

ON EQUILIBRIUM POINTS OF LOGARITHMIC AND NEWTONIAN POTENTIALS

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ABSTRACT

Let $f(z) = \sum_{j=1}^{\infty} a_j/(z-z_j)$, where $z_j \neq 0$ and $\sum_{j=1}^{\infty} |a_j|/|z_j| < \infty$. Then f can be realized as the complex conjugate of the gradient of a logarithmic potential or, for integral a_j , as the logarithmic derivative of a meromorphic function. We investigate conditions on a_j and z_j that guarantee that f has zeros. In the potential theoretic setting, this asks whether certain logarithmic potentials with discrete mass distribution have equilibrium points.

1. Introduction

In this paper we will discuss the following problem. Consider the meromorphic function

$$f(z) = \sum_{j=1}^{\infty} \frac{a_j}{z-z_j}, \quad \sum_{j=1}^{\infty} \frac{|a_j|}{|z_j|} < \infty \quad (1.1)$$

where $z_j \neq 0$. Under what conditions on a_j and z_j can one guarantee that f has zeros?

This question is an extended version of a problem proposed by L. Rubel [2, Problem 7.78]. We shall suppose that $a_j \in \mathbb{R}$. Then the problem has the following physical interpretation: if we imagine particles (really charged wires perpendicular to the complex plane) placed at each point z_j and having the charges a_j then these generate a logarithmic potential given by

$$u(z) = \sum_{j=1}^{\infty} a_j \log \left| 1 - \frac{z}{z_j} \right|. \quad (1.2)$$

The gradient of this potential is the complex conjugate of f . Thus the question is whether such a field must always have an equilibrium point, that is a point upon which a free electron (or wire) once placed would remain.

This question and the corresponding problem in \mathbb{R}^n , $n \geq 3$ stimulate an interesting area of research—distribution of critical points of potentials generated by an infinite discrete distribution of charge. This paper contains some preliminary results in this area and some new questions.

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2. Formulation of results

If the a_j are integers, we may consider the meromorphic function

$$F(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right)^{a_j} \quad (2.1)$$

which is of order at most one, minimal type. (It belongs to the convergence class of order 1). Then

$$f(z) = \frac{F'(z)}{F(z)}. \quad (2.2)$$

If the z_j are distinct and $a_j = 1$ for all j then (2.2) implies that f has infinitely many zeros. Indeed in this case the zeros of F' are the zeros of f and since F' is also of order at most one, minimal type, F' has infinitely many zeros.

THEOREM 2.1. *If the a_j are positive integers then f has an infinite set of zeros.*

This follows immediately from the following.

THEOREM 2.2. *Let F be a transcendental entire function of order at most one, minimal type. Then F'/F has an infinite set of zeros.*

In the opposite direction we have

EXAMPLE 2.3. For every $\rho \geq 1$ there exists an entire function F of order ρ such that F'/F has no zeros.

A. A. Goldberg and W. K. Hayman brought the following result to our attention.

THEOREM 2.4. *If $a_j = \pm 1$, then f has an infinite set of zeros.*

This is a consequence of a result of F. Nevanlinna [10]. Under the conditions of Theorem 2.4 the function F defined by (2.1) has only simple zeros and poles. So if f has a finite set of zeros, then F has only a finite number of critical points. In [10] all functions of finite order with this property were described. They satisfy differential equations of the form

$$\frac{F'''}{F'} - \frac{3}{2} \left(\frac{F''}{F'}\right)^2 = R, \quad (2.3)$$

where R is a rational function. Asymptotic integration shows that such functions F have normal type and order $n/2$, where n is an integer, $n \geq 2$.

If the a_j are only supposed to be integers then f may have no zeros. The simplest example is the following. Take

$$F(z) = \tan^2 \sqrt{z},$$

which is a meromorphic function of order $1/2$, normal type. Then

$$f(z) = \frac{F'(z)}{F(z)} = \frac{2}{\sqrt{z} \sin 2\sqrt{z}} \neq 0$$

and f has the form (1.1). A similar example was communicated to the authors by W. K. Hayman. This example has also come up in the work of S. Bank and I. Laine [1] on ordinary differential equations in the complex domain.

EXAMPLE 2.5. For every $\rho \geq 1/2$ there exists a meromorphic function F of order ρ such that F'/F has no zeros.

On the other hand we have the following theorem

THEOREM 2.6. *If F is a transcendental meromorphic function of order at most $1/2$, minimal type then its logarithmic derivative f has an infinite set of zeros.*

Let us state some conjectures based on the above theorems and examples.

CONJECTURE 2.7. *If the $a_j > 0$ then the function (1.1) has zeros. Equivalently, subharmonic functions of the form (1.2) have critical points.*

CONJECTURE 2.8. *If the a_j are real and*

$$\sum_{j:|z_j| \leq r} |a_j| = o(\sqrt{r}) \quad (2.4)$$

then the function (1.1) has zeros. Equivalently, δ -subharmonic functions of the form (1.2) of order at most $1/2$, minimal type have critical points.

CONJECTURE 2.9. (W. K. Hayman) *If the a_j are integers and $a_j \geq -1$ then the function (1.1) has zeros.*

It is fairly easy to see that Conjecture 2.9 is equivalent to the following simply stated problem: *if F is meromorphic of order at most one, convergence class, then F' has zeros.* We remark that by Theorem 2.6 the problem is solved for order at most $1/2$, minimal type.

We can partially prove Conjecture 2.7 with the additional assumption (2.4).

THEOREM 2.10.¹ *Under the conditions $a_j > 0$ and (2.4) the function (1.1) has an infinite set of zeros.*

Theorem 2.10 may be compared with Keldysh's theorem [see 5, Chapter 5, Theorem 6.2] that if

$$\sum_{j=1}^{\infty} |a_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} a_j \neq 0$$

then the function (1.1) has zeros and moreover,

$$\delta(0, f) = 0.$$

Here $a_j \in \mathbb{C}$ and $\delta(0, f)$ is the usual Nevanlinna deficiency of f at 0.

Our proof of all mentioned results except Theorem 2.10 require some complex analysis. So in the space \mathbb{R}^m , $m \geq 3$, we are able to prove only the following weak result.

THEOREM 2.11. *Let $x_k \in \mathbb{R}^m$, $m \geq 3$ be distinct, nonzero points, $x_k \rightarrow \infty$. Then the function*

$$u(x) = \sum_{k=1}^{\infty} \frac{a_k}{|x - x_k|^{m-2}}, \quad a_k \geq 1, \quad \sum_{k=1}^{\infty} \frac{a_k}{|x_k|^{m-2}} < \infty \quad (2.5)$$

has an infinite set of critical points.

¹ Essentially the same theorem was proved independently by J. Miles.

CONJECTURE 2.12. If $x_k \in \mathbb{R}^m$

$$\sum_{k=1}^{\infty} |x_k|^{-1+m} < \infty,$$

then the function

$$u(x) = \sum_{k=1}^{\infty} \left(\frac{1}{|x - x_k|^{m-2}} - \frac{1}{|x_k|^{m-2}} \right)$$

has an infinite set of critical points.

3. Preliminaries

Let us begin by recalling some topological properties of a meromorphic function f [8, 11]. Suppose $f(z_0) = w_0$ and $f'(z_0)$ is nonzero and finite. Then we can analytically continue the branch of f^{-1} taking w_0 to z_0 in a neighbourhood of w_0 . There are two types of obstacles preventing a further continuation of f^{-1} . The first of these arises from critical points, that is points z where $f'(z) = 0$. Here the point $f(z)$ is called a *critical value*. The second obstacle arises from asymptotic values of f . A point a is called an *asymptotic value* if there exists a curve Γ tending to infinity such that $f(z)$ tends to a along Γ . If a domain V contains no critical or asymptotic values then every branch of f^{-1} can be analytically continued throughout V .

To classify the singularities of f^{-1} , fix $a \in \mathbb{C}$ and take a set A of connected neighbourhoods of the point a such that $\bigcap \{N : N \in A\} = a$. Then for every $N \in A$ choose a component V_N of the preimage $f^{-1}(N)$ in such a way that $N_1 \subset N_2 \Rightarrow V_{N_1} \subset V_{N_2}$. There are two possibilities:

1. $\bigcap \{V_N : N \in A\}$ is a point $z \in \mathbb{C}$. This point may be non-critical or critical. In the latter case we say that f^{-1} has an *algebraic singularity* (or algebraic branch point) over a .

2. $\bigcap \{V_N : N \in A\} = \emptyset$. In this case we say that our choice $N \mapsto V_N$ defines a *transcendental singularity* of f^{-1} over a . This transcendental singularity is called *direct* if some V_{N_0} contains no preimages of the point a . (Then V_N contains no preimages of a for all $N \subset N_0$). If all V_N contain preimages of a then each of them contains an infinite number of these preimages. Such a singularity is called *indirect*. Note that f^{-1} has a transcendental singularity over a if and only if a is an asymptotic value.

We need the following theorem.

DENJOY-CARLEMAN-AHLFORS THEOREM. *Let f be a meromorphic function and suppose that f^{-1} has $n \geq 2$ direct transcendental singularities. Then*

$$\liminf_{r \rightarrow \infty} r^{-n/2} T(r, f) > 0. \quad (3.1)$$

If f is entire (3.1) is equivalent to

$$\liminf_{r \rightarrow \infty} r^{-n/2} \log M(r, f) > 0. \quad (3.2)$$

If an entire function f has $n \geq 1$ distinct finite asymptotic values then (3.2) holds.

We need also the following elementary proposition concerning isolated singularities of f^{-1} (c.f. [11, p. 283]).

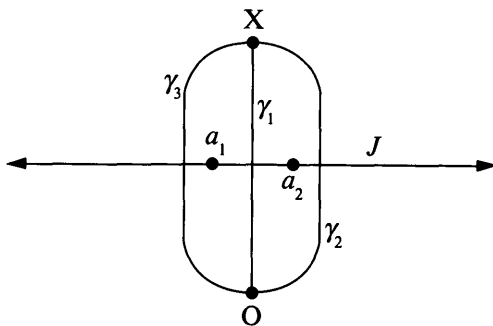
PROPOSITION 3.1. *Let f be a meromorphic function in \mathbb{C} . Suppose there exists a point a and a disk N centred at a such that some branch of f^{-1} can be analytically continued throughout $N \setminus \{a\}$. Let V be the corresponding component of $f^{-1}(N \setminus \{a\})$. Then there are two possibilities.*

(i) *For some z_0 with $f(z_0) = a$, $V \cup \{z_0\}$ is a bounded Jordan domain, f maps n to 1 from V to $N \setminus \{a\}$ and $f(\partial V) = \partial N$. (Note that z_0 may or may not be a critical point.)*

(ii) *V is an unbounded, simply connected domain, and ∂V is a simple curve which tends to ∞ in both directions. The point a has no preimages in V .*

We note that if (ii) holds then V is a neighbourhood of a special type of direct transcendental singularity called a *logarithmic singularity* or *logarithmic branch point*.

We shall use the notion of *line complex* [11, 5, 12]. Let us recall the main ideas here. Suppose that there exists a finite set $A = \{a_1, \dots, a_q\}$ containing all critical and asymptotic values of f . It follows from Proposition 3.1 that all possible singularities of f^{-1} are algebraic and logarithmic branch points. Consider a Jordan curve J passing through the points a_1, \dots, a_q . This divides the sphere $\bar{\mathbb{C}}$ into two components. Choose two points, one in each component and denote them by X and O . The curve J is divided by the points a_1, \dots, a_q into q open arcs l_1, \dots, l_q . For every k connect the points X and O by a simple curve γ_k such that the only point of intersection of γ_k with J lies on l_k , and all the γ_k are disjoint. We get a graph γ with two vertices X and O and q edges γ_k . In the following picture $q = 3$ and $a_3 = \infty$.



The full preimage $\Gamma = f^{-1}(\gamma) \subset \mathbb{C}$ is called a *line complex* (Streckenkomplexe) and has the following properties, which are almost evident.

1. Γ is connected.
2. Γ has vertices of two types: X and O . Each edge connects two vertices of different types.
3. The number of edges adjacent to each vertex is equal to q .

It is known (and easy to prove using the Uniformization Theorem) that every graph with these properties embedded in the plane arises from some meromorphic function in the plane or in the unit disk having a finite set of critical and asymptotic values.

The faces of Γ (which means connected components of $\mathbb{C} \setminus \Gamma$) are mapped to the neighbourhoods of the points a_k (bounded by γ_k and γ_{k-1}). So we may mark each face which is mapped to a neighbourhood of a_k by the symbol a_k . It is easy to see that two faces marked by the same symbol have disjoint closures. Each face is bounded by an

even number of edges. Each $2n$ -gonal face contains one a_k -point of multiplicity n . A logarithmic singularity corresponds to the face bounded by an infinite number of edges which forms the following simple chain.

$$-O-X-O-X-O-X-O-X-O-X-$$

Sometimes we shall consider a reduced line complex which means that we delete all 2-gons. A reduced line complex contains at most one edge connecting a pair of vertices.

4. Proof of Theorem 2.2

Suppose that F'/F has a finite set of zeros. So $F'(z) = 0 \Rightarrow F(z) = 0$ with at most a finite number of exceptions. It follows that the set of critical values of F is finite. By the Denjoy–Carleman–Ahlfors Theorem, the set of asymptotic values is also finite. Since F is transcendental, Proposition 3.1 implies that there is a logarithmic branch point over ∞ . Further since there are only finitely many finite singularities of f^{-1} , any finite asymptotic value is also a logarithmic branch point. Consequently by the Denjoy–Carleman–Ahlfors Theorem and the order restriction on F , F can have no finite asymptotic value. That this is impossible follows from the following lemma with $a_1 = 0$.

LEMMA 4.1. *Let F be transcendental, entire with finitely many critical values a_1, \dots, a_n . Suppose $E = \{z: F(z) = a_i, F'(z) = 0, i = 2, \dots, n\}$ is a finite set. Then F has a finite asymptotic value.*

Proof. If E is empty, we get an immediate contradiction from Proposition 3.1 since it implies that a_1 is a logarithmic singularity and hence an asymptotic value. Suppose F has no finite asymptotic value. Then there exists $R > 0$ such that $U = \{z: |z| < R\}$ contains all the finite singularities of F^{-1} . Since any branch of F^{-1} can be continued analytically without restriction in $\mathbb{C} \setminus \bar{U}$, it follows from Proposition 3.1 that any component V of $F^{-1}(\mathbb{C} \setminus \bar{U})$ corresponds to a logarithmic singularity at infinity. Each component of $F^{-1}(\partial U)$ is a boundary component of such a domain V and hence, in this case, coincides with ∂V . Thus ∂V is a Jordan curve going to infinity in both directions. Fix a point b in U and insert n line segments from b to ∂U . This divides U into n disjoint regions C_1, \dots, C_n each bounded by consecutive rays, R_i and R_{i+1} (throughout, $n+1$ will be identified with 1). By moving b if necessary we can arrange that $a_i \in C_i$.

By Proposition 3.1 and the assumption that F has no finite asymptotic value, every component of $F^{-1}(C_i)$ is bounded and consequently, $F^{-1}(C_i)$ has infinitely many components for each $i, i = 1, \dots, n$. Since E is finite, there exist components B_i of $F^{-1}(C_i)$ such that F' is not zero in $\bar{B}_i, i = 2, \dots, n$ and $\partial B_i \cap \partial B_{i+1}$ maps into R_{i+1} . Thus, since there are no finite asymptotic values of F , the monodromy theorem gives that F^{-1} is 1-1 from $\bar{C}_2 \cup \dots \cup \bar{C}_n$ onto $\bar{B}_2 \cup \dots \cup \bar{B}_n$. Since the singularities of F^{-1} in \bar{C}_1 lie over a_1 , we can continue this branch of F^{-1} , to all of $U \setminus \{a_1\}$. Again by Proposition 3.1, the resulting component V of $F^{-1}(U)$ is a bounded region with $F(\partial V) = \partial U$. But, as mentioned above, every component of $F^{-1}(\partial U)$ is a simple Jordan curve extending to infinity in both directions. This gives the desired contradiction. The lemma, and hence Theorem 2.2, is now proved.

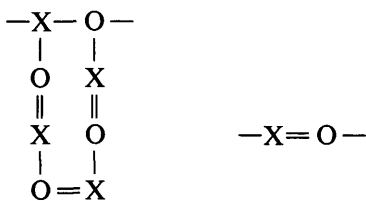
We present a more elegant (but essentially identical) proof of Lemma 4.1 using the notion of line complexes. Since our examples depend heavily on this notion, the two proofs may serve to help the reader better to understand the next section.

Consider the reduced line complex corresponding to F . It contains a face D_0 corresponding to the logarithmic singularity over $a_q = \infty$ which is bounded by a simple chain. The line complex also may contain a finite number of faces, say D_1, \dots, D_n marked by the points a_2, \dots, a_{q-1} and an infinite set of faces $D_j, j \geq n+1$, marked by $a_1 = 0$. Take two adjacent edges w_1 and w_2 on ∂D_0 which do not belong to the boundaries of D_1, \dots, D_n . One of them, say w_1 , does not belong to the boundary of any $D_j, j \geq n+1$, because the closures of $D_j, j \geq n+1$ are pairwise disjoint. So there is a face D^* adjacent to D_0 along w_1 such that D^* is not a $D_j, j \geq 0$. Since all faces corresponding to algebraic branch points are listed in the sequence $D_j, j \geq 1, D^*$ is a face corresponding to some finite logarithmic branch point other than that of D_0 . This logarithmic branch point gives the required asymptotic value.

5. Examples

Our examples are based on the theory of Riemann surfaces with almost periodic ends which was developed by A. A. Goldberg [4] toward the solution of the inverse problem of Nevanlinna theory. An excellent self-contained presentation of the theory is in [5, Chapter 7].

Construction of Example 2.3. If $\rho = 1$, we may take $F(z) = e^z$; so we suppose now that $\rho > 1$. Fix a natural number $n \geq 2$ and consider the following two graphs called Block 1 and Block 2. Block 1 contains $2n$ vertices.



BLOCK 1 ($n=4$) BLOCK 2

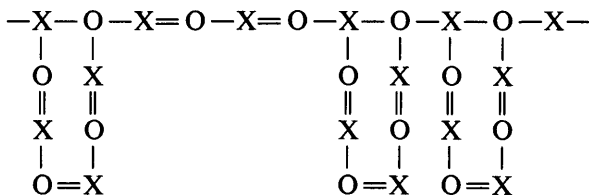
Then fix a number $x, 0 < x < 1$ and find a sequence Σ of natural numbers with the property

$$\#(\Sigma \cap [1, m]) = mx + O(1), \quad m \rightarrow \infty. \tag{5.1}$$

Now construct the line complex Γ as follows: take the sequence of blocks $(B_k), -\infty < k < \infty$ such that

- B_k is Block 1 if $k \in \Sigma,$
- B_k is Block 2 if $k \notin \Sigma.$

Then connect each B_k to B_{k+1} identifying the right free edge of B_k to the left free edge of B_{k+1} . We get the following line complex when $n = 4$ and $\Sigma = \{1, 4, 5, \dots\}$.



All conditions for this graph to be a line complex are satisfied. The corresponding function is ramified only over three points which we may choose to be 0, 1 and ∞ . Let us mark the upper infinite polygonal face by 1, the lower one by ∞ and all the $2n$ -gonal faces by 0.

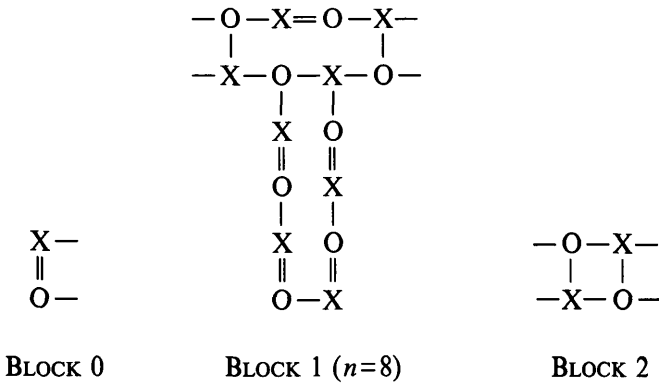
It follows from the theory of almost periodic ends that the constructed line complex corresponds to a function F meromorphic in \mathbb{C} and having normal type of order

$$\rho = 1 + \left(\frac{\log((n-1)x+1)}{2\pi} \right)^2 \tag{5.2}$$

[5, Theorem 7.1, (7.17)]. It is easy to see that one can satisfy (5.11) with given $\rho > 1$ by choosing a natural number $n \geq 2$ and a positive $x < 1$.

From the line complex we see that the function F is entire since there are no faces with a finite number of edges marked by ∞ . All critical points (which correspond to $2n$ -gonal faces) are mapped by F to 0. So our example has all the desired properties.

Construction of Example 2.5. Fix a natural number $n \geq 5$ and consider the following three blocks:

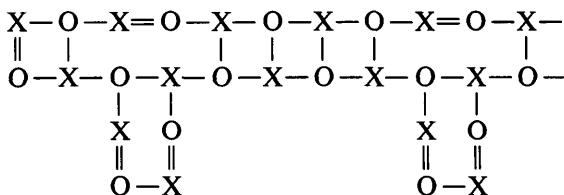


Block 1 has two faces: one of them is an 8-gon (above) and the other is a $2(n-3)$ -gon (below).

Fix a number x , $0 < x < 1$ and choose a sequence of natural numbers Σ satisfying (5.1). Then construct the line complex Γ as follows: take a sequence of blocks (B_k) , $k = 0, 1, 2, \dots$ such that

- B_0 is Block 0,
- B_k is Block 1 if $k \in \Sigma$,
- B_k is Block 2 in all other cases.

Connect B_k to B_{k+1} by identifying the right pair of free edges of B_k to the left pair of free edges of B_{k+1} . We get a line complex like this:



In the picture $n = 6$ and $\Sigma = \{1, 3, \dots\}$. Mark all faces of Γ which are polygons with a finite number of sides by 0 and ∞ such that adjacent faces have different markers. The only infinite face will be marked by 1. It follows from the theory of almost periodic ends that there is a function F meromorphic in \mathbb{C} which corresponds to Γ . The only ramified values of f are 0 and ∞ . The order of F is given by [5, Chapter 7, (7.17)]:

$$\rho = \frac{1}{2} \left(1 + \left(\frac{1}{\pi} \log \left(\frac{(n-3)x+1}{x+1} \right) \right)^2 \right). \quad (5.3)$$

It is easy to see that one can get an arbitrary order $\rho > 1/2$ by choosing appropriate n and x .

6. Proof of Theorem 2.6

We use a theorem by A. Huber [7] (see also [9] for another proof).

HUBER'S THEOREM. *Let g be a transcendental entire function. Then there exists a curve Γ tending to infinity such that*

$$\int_{\Gamma} |g(z)|^{-1} |dz| < \infty.$$

Let us suppose that Theorem 2.6 is false and hence that F'/F has finitely many zeros. We write

$$\frac{F'}{F} = \frac{P}{g},$$

where P is a polynomial and g is a transcendental entire function of order at most $\frac{1}{2}$, minimal type. We write $g = P_1 g_1$ where P_1 is a polynomial of the same degree as P and g_1 is entire. By Huber's Theorem applied to g_1 , there exists a path Γ from zero to infinity such that F approaches a finite asymptotic value a as $z \rightarrow \infty$ along Γ .

By the $\cos \pi\rho$ -theorem [3] there exists a sequence of circles $\{z: |z| = r_n\}$, $r_n \rightarrow \infty$ such that

$$\left| \frac{F'}{F}(r_n e^{i\theta}) \right| = \left| \frac{P}{g}(r_n e^{i\theta}) \right| \rightarrow O(r_n^{-1}) \quad (6.1)$$

uniformly in θ as $n \rightarrow \infty$. By (6.1) and the fact that $F(z) \rightarrow a$ as $z \rightarrow \infty$, $z \in \Gamma$, we have that

$$F(r_n e^{i\theta}) \rightarrow a \quad (6.2)$$

uniformly in θ as $r_n \rightarrow \infty$. Hence F can have no other asymptotic value. Clearly then, we can find a Jordan neighbourhood U of a such that F^{-1} has no singularities in $U \setminus \{a\}$. Thus F has a logarithmic singularity over a . By (6.2) we have that $\{z: |z| = r_n\}$ is mapped into U for large n . But Proposition 3.1 states that $F^{-1}(\partial U)$ is a curve tending to infinity in both directions, so we have a contradiction, which proves the theorem.

7. Proof of Theorem 2.10

The conditions of the theorem imply that the function u given by (1.2) is subharmonic in \mathbb{C} and has at most order $\frac{1}{2}$, minimal type. It follows from the $\cos \pi\rho$ -theorem [3] that there exists a sequence $r_k \rightarrow \infty$ such that

$$m(r_k) := \inf \{u(z) : |z| = r_k\} \rightarrow \infty.$$

Take an arbitrary natural number N and fix r_0 such that at least N of the points z_j are contained in the disk $|z| < r_0$. Let

$$M(r) := \max \{u(z) : |z| = r\}.$$

Fix k so large that

$$m(r_k) > M(r_0) \tag{7.1}$$

and consider the level set

$$L(t) = \{z : u(z) < t\}, \quad -\infty < t \leq M(r_0).$$

If $t = M(r_0)$ then there is a component L_0 of the set $L(t)$ such that

$$D(0, r_0) \subset L_0 \subset D(0, r_k).$$

Here $D(0, r)$ is the disk of radius r centred at 0. If t is very small, the intersection $L(t) \cap L_0$ consists of at least N simply connected components containing the points z_j . As t increases these components may collide only at critical points of u . So u has at least $N - 1$ critical points in $D(0, r_k)$ counting multiplicity. As N was chosen arbitrary large, the theorem is proved.

REMARK. Both Theorem 2.2 and Theorem 2.10 could have been proved a little more concisely (but with less of an elementary flavour) by using Huber’s theorem. Indeed, we could obtain an alternate and immediate proof of Lemma 4.1 (with entire replaced by meromorphic!) giving a finite asymptotic value for F in Theorem 2.2 which would lead to the same contradiction. In Theorem 2.10 Huber’s Theorem gives a path Γ on which the logarithmic potential u is bounded. Since u is subharmonic, a variant of the Denjoy–Carleman–Ahlfors Theorem shows that u has order at least $\frac{1}{2}$, normal type, contradicting (2.4).

8. Subharmonic functions in space. Proof of Theorem 2.11

Under the conditions of this theorem u is a non-negative superharmonic function. We shall essentially repeat the arguments of Section 7 based on (7.1). The only thing we need is a replacement of the $\cos \pi\rho$ -theorem. We shall prove that the conditions of Theorem 2.11 imply that

$$B(r) := \max \{u(x) : |x| = r\} \longrightarrow 0 \tag{8.1}$$

on a sequence of r tending to ∞ .

REMARK. It is known [6] that positive superharmonic functions tend to zero outside a small exceptional set. But this does not imply (8.1). Indeed (8.1) may fail for a general positive superharmonic function, because the function may be equal to ∞ on a ray. As we shall see the crucial condition is $a_k \geq 1$.

We have

$$B(r) \leq \sum_{k=1}^{\infty} \frac{a_k}{\|x\| - \|x_k\|^{m-2}}$$

So all we need is the following lemma.

LEMMA 8.1. *Suppose that $0 < x_1 < x_2 < \dots$, $m \geq 3$ is an integer, $a_k \geq 1$ for all k and*

$$\sum_{k=1}^{\infty} a_k x_k^{2-m} < \infty. \tag{8.2}$$

Set

$$\Psi(x) = \sum_{k=1}^{\infty} a_k |x - x_k|^{2-m}.$$

Then

$$\liminf_{x \rightarrow +\infty} \Psi(x) = 0.$$

Proof. We shall take $m = 3$ for simplicity; the case $m > 3$ can be treated similarly. Set

$$n(r) = \sum_{k: x_k \leq r} a_k$$

Then

$$\sum_{k: x_k \leq r} \frac{a_k}{x_k} = \int_0^r \frac{dn(t)}{t} = \frac{n(r)}{r} + \int_0^r \frac{n(t) dt}{t^2}.$$

It follows from (8.2) that

$$\int_0^{\infty} \frac{n(t)}{t^2} dt < \infty.$$

So there exists a sequence $r_j \rightarrow \infty$ such that

$$n(r_j) = o(r_j / \log r_j), \quad j \longrightarrow \infty. \tag{8.3}$$

Set

$$\phi(x) = \min\{1, 1/|x|\}$$

and

$$\Phi(x) = \sum_{k=1}^{\infty} a_k \phi(x - x_k).$$

Suppose that the lemma is not true. Then

$$\liminf_{x \rightarrow +\infty} \Psi(x) \geq c \tag{8.4}$$

for some c , $0 < c < 1$. We deduce from (8.4) that

$$\liminf_{x \rightarrow +\infty} \Phi(x) \geq c. \tag{8.5}$$

Indeed, if $\text{dist}(x, \{x_k : k \in \mathbb{N}\}) \leq 1$ then $\Phi(x) \geq 1 > c$, and if $\text{dist}(x, \{x_k : k \in \mathbb{N}\}) > 1$ then $\Phi(x) = \Psi(x)$ so in either case we have $\Phi(x) \geq \min\{1, \Psi(x)\}$.

It follows from (8.5) that

$$\int_0^r \Phi(x) dx \geq \frac{1}{2} cr \tag{8.6}$$

for all sufficiently large r .

To obtain the desired contradiction we estimate this integral from above

$$\begin{aligned} \int_0^r \Phi(x) dx &= \sum_{k=1}^{\infty} a_k \int_0^r \phi(x - x_k) dx = \sum_{k=1}^{\infty} a_k \int_{-x_k}^{r-x_k} \phi(t) dt \\ &= \sum_{x_k \leq 2r} a_k \int_{-x_k}^{r-x_k} \phi(t) dt + \sum_{x_k > 2r} a_k \int_{-x_k}^{r-x_k} \phi(t) dt \\ &= \Sigma_1 + \Sigma_2. \end{aligned} \tag{8.7}$$

For Σ_1 and Σ_2 we have

$$\begin{aligned}\Sigma_1 &\leq \sum_{x_k \leq 2r} a_k (\log x_k + \log^+(r - x_k) + 2) \leq 2n(2r) (\log 2r + 2), \\ \Sigma_2 &\leq \sum_{x_k > 2r} a_k (\log x_k - \log(x_k - r)) = \sum_{x_k \geq 2r} a_k \log \left(1 + \frac{r}{x_k - r} \right) \\ &\leq \sum_{x_k \geq 2r} \frac{2ra_k}{x_k} = o(r), \quad r \longrightarrow \infty.\end{aligned}\tag{8.8}$$

Putting this together and substituting $2r = r_j$ where r_j satisfies (8.3), we get

$$\int_0^r \Phi(x) dx \leq 2n(r_j) (\log r_j + 2) + o(r_j) = o(r_j) = o(r),$$

which contradicts (8.6).

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