Generalization of a theorem of Clunie and Hayman

Matthew Barrett and Alexandre Eremenko*

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Abstract

Clunie and Hayman proved that if the spherical derivative $\|f'\|$ of an entire function satisfies $\|f'\|(z) = O(|z|^\sigma)$ then $T(r, f) = O(r^{\sigma+1})$. We generalize this to holomorphic curves in projective space of dimension n omitting n hyperplanes in general position.

MSC 32Q99, 30D15.

Introduction

We consider holomorphic curves $f : \mathbf{C} \to \mathbf{P}^n$; for the general background on the subject we refer to [7]. The Fubini–Study derivative $\|f'\|$ measures the length distortion from the Euclidean metric in $\mathbf{C}$ to the Fubini–Study metric in $\mathbf{P}^n$. The explicit expression is

$$\|f'\|^2 = \|f\|^{-4} \sum_{i<j} |f_i' f_j - f_i f_j'|^2,$$

where $(f_0, \ldots, f_n)$ is a homogeneous representation of f (that is the f_j are entire functions which never simultaneously vanish), and

$$\|f\|^2 = \sum_{j=0}^{n} |f_j|^2.$$

See [3] for a general discussion of the Fubini-Study derivative.

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We recall that the Nevanlinna–Cartan characteristic is defined by

$$T(r,f)=\int_0^r \frac{dt}{t}\left(\frac{1}{\pi}\int_{|z|\leq t}\|f'\|^2(z)dm(z)\right),$$

where dm is the area element in $\mathbf{C}$. So the condition

$$\limsup_{z\to\infty}|z|^{-\sigma}\|f'(z)\|\leq K<\infty \tag{1}$$

implies

$$\limsup_{r\to\infty}\frac{T(r,f)}{r^{2\sigma+2}}<\infty. \tag{2}$$

Clunie and Hayman [4] found that for curves $\mathbf{C}\to\mathbf{P}^1$ omitting one point in $\mathbf{P}^1$, a stronger conclusion follows from (1), namely

$$\limsup_{r\to\infty}\frac{T(r,f)}{r^{\sigma+1}}\leq KC(\sigma). \tag{3}$$

In the most important case $\sigma=0$, a different proof of this fact for $n=1$ is due to Pommerenke [8]. Pommerenke's method gives the exact constant $C(0)$. In this paper we prove that this phenomenon persists in all dimensions.

Theorem. *For holomorphic curves $f:\mathbf{C}\to\mathbf{P}^n$ omitting n hyperplanes in general position, condition* (1) *implies* (3) *with an explicit constant $C(n,\sigma)$.*

In [6], the case $\sigma=0$ was considered. There it was proved that holomorphic curves in $\mathbf{P}^n$ with bounded spherical derivative and omitting n hyperplanes in general position must satisfy $T(r,f)=O(r)$. With a stronger assumption that f omits $n+1$ hyperplanes this was earlier established by Berteloot and Duval [2] and by Tsukamoto [9]. The proof in [6] has two drawbacks: it does not extend to arbitrary $\sigma\geq 0$, and it is non-constructive; unlike Clunie–Hayman and Pommerenke's proofs mentioned above, it does not give an explicit constant in (3).

It is shown in [6] that the condition that n hyperplanes are omitted is exact: there are curves in any dimension n satisfying (1), $T(r,f)\sim cr^{2\sigma+2}$ and omitting $n-1$ hyperplanes.

Preliminaries

Without loss of generality we assume that the omitted hyperplanes are given in the homogeneous coordinates by the equations $\{w_j=0\},\ 1\leq j\leq n$.

We fix a homogeneous representation $(f_0, \ldots, f_n)$ of our curve, where f_j are entire functions, and $f_n = 1$. Then

$$u = \log \sqrt{|f_0|^2 + \ldots + |f_n|^2} \tag{4}$$

is a positive subharmonic function, and Jensen's formula gives

$$T(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta}) d\theta - u(0) = \int_0^r \frac{n(t)}{t} dt,$$

where $n(t) = \mu(\{z : |z| \leq t\})$, and $\mu = \mu_u$ is the Riesz measure of u, that is the measure with the density

$$\frac{1}{2\pi} \Delta u = \frac{1}{\pi} \|f'\|^2. \tag{5}$$

This measure μ is also called Cartan's measure of f. Positivity of u and (2) imply that all f_j are of order at most $2\sigma + 2$, normal type. As $f_j(z) \neq 0,\ 1 \leq j \leq n$ we conclude that

$$f_j = e^{P_j}, \quad 1 \leq j \leq n,$$

where

$$P_j \quad \text{are polynomials of degree at most} \quad 2\sigma + 2. \tag{6}$$

We need two lemmas from potential theory.

Lemma 1. [6] *Let v be a non-negative harmonic function in the closure of the disc $B(a, R)$, and assume that $v(z_1) = 0$ for some point $z_1 \in \partial B(a, R)$. Then*

$$v(a) \leq 2R |\nabla v(z_1)|.$$

We include a proof, suggested by the referee, which is simpler than that given in [6]. Without loss of generality, assume that $a = 0, R = 1, z_1 = 1$. Then Harnack's inequality gives

$$\frac{v(0)}{1 + r} \leq \frac{v(r)}{1 - r} = \frac{v(r) - v(1)}{1 - r}.$$

Passing to the limit as $r \to 1$ we obtain the result.

Lemma 2. *Let v be a non-negative superharmonic function in the closure of the disc $B(a,R)$, and suppose that $v(z_1)=0$ for some $z_1 \in \partial B(a,R)$. Then*

$$|\mu_v(B(a,R/2))| \leq 3R \left| \frac{\partial v}{\partial n}(z_1) \right|.$$

By $|\partial v/\partial n|$ we mean here $\liminf |v(rz_1)|/(R(1-r))$ as $r \to 1-$.

Proof. Function $v(a+Rz)$ satisfies the conditions of the lemma with $R=1$. So it is enough to prove the lemma with $a=0$ and $R=1$. Let

$$w(z) = \int_{|\zeta| \leq 1/2} G(z,\zeta) d\mu_v(\zeta)$$

be the Green potential of the restriction of μ_v onto the disc $|\zeta| \leq 1/2$ that is

$$G(z,\zeta) = \log \left| \frac{1-\overline{\zeta} z}{z-\zeta} \right|.$$

Then $w \leq v$ and $w(z_1) = v(z_1) = 0$ which implies that

$$\left| \frac{\partial v}{\partial n}(z_1) \right| \geq \left| \frac{\partial w}{\partial |z|}(z_1) \right|.$$

Minimizing $|\partial G/\partial |z||$ over $|z|=1$ and $|\zeta|=1/2$ we obtain $1/3$ which proves the lemma.

Proof of the theorem

We may assume without loss of generality that f_0 has infinitely many zeros. Indeed, we can compose f with an automorphism of $\mathbf{P}^n$, for example replace f_0 by $f_0 + cf_1$, $c \in \mathbf{C}$ and leave all other f_j unchanged. This transformation changes neither the n omitted hyperplanes nor the rate of growth of $T(r,f)$ and multiplies the spherical derivative by a bounded factor.

Put $u_j = \log |f_j|$, and

$$u^* = \max_{1 \leq j \leq n} u_j.$$

Here and in what follows max denotes the pointwise maximum of subharmonic functions.

Proposition 1. *Suppose that at some point z_1 we have*

$$u_m(z_1) = u_k(z_1) \geq u_j(z_1)$$

for some $m \neq k$ and all j; $m, k, j \in \{0, \dots, n\}$. Then

$$\|f'(z_1)\| \geq (n+1)^{-1} |\nabla u_m(z_1) - \nabla u_k(z_1)|.$$

Proof.

$$\|f'(z_1)\| \geq \frac{|f'_m(z_1) f_k(z_1) - f_m(z_1) f'_k(z_1)|}{|f_0(z_1)|^2 + \ldots + |f_n(z_1)|^2} \geq (n+1)^{-1} \left| \frac{f'_m(z_1)}{f_m(z_1)} - \frac{f'_k(z_1)}{f_k(z_1)} \right|,$$

and the conclusion of the proposition follows since $|\nabla \log |f|| = |f'/f|$.

Proposition 2. *For every $\epsilon > 0$, we have*

$$u(z) \leq u^*(z) + K(2+\epsilon)^{\sigma+1}(n+1)|z|^{\sigma+1}$$

for all $|z| > r_0(\epsilon)$.

Proof. If $u_0(z) \leq u^*(z)$ for all sufficiently large $|z|$, then there is nothing to prove. Suppose that $u_0(a) > u^*(a)$, and consider the largest disc $B(a, R)$ centered at a where the inequality $u_0(z) > u^*(z)$ persists. If z_0 is the zero of the smallest modulus of f_0 then $R \leq |a| + |z_0| < (1+\epsilon)|a|$ when $|a|$ is large enough.

There is a point $z_1 \in \partial B(a, R)$ such that $u_0(z_1) = u^*(z_1)$. This means that there is some $k \in \{1, \dots, n\}$ such that $u_0(z_1) = u_k(z_1) \geq u_m(z_1)$ for all $m \in \{1, \dots, n\}$. Applying Proposition 1 we obtain

$$|\nabla u_k(z_1) - \nabla u_0(z_1)| \leq (n+1)\|f'(z_1)\|.$$

Now $u_0(z) > u^*(z) \geq u_k(z)$ for $z \in B(a, R)$, so we can apply Lemma 1 to $v = u_0 - u_k$ in the disc $B(a, R)$. This gives

$$u_0(a) - u_k(a) \leq 2R|\nabla u_k(z_1) - \nabla u_0(z_1)| \leq 2R(n+1)\|f'(z_1)\|.$$

Now $R < (1+\epsilon)|a|$ and $|z_1| \leq (2+\epsilon)|a|$, so

$$u_0(a) \leq u^*(a) + K(2+\epsilon)^{\sigma+1}(n+1)|a|^{\sigma+1},$$

and the result follows because $u = \max\{u_0, u^*\} + O(1)$.

Next we study the Riesz measure of the subharmonic function

$$u^* = \max\{u_1, \ldots, u_n\}.$$

We begin with maximum of two harmonic functions. Let u_1 and u_2 be two harmonic functions in $\mathbf{C}$ of the form $u_j = \operatorname{Re} P_j$ where $P_j \neq 0$ are polynomials. Suppose that $u_1 \neq u_2$. Then the set $E = \{z \in \mathbf{C} : u_1(z) = u_2(z)\}$ is a proper real-algebraic subset of $\overline{\mathbf{C}}$ without isolated points. Apart from a finite set of ramification points, E consists of smooth curves. For every smooth point $z \in E$, we denote by $J(z)$ the jump of the normal (to E) derivative of the function $w = \max\{u_1, u_2\}$ at the point z. This jump is always positive and the Riesz measure μ_w is given by the formula

$$d\mu_w = \frac{J(z)}{2\pi}|dz|, \tag{7}$$

which means that μ_w is supported by E and has a density $J(z)/2\pi$ with respect to the length element $|dz|$ on E.

Now let $E_{i,j} = \{z : u_i(z) = u_j(z) \geq u_k(z), 1 \leq k \leq n\}$, and $E = \cup E_{i,j}$ where the union is taken over all pairs $1 \leq i, j \leq n$ for which $u_i \neq u_j$. Then E is a proper real semi-algebraic subset of $\overline{\mathbf{C}}$, and ∞ is not an isolated point of E. For the elementary properties of semi-algebraic sets that we use here see, for example, [1, 5]. There exists $r_0 > 0$ such that $\Gamma = E \cap \{r_0 < |z| < \infty\}$ is a union of finitely many disjoint smooth simple curves,

$$\Gamma = \cup_{k=1}^m \Gamma_k.$$

This union coincides with the support of μ_{u^*} in $\{z : r_0 < |z| < \infty\}$.

Consider a point $z_0 \in \Gamma$. Then $z_0 \in \Gamma_k$ for some k. As Γ_k is a smooth curve, there is a neighborhood D of z_0 which does not contain other curves $\Gamma_j,\ j \neq k$ and which is divided by Γ_k into two parts, D_1 and D_2. Then there exist i and j such that $u^*(z) = u_i(z),\ z \in D_1$ and $u^*(z) = u_j(z),\ z \in D_2$, and $u^*(z) = \max\{u_i(z), u_j(z)\},\ z \in D$. So the restriction of the Riesz measure μ_{u^*} on D is supported by $\Gamma_k \cap D$ and has density $J(z)/(2\pi)$ where

$$|J(z)| = |\partial u_i/\partial n - \partial u_j/\partial n|(z) = |\nabla(u_i - u_j)|(z),$$

and $\partial/\partial n$ is the derivation in the direction of a normal to Γ_k. Taking into account that $u_j = \operatorname{Re} P_j$ where P_j are polynomials, we conclude that there exist positive numbers c_k and b_k such that

$$J(z)/(2\pi) = (c_k + o(1))|z|^{b_k}, \quad z \to \infty, \quad z \in \Gamma_k. \tag{8}$$

Let $b = \max_k b_k$, and among those curves Γ_k for which $b_k = b$ choose one with maximal c_k (which we denote by c_0). We denote this chosen curve by Γ_0 and fix it for the rest of the proof.

Proposition 3. *We have*

$$b \leq \sigma \quad \textit{and} \quad c_0 \leq 3 \cdot 4^\sigma K(n+1).$$

Proof. We consider two cases.

Case 1. There is a sequence $z_n \to \infty,\ z_n \in \Gamma_0$ such that $u_0(z_n) \leq u^*(z_n)$. Then (1) and Proposition 1 imply that

$$J(z_n) \leq (n+1)K|z_n|^\sigma,$$

and comparison with (8) shows that $b \leq \sigma$ and $c_0 \leq K(n+1)/(2\pi)$.

Case 2. $u_0(z) > u^*(z)$ for all sufficiently large $z \in \Gamma_0$. Let a be a point on Γ_0, $|a| > 3r_0$, and $u_0(a) > u^*(a)$. Let $B(a, R)$ be the largest open disc centered at a in which the inequality $u_0(z) > u^*(z)$ holds. Then

$$R \leq |a| + O(1), \quad a \to \infty \tag{9}$$

because we assume that f_0 has zeros, so $u_0(z_0) = -\infty$ for some z_0.

In $B(a, R)$ we consider the positive superharmonic function $v = u_0 - u^*$. Let us check that it satisfies the conditions of Lemma 2. The existence of a point $z_1 \in \partial B(a, R)$ with $v(z_1) = 0$ follows from the definition of $B(a, R)$. The Riesz measure of μ_v is estimated using (7), (8):

$$|\mu_v(B(a, R/2))| \geq |\mu_v(\Gamma_0 \cap B(a, R/2))| \geq c_0 R(|a| - R/2)^b.$$

Now Lemma 2 applied to v in $B(a, R)$ implies that

$$|\nabla v(z_1)| \geq (c_0/3)(|a| - R/2)^b. \tag{10}$$

On the other hand (1) and Proposition 1 imply that

$$|\nabla v(z_1)| \leq K(n+1)(|a| + R)^\sigma$$

Combining these two inequalities and taking (9) into account, we obtain $b \leq \sigma$ and $c_0 \leq 3 \cdot 4^\sigma K(n+1)$, as required.

We denote
$$T^*(r)=\frac{1}{2\pi}\int_{-\pi}^{\pi}u^*(re^{i\theta})d\theta-u^*(0);$$
This is the characteristic of the "reduced curve" $(f_1,\ldots,f_n)$.

Proposition 4.
$$T^*(r)\leq 6\cdot 4^\sigma K\frac{n(n+1)^2}{\sigma+1}r^{\sigma+1}.$$

Proof. By Jensen's formula,
$$T^*(r)=\int_0^r\nu(t)\frac{dt}{t},$$
where $\nu(t)=\mu_{u^*}(\{z:|z|\leq t\})$. The number of curves Γ_k supporting the Riesz measure of u^* is easily seen to be at most $2n(n-1)(\sigma+1)$. The density of the Riesz measure μ_{u^*} on each curve Γ_k is given by (8), where $c_k\leq c_0$ and $b_k\leq b$, and the parameters c_0 and b are estimated in Proposition 3. Combining all these data we obtain the result.

It remains to combine Propositions 2 and 4 to obtain the final result.

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Purdue University, West Lafayette IN 47907 USA