, X) \leq \beta(z)\}$ we obtain

$$\psi_0'(z) \to 0, \quad z \to \infty, \ z \in \mathbf{C} \backslash Y. \tag{2.8}$$

We also have

$$\operatorname{meas}\{\theta \in [0, 2\pi] \colon re^{i\theta} \in Y\} \to 0, \quad r \to \infty.$$

Let $Y' = \psi(Y)$. Using (2.7) we see that also

$$\operatorname{meas}\{\theta \in [0, 2\pi] \colon re^{i\theta} \in Y'\} \to 0, \quad r \to \infty. \tag{2.9}$$

We put $E = F/F'$. As $F'(z) \neq 0$ for all $z \in \mathbf{C}$ by construction, E is entire. As all zeros of F are simple, all residues of F'/F are equal to 1, so $E'(z) = 1$ at every zero z of E, which implies the Bank–Laine property.

First we prove that E is of finite order. In order to do this, we use the standard terminology of Nevanlinna theory; see [7] or [12]. The counting function of the sequence of zeros of g_m and g_n is of order 1, so the counting function of the zeros of U in (2.6) is of finite order. Then (2.7) shows that the counting function of zeros of F, and hence the counting function of the zeros of E, is also of finite order; that is, $\log N(r, 1/E) = O(\log r)$. Similarly, $\log\log m(r, F) = O(\log r)$, so by the Lemma on the logarithmic derivative [7, Chapter 3, Theorem 1.3] we have $\log m(r, 1/E) = \log m(r, F'/F) = O(\log r)$. Thus $\log T(r, E) = O(\log r)$ so that E is of finite order.

Now we estimate more precisely the growth of the Nevanlinna proximity function $m(r, 1/E) = m(r, F'/F)$. The "small arcs lemma" of Edrei and Fuchs [7, Chapter 1, Theorem 7.3] permits us to discard the exceptional set $Y' = \psi(Y)$. Outside of this set we have $\psi'(z) \to 1$ in view of (2.8), therefore

$$\int_{\{\theta\in[0,2\pi]:\, re^{i\theta}\in\mathbf{C}\setminus Y'\}} \left|\log|\psi'(re^{i\theta})|\right| d\theta = o(1), \quad r \to \infty. \tag{2.10}$$

Furthermore, as $h(z) = z^{1/\mu}$, we have

$$\int_0^{2\pi} \left|\log|h'(re^{i\theta})|\right| d\theta = O(\log r), \quad r \to \infty. \tag{2.11}$$

Now we have in $\psi^{-1}(D^+\setminus Y)$

$$\frac{F'}{F} = \left(\frac{g_m'}{g_m} \circ h \circ \psi^{-1}\right)(h' \circ \psi^{-1})(\psi^{-1})'. \tag{2.12}$$

According to (2.10) and (2.11), the contribution of h' and $(\psi^{-1})'$ to $m(r, F'/F)$ is $O(\log r)$. Using the explicit form of g_m'/g_m we obtain, outside small neighborhoods of the zeros of g_m whose contribution can be neglected again by the small arcs lemma of Edrei and Fuchs,

$$\log^+\left|\frac{g_m'(z)}{g_m(z)}\right| \sim \operatorname{Re}^+ z, \quad z \to \infty. \tag{2.13}$$

Now the image of the circle $\{z \colon |z| = r\}$ under $h(z) = z^{1/\mu}$ is the part of the logarithmic spiral which connects two points on the negative real axis and intersects the positive real axis at $r^{1/\operatorname{Re}\mu}$; cf. Figure 1. By (2.7), the image of this circle under $h \circ \psi^{-1}$ is an arc close to this part of the logarithmic spiral. It now follows from (2.10), (2.11), (2.12) and (2.13) that the part of $m(r, F'/F)$ which comes from $\psi^{-1}(G^+\setminus Y)$ has order

$$\rho = \frac{1}{\operatorname{Re}\mu} = 1 + \frac{\log^2 k}{4\pi^2}.$$

The other part which comes from $\psi^{-1}(G^-\setminus Y)$ is similar, and the contribution of Y' is negligible in view of (2.9). So $m(r, 1/E) = m(r, F'/F)$ has order ρ.

Now (1.3) says that

$$4A = -2\frac{E''}{E} + \left(\frac{E'}{E}\right)^2 - \frac{1}{E^2}.$$

It follows from the lemma on the logarithmic derivative that

$$m(r, A) = 2m\left(r, \frac{1}{E}\right) + O(\log r).$$

Thus A also has order ρ.

3 Remarks

Remark 1. To prove that $\rho(A) = \rho$ it was sufficient to determine the growth of $m(r, 1/E)$. To show that $\rho(E) = \rho$ we also have to estimate the counting function of the zeros of E. In order to do so we note that $N(r, 1/g_m) = O(r)$ and $N(r, 1/g_n) = O(r)$. Hence $N(r, 1/U) = O(r^\rho)$ and thus (2.7) implies that

$$N\left(r, \frac{1}{E}\right) = N(r, F) = O(r^\rho).$$

Altogether we see that $\rho(E) = \rho = \rho(A)$, as stated in the introduction.

We note that $\rho(A) < 1$ implies that $\rho(E) > 1$, as follows from any of the following inequalities [13, Theorem 12.3.1]:

$$\rho(A) + \rho(E) \geq 2, \quad \frac{1}{\rho(A)} + \frac{1}{\rho(E)} \leq 2 \quad \text{and} \quad \rho(A)\rho(E) \geq 1.$$

Moreover, it can be deduced from (1.3) that if $\rho(A) < 1$, then $\lambda(E) = \rho(E)$; see [13, p. 442].

As our method yields examples with $\rho(E) = \rho(A)$, it does not seem suitable to give examples with $\rho(A) < 1$. The question whether $\rho(A) \in (\frac{1}{2}, 1)$ implies that $\max\{\lambda(w_1), \lambda(w_2)\} = \infty$ for linearly independent solutions w_1 and w_2 of (1.1) remains open.

Remark 2. We started our construction with two periodic locally univalent functions g_m and g_n and obtained a set of orders ρ which is dense in $[1, +\infty)$. By using almost periodic building blocks instead of g_m and g_n, one can probably achieve any prescribed order greater than 1; cf. [7, Chapter 7, Section 6]. In this case g_m and g_n will not be explicitly known, but their asymptotic behavior can be obtained.

Remark 3. The Bank–Laine functions we have constructed actually satisfy $E(z) = 1$ whenever $E'(z) = 0$. Equivalently, one of the two solutions of (1.1) whose product is E has no zeros while the other one has a finite exponent of convergence.

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