Meromorphic traveling wave solutions of the Kuramoto–Sivashinsky equation

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Abstract

We determine all cases when there exists a meromorphic solution of the ODE

$$\nu w''' + bw'' + \mu w' + w^2/2 + A = 0.$$

This equation describes traveling waves solutions of the Kuramoto-Sivashinsky equation. It turns out that there are no other meromorphic solutions besides those explicit solutions found by Kuramoto and Kudryashov. The general method used in this paper, based on Nevanlinna theory, is applicable to finding all meromorphic solutions of a wide class of non-linear ODE.

Keywords: Kuramoto and Sivashinsky equation, meromorphic functions, elliptic functions, Nevanlinna theory.

The Kuramoto–Sivashinsky equation

$$\phi_t + \nu\phi_{xxxx} + b\phi_{xxx} + \mu\phi_{xx} + \phi\phi_x = 0, \quad \nu, b, \mu \in \mathbf{R}, \quad \nu \neq 0$$

arises in several problems of physics and chemistry [13], and it was intensively studied in the recent years [2, 10, 13, 11, 12, 17, 19]. Solutions of the form of a traveling wave

$$\phi(x,t) = c + w(z), \quad z = x - ct,$$

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satisfy the ordinary differential equation

$$\nu w''' + bw'' + \mu w' + w^2/2 + A = 0, \quad \nu \neq 0, \tag{1}$$

which is the object of our study here. We allow complex values for parameters ν, μ, b and A in the equation (1).

It is known [17, 19] that the general solution of (1) has movable logarithmic branch points, which indicates chaotic behavior. However, for some values of parameters ν, b, μ and A, physically meaningful one-parametric families of meromorphic solutions were found in [13, 11, 12]. Here and in what follows, "meromorphic function" means a function meromorphic in the complex plane $\mathbf{C}$. In [19] the possibility of existence of other meromorphic solutions, except those found in [11, 12, 13] is discussed. All known meromorphic solutions of the equation (1) are elliptic functions or their degenerations. More precisely, let us say that a meromorphic function f belongs to the class W if f is a rational function of z, or a rational function of $\exp(az)$, $a \in \mathbf{C}$, or an elliptic function. The letter W is chosen for Weierstrass who proved that only these functions can satisfy an algebraic addition theorem.

In this paper we will show that for any choice of parameters, such that $\nu \neq 0$, all meromorphic solutions of the equation (1) belong to the class W. Moreover, there are no meromorphic solutions except those found in [13, 11, 12].

The crucial fact about (1) used here is the following

Uniqueness Property: *there is exactly one formal meromorphic Laurent series with a pole at zero that satisfies the equation.*

To check this we substitute the series

$$w(z) = \sum_{k=m}^{\infty} c_k z^k \quad \text{with} \quad m < 0,\ c_m \neq 0 \tag{2}$$

into the equation (1), and obtain $m = -3$, $c_{-3} = 120\nu \neq 0$, and the rest of the coefficients c_k are determined uniquely (see, for example, [2, 4]). The principal part of the expansion is

$$w(z) = 120\nu z^{-3} - 15bz^{-2} + \left(\frac{60\mu}{19} - \frac{15b^2}{76\nu}\right) z^{-1} + \dots. \tag{3}$$

Theorem 1. *All meromorphic solutions w of the equation* (1) *belong to the class W. If for some values of parameters such solution w exists, then all*

other meromorphic solutions form a one-parametric family $w(z-z_0),\ z_0 \in \mathbf{C}$. *Furthermore,*
(i) Elliptic solutions exist only if $b^2 = 16\mu\nu$. *They are of order* 3 *and have one triple pole per parallelogram of periods.*
(ii) All exponential solutions have the form $P(\tan kz)$, *where* P *is a polynomial of degree at most* 3 *and* $k \in \mathbf{C}$.
(iii) Non-constant rational solutions occur if and only if $b = \mu = A = 0$ *and they have the form* $w(z) = 120\nu(z - z_0)^{-3},\ z_0 \in \mathbf{C}$.

Statements (i)-(iii) permit to find all values of parameters when meromorphic solutions occur, as well as solutions themselves, explicitly. It turns out that there are no other elliptic solutions except those found by Kudryashov in [12] (see also [19]). This fact was recently independently established by Hone [10]. Similarly, it follows from (ii) that there are no other exponential solutions except those found by Kuramoto–Tsuzuki [13] and Kudryashov [11].

Our Theorem 1 does not exclude the existence of other "explicit" solutions, but it implies that all solutions except those lited in (i)-(iii) have more complicated singularities, other than poles, like branching points, or essential isolated singularities in $\mathbf{C}$, or non-isolated singularities.

We will see that the proof of Theorem 1 is of very general character, and applies to many other equations which have the uniqueness property of formal Laurent solutions stated above. In [4] the author proved a similar result about the generalized Briot–Bouquet equation $F(w^{(k)}, w) = 0$, where F is a polynomial in two variables and k is odd. If k is even, the equation does not have the uniqueness property, as stated above. However, the conjecture that all meromorphic solutions of all generalized Briot–Bouquet equations belong to the class W is plausible, and recently Tuen Wai Ng informed the author that he made a progress towards this conjecture.

It is desirable to search other interesting ODE's with this uniqueness property. The method proposed here will permit to find all their meromorphic solutions. We also mention that for any given algebraic ODE, the uniqueness property can be checked with an efficient algorithm explained in [1].

The proof of Theorem 1 can be based on any of the two standard tools of analytic theory of differential equations, Nevanlinna theory or Wiman–Valiron theory, see [18, Chap. V] and [7, Chap. VI]. We choose Nevanlinna theory here as a more general method. For convenience of a reader unfamiliar with this theory we include the appendix with definitions and statements of the results we use.

Proof of Theorem 1. We write equation (1) as

$$L(w) = w^2 - 2A, \quad \text{where} \quad L(w) = 2(\mu w''' + b w'' + \mu w'). \tag{4}$$

Let w be a meromorphic solution of (1). The symbols O and o in our formulas refer to asymptotics when $r \to \infty$, $r \notin E$, where $E \subset [0, \infty)$ is a set of finite measure.

We consider two cases.

Case 1. w has finitely many poles (possibly none). Then the Nevanlinna characteristic $t(r, L(w))$ can be estimated as follows:

$$\begin{aligned} T(r, L(w)) &= m(r, L(w)) + O(\log r) \\ &\le m(r, L(w)/w) + m(r, w) + O(\log r) \\ &\le (1 + o(1)) T(r, w) + O(\log r), \end{aligned}$$

where we used property (13) and the Lemma on the Logarithmic derivative (see the Appendix) to estimate $m(r, L(w)/w)$. On the other hand, $T(r, w^2 - 2A) = 2T(r, w) + O(1)$ (Appendix, (9), (10)). So (4) gives

$$T(r, w) = O(\log r),$$

thus w is a rational function.

If z_0 and z_1 are two poles of w in $\mathbf{C}$ then both $w(z + z_0)$ and $w(z + z_1)$ are solutions of (1) with a pole at zero, thus $w(z) \equiv w(z - z_1 + z_0)$ by the uniqueness property, and we conclude that w is periodic. This is a contradiction because the only periodic rational functions are constants, and they do not have poles.

If w has one pole in $\mathbf{C}$, then $w(z) = c(z - z_0)^{-3} + P(z)$, where P is a polynomial. Substituting this to our equation, we conclude that $P = 0$, $b = \mu = A = 0$ and $c = 120\nu$. This gives (iii).

Case 2. w has infinitely many poles. Arguing as above we conclude that for every pair of poles z_0 and z_1, the difference $z_0 - z_1$ is a period of w. So the set of all poles is of the form $z_0 + \Gamma$ where Γ is a non-trivial discrete subgroup of $(\mathbf{C}, +)$. Thus Γ is isomorphic to either $\mathbf{Z}$ or $\mathbf{Z} \times \mathbf{Z}$, and we consider each case separately.

If Γ is isomorphic to $\mathbf{Z} \times \mathbf{Z}$ then w is elliptic and there is exactly one pole per period. From (3) we conclude that all poles are of order 3. The residues

at these poles should be zero, so we obtain from (3) $b^2 = 16\mu\nu$. This proves (i).

Now we consider the remaining case when Γ is isomorphic to $\mathbf{Z}$. Then $\mathbf{C}/\Gamma = \mathbf{C}^* = \mathbf{C}\backslash 0$, and w is a simply periodic meromorphic function, so it factors as $R(\exp(az))$, where R is a meromorphic function in $\mathbf{C}^*$, having exactly one pole in $\mathbf{C}^*$. Our goal is to prove that R is rational,

Making the change of the independent variable $\zeta = \exp(az)$ in (1) we obtain

$$a^3\nu\zeta^3 R''' + (3a^3\nu + a^2 b)\zeta^2 R'' + (a^3\nu + a^2 b + a\mu)\zeta R' = R^2/2 - A. \qquad (5)$$

Now we argue exactly as in Case 1, denoting the left hand side of (5) by $L(R)$. As R has only one pole, the Lemma on the Logarithmic Derivative implies

$$T(r, L(R)) \leq (1 + o(1))T(r, R) + O(\log r),$$

but $T(r, R^2/2 - A) = 2T(r, R) + O(1)$, so, by (5), $T(r, R) = O(\log r)$, and thus R has no essential singularity at ∞. Applying the same argument to $R(1/\zeta)$, we conclude that R has no essential singularity at zero. So R is rational.

Now it is easy to see from (5) that R cannot have a pole at ∞ (if $R(\zeta) \sim c\zeta^d$), $d > 0$ then the right hand side has order ζ^{2d} while the left hand side has order at most ζ^d). Similar argument shows that R cannot have a pole at zero.

Thus R has only one pole in $\overline{\mathbf{C}}$, and this pole has to be of order 3 by (3). So we obtain statement (ii).

This completes the proof.

Conclusions and generalizations.

The method of this paper permits the following generalization. Consider an algebraic autonomous differential equation

$$\sum a_j w^{j_0}(w')^{j_1} \dots (w^{(k)})^{j_k} = 0, \qquad (6)$$

where $j = (j_0, \dots, j_m)$ is a multi-index, and a_j are constants. The number $j_0 + \dots + j_k$ is called the degree of a monomial. Uniqueness Property can be replaced by the following

Finiteness Property. *There are only finitely many formal Laurent series of the form* (2) *that satisfy the equation.*

For any given equation, Finiteness Property can be verified either by substituting to the equation a Laurent series with undetermined coefficients or by an algorithm in [2].

Theorem 2. *Suppose that* (6) *has the finiteness property, so that the equation is satisfied by finitely many Laurent series* ϕ_n, $1 \leq n \leq p$ *of the form* (2). *If in addition* (6) *has only one monomial of top degree, then all meromorphic solutions belong to the class* W. *Each solution is either*
a) an elliptic with at most p *poles per parallelogram of periods, or*
b) has the form $R(e^{az})$, *where* R *is a rational function with at most* p *poles in* $\mathbf{C}^*$, *or*
c) is a rational function R *with at most* p *poles in* $\mathbf{C}$.

Nevanlinna and Wiman–Valiron theories usually give only necessary conditions for existence of meromorphic solutions of non-linear ODE. However, sometimes these necessary conditions are so strong that they permit to find or classify all meromorphic solutions. For example, all meromorphic solutions of the differential equations $F(w', w, z) = 0$, where F is a polynomial and $w = w(z)$ were classified in [5, 6] in this way.

In combination with the Finiteness Property, Nevanlinna theory permits to make a strong conclusion that all meromorphic solutions belong to the class W, and moreover, to give a priori bounds for degrees of these meromorphic solutions, as in statements (i)-(iii) of our Theorem 1. Having established such bounds one can plug the solution with indetermined coefficients into the equation, and find all meromorphic solutions explicitly. Such computation can be hard, but in principle it can be always done in finitely many steps.

Other instances known to the author when such method was aplied successfully are the paper on Briot–Bouquet-type equations [4] mentioned above, and [3] were all meromorphic solutions of the equation

$$w''w - (w')^2 + aw'' + bw' + cw + d \tag{7}$$

were found. The method of [3] is a combination of the Finiteness Property and Wiman–Valiron theory. Solutions of (7) do not have poles, but for generic parameters the following version of the Finiteness Property holds: there are at most two holomorphic solutions w with $w(0) = 0$ in a neighborhood of 0.

Appendix.

Good general introductions to Nevanlinna theory can be found in [18], which contains a chapter on analytic theory of differential equations, and

[14, Ch.VI] The modern development is described in [8, 9]. Nevanlinna's own books are [15, 16].

Let $f \not\equiv 0$ be a meromorphic function in a punctured neighborhood of infinity $\{z : r_0 \le |z| < \infty\}$. Let $n(r, f)$ be the counting function of poles, that is $n(r, f)$ is the number of poles in the ring $r_0 \le |z| \le r$, counting multiplicity. We set for $r > r_0$

$$N(r, f) = \int_{r_0}^{r} \frac{n(t, f)}{t} dt, \tag{8}$$

and

$$m(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| \, d\theta,$$

where $x^+ = \max\{x, 0\}$. The Nevanlinna characteristic is defined by

$$T(r, f) = m(r, f) + N(r, f).$$

Using another number r_0 in the definition of $N(r, f)$ adds to the characteristic $O(\log r)$ as $r \to \infty$ and we will see that such summands are negligible when f has an essential singularity at infinity.

The characteristic $T(r, f)$ is a non-negative function, and

1. If the singularity of f at infinity is essential then $T(r, f)$ is increasing and $T(r, f)/\log r \to \infty$ as $r \to \infty$. If the infinite point is a removable or a pole, we have $T(r, f) = O(\log r)$.

2. The algebraic properties of $T(r, f)$ are similar to the properties of the degree of a rational function:

$$T(r, fg) \le T(r, f) + T(r, g), \tag{9}$$

$$T(r, f^n) = nT(r, f), \tag{10}$$

$$T(r, f + g) \le T(r, f) + T(r, g) + O(1), \tag{11}$$

$$T(r, 1/f) = T(r, f) + O(1). \tag{12}$$

Here we assume that the same r_0 was used in the definition of $T(r, f)$ and $T(r, g)$. Properties (9–11) are elementary and follow from the similar properties of $N(r, f)$ and $m(r, f)$, for example,

$$m(r, fg) \le m(r, f) + m(r, g). \tag{13}$$

Property (12) is the restatement of the Jensen formula, which is fundamental for the whole subject. These properties show that $T(r, f)$ can be considered as a generalization of the degree of a rational function to functions of "infinite degree", that is to meromorphic functions which have an essential singularity at infinity. For such functions, the "generalized degree" $T(r, f)$ is an increasing function rather than a number. If f is a rational function and $f(0) \neq \infty$ we can take $r_0 = 0$ in the definition of $N(r, f)$. Then it is easy to see that $T(r, f) = \deg f \log r + O(1)$.

For applications to differential equations, the most important property is

The Lemma on the Logarithmic Derivative

$$m(r, f'/f) = O(\log T(r, f) + \log r), \quad r \to \infty,\ r \notin E,$$

where E is some exceptional set of finite length. The term $\log r$ can be omitted if f has no essential singularity at infinity. The exceptional set E may indeed occur but it does not hurt in most applications. From now on all our asymptotic relations have to be understood with $r \to \infty,\ r \notin E$.

As the differentiation increases the orders of poles by a factor at most 2, we obtain $N(r, f') \leq 2N(r, f)$. Combined with the Lemma on the Logarithmic Derivative, and property (13) this gives

$$\begin{aligned} T(r, f') &= N(r, f') + m(r, f') \leq 2N(r, f) + m(r, ff'/f) \\ &\leq 2N(r, f) + m(r, f) + m(r, f'/f) \leq (2 + o(1))T(r, f). \end{aligned}$$

Thus $T(r, f^{(n)}) = O(T(r, f))$. If f has no poles, we obtain

$$T(r, f') = m(r, f') = m(r, ff'/f) \leq m(r, f) + m(r, f'/f) \leq (1 + o(1))T(r, f),$$

and, by induction,

$$T(r, f^n) \leq (1 + o(1))T(r, f).$$

Finitely many poles contribute $O(\log r)$ to $N(r, f)$, so for functions with finitely many poles we have

$$T(r, f^n) \leq (1 + o(1))T(r, f) + O(\log r).$$

Similarly, if $L(f)$ is a linear differential polynomial of f with rational coefficients, and f has finitely many poles, we obtain

$$T(r, L(f)) \leq (1 + o(1))T(r, f) + O(\log r).$$

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