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ON THE CHARACTERIZATION OF A RIEMANN SURFACE BY ITS SEMIGROUP OF ENDOMORPHISMS

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ABSTRACT. Suppose D_1 and D_2 be Riemann surfaces which have bounded nonconstant holomorphic functions. Denote by $E(D_i)$, $i = 1, 2$, the semigroups of all holomorphic endomorphisms. If $\phi: E(D_1) \to E(D_2)$ is an isomorphism of semigroups then there exists a conformal or anticonformal isomorphism $\psi: D_1 \to D_2$ such that ϕ is the conjugation by ψ. Also the semigroup of injective endomorphisms as well as some parabolic surfaces are considered.

1. INTRODUCTION

For a Riemann surface D denote by $E(D)$ its semigroup of endomorphisms, i.e. the set of all holomorphic maps $f: D \to D$ with the operation of composition $\circ$.

A well-known theorem by L. Bers states that two plane domains are conformally or anticonformally equivalent iff their rings of holomorphic functions are isomorphic [1, 3]. L. Rubel raised the question whether the conformal type of a Riemann surface D can be recovered from the algebraic structure of $E(D)$. A similar question in topological context (recovering a topological space from the algebraic structure of its semigroup of continuous self-maps) has been extensively studied (a survey is [6]).

The following example was pointed out by A. Hinkkanen [4]. Let $D = \bar{\mathbf{C}}\backslash E$, $4 \leq \operatorname{card} E \leq \infty$ and E is in general position, i.e. the only conformal automorphism of D is the identity. Then $E(D)$ consists of the identity and constants. Indeed, every $f \in E(D)$ can be extended to an element of $E(\bar{\mathbf{C}})$ by the Picard theorem. So f is a rational function such that $F^{-1}(E) \subset E$. As $\operatorname{card} E \geq 3$ one can easily prove that f is a Möbius transformation or constant. The only possible Möbius transformation is the identity. It is easy to see that all semigroups of that kind are isomorphic. So even two topologically different Riemann surfaces can have isomorphic semigroups of endomorphisms.

On the other hand one may easily characterize the conformal type of $\bar{\mathbf{C}}$, $\mathbf{C}$ and $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$ by their semigroups of endomorphisms. The semigroup of a torus $\mathbf{T}$ determines its topology but not its (anti-)conformal type. Parabolic surfaces are treated in §4.

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The main result of this note is

Theorem 1.[1] *Let D_1 and D_2 be Riemann surfaces which admit bounded nonconstant holomorphic functions. Suppose that $\phi : E(D_1) \to E(D_2)$ is an isomorphism of semigroups. Then there exists a conformal or anticonformal isomorphism $\psi : D_1 \to D_2$ such that*

$$\phi f = \psi \circ f \circ \psi^{-1}. \tag{1}$$

We also may consider the smaller semigroup $E_0(D)$ of all univalent holomorphic maps D into D.

Theorem 2. *Let D_1 and D_2 be bounded plane domains. If $\phi : E_0(D_1) \to E_0(D_2)$ is an isomorphism of semigroups then there exists a conformal or anticonformal isomorphism $\psi : D_1 \to D_2$ which satisfies* (1).

We prove Theorems 1 and 2 in §§2 and 3 respectively. Section 4 is devoted to some parabolic surfaces. For the convenience of the reader we have collected some known facts in appendices. Good references for Appendix 5.2 are the classical papers by Fatou [2] and Julia [5].

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2. Proof of Theorem 1

We will prove first that ϕ maps constants to constants. This will allow us to construct a bijection $\psi : D_1 \to D_2$. Then we will prove that ψ is continuous and after that that ψ is (anti-)conformal.

2.1. **Definition of constants and construction of ψ.** The result of this subsection is well known to the specialists in semigroup theory (the earliest reference is probably [7]) but we include the proof for completeness.

Denote by $C = C(D)$ the subsemigroup of $E(D)$ which consists of constant endomorphisms. Let $c_z \in C$ be the constant function which maps D to $z \in D$. The subset $C(D) \subset E(D)$ may be characterized using only the semigroup structure:

$$c \in C \quad \text{iff} \quad \forall (f \in E(D)), \qquad c \circ f = c.$$

So ϕ induces a bijection of $C(D_1)$ onto $C(D_2)$.

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Define $\psi : D_1 \to D_2$ by

$$\psi(z) = w \quad \text{iff} \quad \phi c_z = c_w; \qquad z \in D_1,\ w \in D_2. \tag{2}$$

So the relation

$$f(z) = w; \qquad f \in E(D),\ z, w \in D, \tag{3}$$

is equivalent to

$$f \circ c_z = c_w. \tag{4}$$

[1] Conjectured by L. Rubel for bounded plane domains.

Applying ϕ to both sides of (4) we obtain $\phi f \circ \phi c_z = \phi c_w$. By (2) this is equivalent to $\phi f \circ c_{\psi(z)} = c_{\psi(w)}$, or, using (3) and (4)

$$(\phi f)(\psi(z)) = \psi(w) = \psi(f(z)).$$

So $\phi f = \psi \circ f \circ \psi^{-1}$, which proves (1).

2.2. **Definition of good elements and continuity of ψ.** Call an element $f \in E(D)$ good if some iterate f^n of f has relatively compact image in D, f has a fixed point in D and f is univalent in a neighborhood of this fixed point.

Remarks. The existence of a fixed point in D follows from compactness of the image. Every element of $E(D)$ other than identity has at most one fixed point in D. If f has a fixed point $z_0 \in D$ then $f'(z_0) = \lambda$ is defined (does not depend on local coordinate) and $|\lambda| < 1$. The function f is univalent in some neighborhood of z_0 iff $\lambda \neq 0$. For all these facts see Appendix 5.1. As D admits bounded nonconstant holomorphic functions, it is easy to see that every point $z \in D$ is the fixed point of some good $f \in E(D)$.

Let us show how to say that an element $f \in E(D)$ is good using only the semigroup structure:

(i) f has a fixed point $c^0 \in C$ iff $f \circ c^0 = c^0$;

(ii) some iterate of f has relatively compact image iff

$$\forall (c' \in C \setminus \{c^0\}) \exists (n \in \mathbf{N}) \forall (c'' \in C), \qquad f^n \circ c'' \neq c'. \tag{5}$$

Indeed, (5) means exactly that

$$\bigcap_{n \in \mathbf{N}} f^n(D) = \{z_0\}, \tag{6}$$

where z_0 is the fixed point $c^0 = c_{z_0}$. If some $f^n(D)$ is compact, then we have (6) because f strictly decreases the Poincaré distance (Appendix 5.1). On the other hand if we have (6) then $\{f^n(D)\}$ forms a fundamental set of neighborhoods of z_0. So at least one of the domains $f^n(D)$ is relatively compact.

Note that we always have $f^{n+1}(D) \subset f^n(D)$.

(iii) To say that f is univalent in a neighborhood of z_0 is the same as

$$\exists (n \in \mathbf{N}) \quad \forall (c' \in C) \quad \forall (c'' \in C)$$
$$(f^{n+1} \circ c' = f^{n+1} \circ c'') \Rightarrow (f^n \circ c' = f^n \circ c'').$$

(This is equivalent to: $\exists n$ such that f is injective on $f^n(D)$.)

Now it is easy to prove that ψ is continuous. Take $z_0 \in D_1$ and $w_0 = \psi(z_0) \in D_2$. Let $f \in E(D_1)$ be a good element which fixes z_0. Then $\phi f = g$ is a good element in $E(D_2)$ which fixes w_0. We also have $\psi(f^n(D_1)) = g^n(D_2)$. So ψ maps a fundamental set of neighborhoods of z_0 to a fundamental set of neighborhoods of w_0. Thus ψ is continuous and we conclude that ψ is a homeomorphism.

2.3. **Proof that ψ is conformal or anticonformal.** Fix an arbitrary point $z_0 \in D_1$ and put $w_0 = \psi(z_0) \in D_2$. Take a good element $f \in E(D_1)$ which fixes z_0. Then $g = \phi f = \psi \circ f \circ \psi^{-1}$ is a good element of $E(D_2)$ which fixes w_0.

Denote by $P(f)$ the set of all $h \in E(D_1)$ which are permutable with f, i.e., $h \circ f = f \circ h$. This is a subsemigroup of $E(D_1)$. We use the following description of $P(f)$ (see Appendix 5.2):

Denote by S the group of all linear self-maps of the field $\mathbf{C}$. (The elements of S are $z \mapsto \lambda z$, $\lambda \in \mathbf{C}^*$. The group S is isomorphic to the multiplicative group $\mathbf{C}^*$.) There is a neighborhood O_1 of z_0 and a local coordinate $F : (O_1, z_0) \to (\mathbf{C}, 0)$ which conjugates $P(f)$ to some subsemigroup $S_1 \subset S$. In other words $s(h) = F \circ h \circ F^{-1} \in S$ if $h \in P(f)$ and $h \mapsto s(h)$ is an isomorphism of semigroups $P(f) \to S_1$.

Similarly consider a local coordinate $G : (O_2, w_0) \to (\mathbf{C}, 0)$, $w_0 \in O_2 \subset D_2$, which conjugates $P(g)$ to a subsemigroup $S_2 \subset S$.

It is important (see Appendix 5.2) that S_1 and S_2, when considered as subsets of $\mathbf{C}^*$ contain some punctured neighborhoods of 0.

Now form the function $V_{z_0} = G \circ \psi \circ F^{-1}$ which maps a neighborhood of 0 to some neighborhood of 0 and conjugates S_1 to S_2. We use the following elementary lemma which will be proved in Appendix 5.3.

Lemma 1. *Let S_1 and S_2 be subsemigroups of the multiplicative group $\mathbf{C}^*$ both containing some punctured neighborhoods of 0. If V is a continuous injective map in a neighborhood of 0 which conjugates S_1 to S_2 then*

$$V(z) = a z^A \bar{z}^B, \tag{7}$$

where $a \in \mathbf{C}^*$, $A, B \in \mathbf{C}$ and $A - B = \pm 1$.

Note that V given by (7) is differentiable (as a function from $\mathbf{R}^2$ to $\mathbf{R}^2$) and nondegenerate in $\mathbf{C}^*$. It is differentiable and nondegenerate at 0 iff $A + B = 1$. In this latter case V is (anti-)conformal because $A + B = 1$ and $A - B = \pm 1$ imply $A = 1$ or $B = 1$.

We conclude that ψ is differentiable and nondegenerate in $O_1 \backslash \{z_0\}$. It follows that for arbitrary $z_1 \in O_1 \backslash \{z_0\}$ the function V_{z_1} is differentiable and nondegenerate. So V_{z_1} is (anti-)conformal and this implies that ψ is (anti-)conformal.

3. Proof of Theorem 2

We only need to construct a bijective map $\psi : D_1 \to D_2$ which satisfies (1). The proof that ψ is (anti-)conformal is exactly the same as in Theorem 1.

Suppose that $D \subset \mathbf{C}$ is a bounded domain and E_0 is the semigroup of univalent holomorphic self-maps of D. Note first that $\{f(D) : f \in E_0\}$ is a base of topology in D which means that every open set in D is a union of some $f(D)$. To see this it is sufficient to consider only the affine maps $z \mapsto az + b$ in E_0.

Define a partial order on E_0 by setting $f \preceq g$ if $f = g \circ h$ for some $h \in E_0$. Then $f \preceq g$ is equivalent to $f(D) \subset g(D)$. For every subset $A \subset E_0$ denote

$$\alpha(A) = \bigcup \{f(D) : f \in A\}. \tag{8}$$

We are going to define a family L of subsets $A \subset E_0$ which is in bijective correspondence with the family T of all nonempty open subsets of D via the map $A \mapsto \alpha(A)$.

For every $f \in E_0$ define $H(f) = \{g : g \preceq f\}$. A set $A \subset H(f)$ is called fundamental for f if for every $g \preceq f$ there exists $h \in A$ such that $h \preceq g$. It is easy to see that A is fundamental for f iff $\{g(D) : g \in A\}$ is a base of topology on $f(D)$.

Let $B \subset E_0$ be an arbitrary subset. Define $H(B)$ as the set of all $f \in E_0$ which have fundamental sets $A \subset \bigcup_{g \in B} H(g)$. Then we have

$$H(B) = \left\{ f \in E_0 : f(D) \subset \bigcup_{g \in B} g(D) \right\}. \tag{9}$$

Note that $B \subset H(B)$.

To prove $\subset$ in (9) take $f \in H(B)$ and fix a fundamental set A for f such that $A \subset \bigcup_{g \in B} H(g)$. We have $f(D) = \bigcup_{h \in A} h(D)$ so $f(D) \subset \bigcup_{g \in B} g(D)$.

To prove $\supset$ in (9) let $f(D) \subset \bigcup_{g \in B} g(D)$. Take $A_g = \{h : h(D) \subset f(D) \bigcap g(D)\} \subset H(g)$, $g \in B$. Then $A := \bigcup_{g \in B} A_g \subset \bigcup_{g \in B} H(g)$. Let us prove that A is a fundamental set for f. Take any $h \preceq f$. Then $h(D) \subset f(D) \subset \bigcup_{g \in B} g(D)$ so there is a g_1 in B such that $h(D) \cap g_1(D) \neq \varnothing$. It follows that there is an $h_1 \in A_{g_1} \subset A$ such that $h_1 \preceq h$. This proves (9).

As a corollary of (9) we remark that

$$\alpha(H(B)) = \alpha(B). \tag{10}$$

Denote $L := \{H(B) : B \subset E_0\}$. Then α defined in (8) gives a bijection between L and the family T of all nonempty open subsets of D. To prove this construct the inverse for α. Let $O \subset D$ be an open set. Put $B = \{f \in E_0 : f(D) \subset O\}$. Then $B = H(B) \in L$ and $\alpha(B) = O$. So α is surjective. From (9) it follows that α is injective because $\{f(D) : f \in E_0\}$ is a base of topology.

Every $f \in E_0$ defines a map $\tilde{f} : E_0 \to E_0$, $\tilde{f}(g) = f \circ g$. This map preserves the order. Note that $\tilde{f}$ does not map L to L, however we define a map $f^* : L \to L$ as follows:

$$f^*(H(B)) = H(\tilde{f}(B)). \tag{11}$$

This definition is unambigous because $H(B_1) = H(B_2)$ means

$$\bigcup_{g \in B_1} g(D) = \bigcup_{g \in B_2} g(D)$$

in view of (9), so

$$\bigcup_{g \in B_1} f \circ g(D) = \bigcup_{g \in B_2} f \circ g(D)$$

which implies (again by (9)) that $H(\tilde{f}(B_1)) = H(\tilde{f}(B_2))$. For every set $A \subset E_0$ we have by (8):

$$\alpha(\tilde{f}(A)) = \alpha(\{f \circ g : g \in A\}) = \bigcup_{g \in A} f \circ g(D) = f(\alpha(A)). \tag{12}$$

Applying α to (11) and using (10), (12) we get

$$\alpha(f^*(H(B))) = \alpha(H(\tilde{f}(B))) = \alpha(\tilde{f}(B)) = f(\alpha(B)) = f(\alpha(H(B))).$$

So if we define $F: T \to T$ by $F(O) = \{f(z): z \in O\}$ then

$$f^* = \alpha^{-1} \circ F \circ \alpha. \tag{13}$$

Call an element $A \in L$ maximal if for every $B \in L$ the inclusion $A \subset B$ implies $A = B$ or $B = E_0$. The map α preserves the inclusion so to maximal elements of L correspond maximal open sets in D. It is easy to see that maximal open sets are the sets of the form $D\backslash\{z\}$ for some point $z \in D$. Denote by M the set of maximal elements of L. There is a natural bijection $\beta : M \to D$. If $A \in L$ and $B \subset M$ then $\alpha(A)$ is a neighborhood of the point $\beta(B)$ iff A is not a subset of B. Consider now two domains D_1 and D_2 and an isomorphism $\phi : E_0(D_1) \to E_0(D_2)$. We use the notations $L_i, M_i, \alpha_i, \beta_i, T_i, i = 1, 2$, introduced before. The isomorphism ϕ preserves the order relation in E_0 and inclusion relations between subsets. So

$$\psi = \beta_2 \circ \phi \circ \beta_1{}^{-1} : D_1 \to D_2$$

is a bijection. Define also

$$\Psi = \alpha_2 \circ \phi \circ \alpha_1^{-1}$$

which is a bijection between T_1 and T_2. These maps are consistent: if $O \subset D_1$ is a neighborhood of $z \in D_1$ then $\Psi(O) \subset D_2$ is a neighborhood of $\psi(z) \in D_2$. So ψ is a homeomorphism and

$$\Psi(O) = \{\psi(z) : z \in O\}. \tag{14}$$

Now let $f \in E_0(D_1)$. Applying ϕ to $\tilde{f}(g) = f \circ g$ we obtain $\phi(\tilde{f}(g)) = \phi f \circ \phi g = \widetilde{\phi f}(\phi g)$ so

$$\widetilde{\phi f} = \phi \circ \tilde{f} \circ \phi^{-1}. \tag{15}$$

From the definition (11) and (15) it follows that

$$(\phi f)^*(H(B)) = H(\widetilde{\phi f}(B)) = H(\phi \circ \tilde{f} \circ \phi^{-1}(B)) = \phi \circ f^* \circ \phi^{-1}(H(B)).$$

So

$$(\phi f)^* = \phi \circ f^* \circ \phi^{-1}. \tag{16}$$

Finally from (13) and (16) it follows that

$$\begin{aligned}\Psi \circ F \circ \Psi^{-1} &= \alpha_2 \phi \alpha_1{}^{-1} \alpha_1 f^* \alpha_1{}^{-1} \alpha_1 \phi^{-1} \alpha_2 \\ &= \alpha_2 \phi f^* \phi^{-1} \alpha_2{}^{-1} = \alpha_2 (\phi f)^* \alpha_2{}^{-1} = \phi F.\end{aligned}$$

So for every open set O we have $\Psi(f(O)) = (\phi f)(\Psi(O))$. In view of (14) this is equivalent to $\psi \circ f = \phi f \circ \psi$. This proves (1) and Theorem 2 follows.

4. Elliptic and parabolic surfaces

Denote $D_0 = \bar{\mathbf{C}}$, $D_1 = \mathbf{C}$, $D_2 = \mathbf{C}^* = \mathbf{C}\backslash\{0\}$.

Theorem 3. *Let D be a Riemann surface. If $E(D)$ is isomorphic to $E(D_k)$, $0 \leq k \leq 2$, then D is conformally equivalent to D_k.*

Proof. Remark that $E(D_0)$ is the set of all rational functions, $E(D_1)$ is the set of all entire functions and $E(D_2)$ is the set of all functions holomorphic in $\mathbf{C}^*$ which do not take the value 0.

Recall that the notions of constant and the value of an endomorphism at a point are expressible in terms of the algebraic structure of $E(D)$ (see §2.1). So if $E(D)$ is isomorphic to one of the three semigroups $E(D_k)$, $0 \le k \le 2$, then D is conformally equivalent to one of the three surfaces D_k, $0 \le k \le 2$, because these are the only surfaces which have endomorphisms with more then one fixed point.

Now $E(D_k)$, $k = 1, 2$, contain elements for which some point has an infinite set of preimages (for example $z \mapsto e^z$) and $E(D_0)$ does not contain such elements. So $E(D_0)$ is not isomorphic to $E(D_k)$, $k = 1, 2$. Finally, the difference between $E(D_1)$ and $E(D_2)$ is that all $f \in E(D_2)$ are surjective (by the Picard theorem) while some $f \in E(D_1)$ are not (an example is again provided by the exponential function). The theorem is proved. □

5. Appendix

5.1. **The Poincaré metric.** Let D be a hyperbolic Riemann surface, i.e., the universal covering of D is the unit disk $\mathbf{U}$. Then $D = \mathbf{U}/\Gamma$ where Γ is a discrete group of conformal automorphisms of $\mathbf{U}$. There is a conformal Riemannian metric $|dz|/(1-|z|^2)$ on $\mathbf{U}$ which is invariant under all conformal automorphisms. So this metric may be pulled down to D. We obtain a Riemann metric on D which is called the Poincaré metric. Denote by ρ the distance in the Poincaré metric in D. The invariant form of the Schwarz lemma states that $\rho(f(z), f(w)) \le \rho(z, w)$ for every z and w in D and $f \in E(D)$. This inequality is strict for every z and w unless f is a covering. In this latter case we have equality for all z and w that are close enough to each other.

If $f(D)$ is relatively compact then f cannot be a covering so f strictly decreases the Poincaré distance. It follows that the sequence $f(D) \supset f^2(D) \supset \cdots$ has one point of intersection and this point z_0 is the unique attractive fixed point of f in D. (Attractive means $|f'(z_0)| < 1$. The derivative at a fixed point does not depend on the choice of local coordinate.)

5.2. **The structure of the semigroup** $P(f)$ **for good** $f \in E(D)$. Let f be an element of $E(D)$ for some hyperbolic Riemann surface D. Suppose f is good (the definition is in 2.2). Then f has a fixed point, $f(z_0) = z_0$ and $0 < |f'(z_0)| < 1$.

(a) If $f \circ g = g \circ f$ then g fixes z_0. Indeed, $f \circ g(z_0) = g \circ f(z_0) = g(z_0)$. So f fixes $g(z_0)$. But the fixed point of f is unique. So $g(z_0) = z_0$. Furthermore, it is easy to see by substituting formal power series to the equation $f \circ g = g \circ f$ that $g'(z_0) \ne 0$.

(b) Let $f'(z_0) = \lambda$, $0 < |\lambda| < 1$. Then the Schröder functional equation

$$F \circ f = \lambda F, \quad F(z_0) = 0, \quad F'(z_0) = 1 \tag{17}$$

has a unique normalized solution F which is holomorphic in a neighborhood of z_0. (There is a unique formal series $F(z) = (z - z_0) + a_2(z - z_0)^2 + \cdots$ which satisfies (17); the convergence is proved by the majorant method.)

The solution F of Schröder equation may be continued analytically to the whole domain D. To prove this suppose that F is originally defined in the neighborhood O of z_0. There is a natural number n such that $f^n(D) \subset O$ (Here we use that f is good so $f^k(D)$ is relatively compact for some k.) Then define $F = \lambda^{-n} F \circ f^n$ in D. In view of (17) we get the analytic continuation of F.

Remark 1. If f is good then the Schröder function F is bounded in D.

Remark 2. If f is univalent in D then F is univalent too. This follows from the procedure of continuation of F if we take into account that F is univalent in O.

(c) If g is permutable with f as above then g has the same Schröder function F [5]. To prove this denote by G the Schröder function of g. It satisfies

$$G \circ g = \mu G, \quad G(z_0) = 0, \quad G'(z_0) = 1, \tag{18}$$

where $\mu = g'(z_0)$. Set $H = \lambda^{-1} G \circ f$. Then $H(z_0) = 0$, $H'(z_0) = 1$, and

$$H \circ g = \lambda^{-1} G \circ f \circ g = \lambda^{-1} G \circ g \circ f = \mu \lambda^{-1} G \circ f = \mu H$$

in view of $f \circ g = g \circ f$ and (18). So $H \circ g = \mu H$ and H is a normalized solution of the Schröder equation (18). But such a solution is unique, so $H = G$ which means by the definition of H that $G \circ f = \lambda G$. So G is a normalized solution of (17) and it follows from the uniqueness of such a solution that $G = F$.

(d) We have proved that all $h \in E(D)$ which are permutable with f satisfy $F \circ h = s(h) \circ F$ where $s(h)$ is the linear map $s(h)\colon z \mapsto h'(z_0)z$. So F conjugates the semigroup $P(f)$ with some subsemigroup S_1 of the multiplicative group $\mathbf{C}^*$.

(e) Let us prove that S_1 contains all elements of $\mathbf{C}^*$ which are close enough to zero. In view of Remark 1 above F is bounded in D. So if $\mu \in \mathbf{C}^*$ and $|\mu|$ is sufficiently small then $g_\mu := F^{-1} \circ (\mu F)$ is a well-defined element of $E(D)$. It is evident that $g_\mu \in P(f)$. Our final remark is that if $f \in E_0(D)$ then F is univalent by Remark 2 above and $g_\mu \in E_0(D)$.

5.3. **Proof of Lemma 1.** Consider S_1 and S_2 as subsets of $\mathbf{C}^*$. Denote by $Q(a) \in S_2$ the element conjugate to $a \in S_1$, i.e., $Q(a) = V^{-1} \circ a \circ V$. Then Q is a homeomorphism $S_1 \to S_2$ with the property

$$Q(ab) = Q(a)Q(b), \qquad a, b \in S_1.$$

Extend Q to $\mathbf{C}^*$. If $a \in \mathbf{C}^*$ take a $b \in S_1$ such that $ab \in S_1$. This is possible because S_1 contains an annulus $0 < |z| < r_0$. Then set $Q(a) := Q(ab)/Q(b)$. It is easy to see that this definition is unambigous, i.e., $Q(a)$ does not depend on the choice of b. It follows that $Q\colon \mathbf{C}^* \to \mathbf{C}^*$ is a continuous homomorphism of multiplicative groups, and Q is injective in $0 < |z| < r_0$. Consider the universal covering $\exp\colon \mathbf{C} \to \mathbf{C}^*$ and denote by $Q^*\colon \mathbf{C} \to \mathbf{C}$ the lifting of Q.

Then Q^* is continuous and satisfies

$$Q^*(a+b) = Q^*(a) + Q^*(b), \qquad a, b \in \mathbf{C},$$

and

$$Q^*(z+2\pi i) = Q^*(z) \pm 2\pi i, \qquad z \in \mathbf{C}.$$

It easily follows that $Q^*(z) = Az + B\bar{z}$, where $A - B = \pm 1$. So

$$Q(z) = z^A \bar{z}^B, \qquad A, B \in \mathbf{C}, \ A - B = \pm 1.$$

Now we have

$$V \circ s_\lambda \circ V^{-1} = s_{Q(\lambda)},$$

where $s_\lambda \in S$, $s_\lambda : z \mapsto \lambda z$. So $V(\lambda z) = \lambda^A \bar{\lambda}^B V(z)$ for all small z and λ, which implies $V(z) = az^A\bar{z}^B$ for some $a \in \mathbf{C}^*$.

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