CO-AXIAL MONODROMY

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We consider a surface S homeomorphic to the sphere and equipped with a Riemannian metric of constant curvature 1 with finitely many conic singularities with angles $\alpha_1, \dots, \alpha_n$.

We measure angles in turns: 1 turn = 2π radians.

The question is: What angles are possible?

Necessary conditions:

$$\sum_{j=1}^{n} (\alpha_j - 1) + 2 > 0 \quad \text{(Gauss–Bonnet)},$$

$$d_1(\mathbf{Z}_o^n, \alpha - \mathbf{1}) \geq 1 \quad \text{(Closure condition)}.$$

Here $\alpha = (\alpha_1, \dots, \alpha_n)$, $\mathbf{Z}_o^n$ is the set of integer lattice points with *odd* sums of coordinates, and d_1 is the ℓ_1 distance.

The standard metric (of area 1) on the sphere is

$$\rho_0(z)|dz| = \frac{|dz|}{\sqrt{\pi}(1+|z|^2)}.$$

Then our metric $\rho(z)|dz|$ has density $\exp(v/2)$ with respect to ρ_0, where

$$\Delta_{\rho_0} v + 2e^v - 8\pi = 4\pi \sum_{j=1}^{n} (\alpha_j - 1)\delta_{a_j}.$$

Our problem is to find out for which α_j this equation is solvable, *with some* a_j.

Necessity of the Closure condition is due to Mondello and Panov (2016). They also proved that the Gauss–Bonnet and the Closure condition with *strict inequality* are sufficient.

Developing map is a multi-valued function

$$S\backslash\{\text{singularities}\} \to \overline{\mathbf{C}},$$

where $\overline{\mathbf{C}}$ is the sphere equipped with the standard spherical metric (of curvature 1), and f is a local isometry away from the singularities. So f is analytic with respect to the conformal structure on S induced by the metric, and the monodromy group of f consists of rotations of $\overline{\mathbf{C}}$.

The metric is recovered from the developing map by the formula

$$\rho(z)|dz| = \frac{2|f'|}{1+|f|^2}|dz|.$$

The monodromy is called *co-axial* if it is a subgroup of $SO(2)$.

Mondello and Panov proved that if the Closure condition holds with equality, then the monodromy must be co-axial.

Thus it remains to obtain a necessary and sufficient condition on the angles for metrics with co-axial monodromy.

α is called *admissible* if a co-axial metric with such angles exists. It does not have to be unique.

Theorem 1. *Suppose wlog that $\alpha_1, \ldots \alpha_m$ are not integers, while $\alpha_{m+1}, \ldots, \alpha_n$ are integers. For α to be admissible it is necessary that there exist $\epsilon_j \in \{\pm 1\}$ and integer $k' \geq 0$ such that:*

$$\sum_{j=1}^{m} \epsilon_j \alpha_j = k', \quad \text{and the number}$$

$$k'' := \sum_{j=m+1}^{n} \alpha_j - n - k' + 2 \quad \text{is non-negative and even.}$$

If the coordinates of the vector $\mathbf{c} := (\alpha_1, \ldots, \alpha_m, \underbrace{1, \ldots, 1}_{k'+k'' \text{ times}})$ *are incommensurable then these two conditions are also sufficient.*

If the coordinates of the vector $\mathbf{c}$ are commensurable, then $\mathbf{c} = \eta \mathbf{b}$ where coordinates of $\mathbf{b}$ are integers whose g.c.d. is 1.

In this case there is an additional necessary condition

$$2 \max_{m+1 \leq j \leq n} \alpha_j \leq \sum_{j=1}^{q} |b_j|, \quad q = m + k' + k'',$$

and all these three conditions together are sufficient.

Corollary. *When $n > 2$, a coaxial metric must have some integer angles whose sum is at least $n + k' - 2$ nd has the same parity as $n + k'$, where k' is an alternating sum of non-integer angles.*

This theorem generalizes the previous results: for $m = 0$ (easy and well-known),

for $n = 2$ (Troyanov, 1989; in this simple case the necessary and sufficient condition is $\alpha_1 = \alpha_2$),

for $n = 3$ (Eremenko, 2004),

for $m = 2$ (Eremenko, Gabrielov, Tarasov, 2014),

and for $m = n$ (S. Dey, 2017),

and completes the description of possible angles.

As the monodromy is co-axial, we have $df/f = Rdz$, where R is a rational function. Assuming that ∞ is not singular we obtain that $R(\infty)$ is a zero of order at least 2.

The singularities are finite zeros and poles, whose residues are not ± 1. Poles with residues ± 1 are not singularities.

For an admissible set of angles α there can be several metrics with these angles. It is not difficult to verify that if α is admissible (so that some metric with these angles exists) then there is also a metric with these angles and with the additional property that *all singularities with integer angles are finite zeros of R.* So that there are no poles of R with integer residues except ± 1.

Thus we can always assume that R is of the form of the form

$$R(z) = \sum_{j=1}^{m} \frac{\epsilon_j \alpha_j}{z - a_j} - \sum_{j=1}^{k'} \frac{1}{z - b_j} + \sum_{j=k'+1}^{k'+k''} \frac{(-1)^j}{z - b_j},$$

the condition that k'' is even comes from the residue theorem. This formula implies that all singularities with integer angles are zeros of R. Notice that we can introduce any number of poles with residues ± 1; they are not singularities of the metric.

Zeros of R are singularities with integer angles: their multiplicities are $\alpha_j - 1$. Since all residues in this formula are determined by the angles, the question is:

Does there exist such a function R with prescribed residues and prescribed **multiplicities of zeros**.

Now restate the problem: *For a given a vector* $(c_1, \ldots, c_q)$ *with* $\sum_j c_j = 0$ *and a given partition of* $q-2 = \sum_{j=1}^{s} \ell_j$, *does there exist a function*

$$R(z) = \sum_{j=1}^{q} \frac{c_j}{z - z_j}$$

with zeros of multiplicities ℓ_j *?*

Theorem 2. *If the* c_j *are incommensurable, such a function* R *exists.*
If $c_j = \eta_j b_j$ *with mutually prime integers* b_j, *then the necessary and sufficient condition for existence of* R *is*

$$2\left(1 + \max_{1 \le j \le s} \ell_j\right) \le \sum_{j=1}^{q} |b_j|.$$

We have to consider the commensurable case first.

Commensurable case. Hurwitz problem. $R = \eta g$,

$$g(z) = \sum_{j=1}^{q} \frac{b_k}{z - a_k}, \quad b_j \quad \text{are mutually prime integers.}$$

Then $g = h'/h$, h is rational, and we are looking for a rational function with prescribed multiplicities of zeros, poles and critical points other than zeros and poles. We have $\deg h = (1/2) \sum_j |b_j|$ and the necessary condition $\ell_j + 1 \leq \deg h$ is evident. Song and Yu (2016) proved that this is also sufficient.

This is a special case of the Hurwitz problem: *when there exist a rational function with given number of critical values and prescribed multiplicities of their preimages*. There is no simple general criterion, but the special case that we need is known.

General case. Consider the real projective space $\mathbf{RP}^{q-2}$ consisting of q-tuples $\mathbf{c} = (c_1, \ldots, c_q)$ with zero sum, modulo proportionality. Let Z be the union of the coordinate hyperplanes $c_j = 0$. Let P be a partition of $q-2$. We say that a point $c \in \mathbf{RP}^{q-2}$ is *P-admissible* if there exists $g(z)$ with residues $\mathbf{c}$ and multiplicities of zeros P. Otherwise $\mathbf{c}$ is *P-exceptional*. A point $\mathbf{c}$ is called *rational* if its coordinates are commensurable.

Proposition. *For every q and P, the set of rational P-exceptional points in $\mathbf{RP}^{q-2}$ is finite.*

Indeed, they satisfy $\sum_{j=1}^{q} |b_j| \leq 2(\max \ell_j + 1)$, and b_j are integers.

Now we try to construct a rational function $R = f'/f$ with prescribed residues and multiplicities of zeros. Suppose that such a function exists. Consider a *flat* metric ρ on $S^* = S \backslash \{\text{singularities}\}$ with developing map $\log f$. The metric space (S^*, ρ) breaks into flat cylinders by the critical level lines of $u = \log |f|$. Semi-infinite cylinders are neighborhoods of the punctures, and the cylinder surrounding a puncture z_j has "waist" $2\pi c_j$. There are also cylinders of finite length, and all cylinders are pasted together along their boundary arcs.

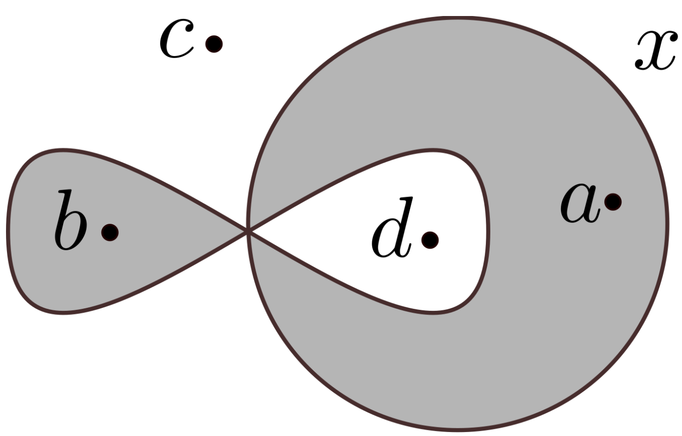
Conversely, if we have such a flat surface, homeomorphic to a punctured sphere, its developing map will be of the form $\log f$, where f is a rational function. The pull-back of the spherical metric under this f will define the metric with conic singularities which we are trying to construct.

To construct such a flat surface, one chooses a scheme of the boundary identifications of cylinders, and prescribes waists to all cylinders, and the lengths of the boundary arcs which are to be identified. The cylinders are pasted together respecting the length on these boundary arcs. Once such a flat surface is constructed, f is recovered by the uniformization theorem.

Example. $q = 4$. The residues are $a, b, -c, -d$ and we want a single critical point of multiplicity 3. The pattern in the figure consists of 4 infinite cylinders whose waists are known. One only need to determine the length of x. We have to find a positive solution to

$$a = x + d, \quad c = x + b.$$

Such an x exists iff $a - c + b - d = 0$ and $x > 0$ if $a > d$ and $c > b$.



This means that there exists a function

$$R(z) = \frac{a}{z - z_1} + \frac{b}{z - z_2} - \frac{c}{z - z_3} - \frac{d}{z - z_4}$$

having a single double zero in the plane, and corresponding to the picture above, if and only if $a + b - c - d = 0$ and $a > d, c > b$.

"Corresponding to the picture above" means that R is the conjugate gradient of a potential u whose level lines have topology shown in the picture. The picture in the case is actually unique up to exchange of letters $a \leftrightarrow b$ and/or $c \leftrightarrow d$.

The possibility of the construction that we outlined depends on the ability to choose the waists of all cylinders and the lengths of the arcs to be pasted together. The waists of the semi-infinite cylinders are prescribed (they are the prescribed residues $\mathbf{c}$.

This leads to a set of equations and inequalities describing the P-*exceptional* vectors $\mathbf{c}$.

These equations and inequalities are of the form

$$A_j(c_1, \ldots, c_q) = 0, \quad B_j(c_1, \ldots, c_q) > 0$$

with some *linear* functions A_j, B_j with *integer coefficients*. We conclude that the set of P-exceptional points $\mathbf{c}$ is a *rational polyhedron* in $\mathbf{RP}^{q-2}$. But we know from the consideration of the commensurable case that this rational polyhedron contains only finitely many rational points.

A rational polyhedron containing finitely many rational points must be finite and must consist of only rational points!

This completes the proof in the general case.

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