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ON SOME FUNCTIONAL EQUATIONS CONNECTED WITH ITERATION OF RATIONAL FUNCTIONS

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ABSTRACT. Methods in the theory of iteration of rational functions are used to investigate the functional equations $f(g) = g(f)$ and $G(g) = f(G)$, where f and g are rational functions, and G is a meromorphic function on the complex plane with an essential singularity at infinity.

Introduction

The creators of the theory of iteration of rational functions, namely, Fatou, Julia, and Ritt, regarded it first of all as a method for investigating functional equations ([1]–[3]). At the beginning of the 1980's this theory went through a period of stormy development that was connected with the use of new methods in geometric function theory and the theory of dynamical systems. The questions put forth have in the first place originated from the theory of dynamical systems (regular and chaotic behavior, bifurcation, structural stability, etc.), and applications to functional equations have received less attention. The goal of this paper is to study two classical functional equations with the help of new methods in the theory of iteration. The first of these is

$$f_1 \circ f_2 = f_2 \circ f_1. \tag{0.1}$$

It is required to find all pairs of commuting rational functions. Fatou ([4], [5]), Julia [6], and Ritt [7](1) devoted thorough investigations to this problem. Before describing their results we present the basic aspects of the theory of iteration of rational functions. They may be found in [8], Chapter VIII, or the surveys [9] and [10]. The classical work [1] continues to be an indispensable source.

Let f be a rational function with $\deg f \geq 2$. Denote by f^n its nth iterate. Functions f and g are said to be conjugate if there exists a linear fractional transformation φ such that $f \circ \varphi = \varphi \circ g$. A set $E \subset \overline{\mathbf{C}}$ is said to be completely invariant if its complete inverse image $f^{-1}E$ coincides with E. A maximal finite completely invariant set $E(f)$ exists and is called the exceptional set. We always have $\operatorname{card} E(f) \leq 2$. Further, if $\operatorname{card} E(f) = 1$, then f is conjugate to a polynomial ($E(f) \ni \infty$ for a polynomial). But if $\operatorname{card} E(f) = 2$, then f is conjugate to $g(z) = z^n$, $n \in \mathbf{Z}\backslash\{0, 1\}$. Obviously, $E(g) = \{0, \infty\}$.

A point z is said to be periodic with period n if $f^n z = z$. The smallest period is called the order of z. A fixed point is a point with period 1. If z is a

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(1)The authors of a number of subsequent papers on this topic were not familiar with [4]–[7]. There is an extensive bibliography on commuting functions (and not just rational ones) in [11].

periodic point of order n, then the number $\lambda = (f^n)'(z)$ is called its multiplier. A periodic point is said to be repelling if $|\lambda| > 1$.

Let $N(f)$ be the maximal open set on which the family $\{f^n\}$ is normal in the Montel sense [8]. Its complement is called the Julia set $J(f) = \overline{\mathbf{C}} \backslash N(f)$. The latter coincides with the closure of the set of repelling periodic points; it is always nonempty, perfect, and completely invariant, and, moreover, $J(f^n) = J(f)$. In particular, the set of repelling periodic points is infinite. On the other hand, as shown by Fatou, the set of nonrepelling periodic points is finite. There can fail to be any repelling fixed points, but there are always repelling points of period 2.

The Julia set is either nowhere dense or coincides with $\overline{\mathbf{C}}$. We remark that $J(f) \neq \overline{\mathbf{C}}$ for a polynomial f, since the family $\{f^n\}$ is normal in a neighborhood of ∞. The same is true for $f(z) = z^{-n}$, $n \in \mathbf{N}\backslash\{1\}$. Thus, $E(f) = \varnothing$ if $J(f) = \overline{\mathbf{C}}$.

Let z_0 be a repelling fixed point of the function f, and let $\lambda = f'(z_0)$. Define $\Lambda: z \mapsto \lambda z$. Then there exists a unique solution of the Poincaré equation

$$F \circ \Lambda = f \circ F, \qquad F(0) = z_0, \qquad F'(0) = 1, \tag{0.2}$$

that is meromorphic in $\overline{\mathbf{C}}$. It is easy to see that the exceptional set $E(f)$ is exactly the set of values not taken by f in $\mathbf{C}$, while all the values in $\overline{\mathbf{C}} \backslash E(f)$ are taken by F infinitely many times.

Following [4] and [6], we impose additional restrictions on the functions f_1 and f_2 in (0.1): $f_1^m \neq f_2^n$ for all $m, n \in \mathbf{N}$. The problem of describing all pairs of functions having a common iterate requires special consideration and will not be discussed here (see [3] and [7] concerning this). We present an outline of the arguments of Fatou and Julia [5], [6] they coincide). It is shown first of all that commuting functions f_1 and f_2 have the same Julia set J and that there exists a repelling periodic point z_0 common for f_1 and f_2. Replacing f_1 and f_2 by certain of their iterates (denote the iterates again by f_1 and f_2), we make z_0 a fixed point. It can then be shown that the Poincaré function F corresponding to z_0 is the same for f_1 and f_2. Thus, the meromorphic function F satisfies the two functional equations

$$F \circ \Lambda_j = f_j \circ F, \qquad \Lambda_j: z \mapsto \lambda_j z, \quad \lambda_j = f_j'(z_0), \quad F(0) = z_0 \qquad j = 1, 2.$$

From the fact that f_1 and f_2 do not have a common iterate it can be deduced that

$$\Lambda_1^n \neq \Lambda_2^m, \qquad m, n \in \mathbf{N}. \tag{0.3}$$

Now let $I = F^{-1}(J)$. It follows from the complete invariance of J with respect to f_1 and f_2 that $\Lambda_j I = I$, $j = 1, 2$. Let Γ be the closed group generated by the transformations Λ_j, $j = 1, 2$. In view of (0.3) Γ is nondiscrete, and hence contains a one-parameter subgroup Γ_1. This imposes strong restrictions on the set I, and thus also on J. There are the following possibilities:

1) $I = \mathbf{C}$, in which case $J = \overline{\mathbf{C}}$.

2) I is nowhere dense and consists of analytic curves (logarithmic spirals, or rays emanating from zero, or circles about zero).

Fatou [5] and Julia [6] thoroughly investigated the second case. It turned out that in this case f_1 and f_2 can be reduced by a conjugacy to the form $f_1(z) = z^m$ and $f_2(z) = z^n$ or to the form $f_1 = T_m$ and $f_2 = T_n$, where T_k is the Tchebycheff polynomial determined by the equation $\cos k\zeta = T_k(\cos\zeta)$.

Thus, the problem of describing commuting functions without common iterates was solved for the case $J \neq \overline{\mathbf{C}}$, in particular for polynomials. Case 1) could not be investigated in the analogous way, because at the time there were no means for describing the chaotic dynamics that takes place in the whole plane in this case.

At the same time, Ritt [7] obtained a complete solution of the problem of commuting functions by a quite different method. Some more pairs, with Poincaré functions expressed in terms of elliptic functions, are added to the indicated pairs f_1, f_2. In no way did Ritt connect the method of his paper with the theory of iteration, and the method has a topological-algebraic character. His proof seems very complicated and devoid of geometric clarity. Ritt writes: "It would be interesting to know whether a proof can also be effected by the use of the Poincaré functions employed by Julia" ([7], p. 400). The article [5] came out a little later, and there Julia already refers to [7]. The search for a proof of Ritt's theorem in the spirit of the ideas of Fatou and Julia is what led to the appearance of the present article. Examples of commuting polynomial mappings $\mathbf{C}^n \to \mathbf{C}^n$ generalizing the polynomials $z \to z^k$ and T_k were constructed in the recent article [12]. A new proof of Ritt's theorem can turn out to be useful for describing all pairs of such mappings.

The new method for investigating (0.1) proved to be applicable to another functional equation

$$G \circ g = f \circ G, \tag{0.4}$$

where g and f are rational functions. In the case when $\deg g = 1$, (0.4) can be reduced either to the Poincaré equation or to the equation $G(z+1) = f \circ G(z)$, which has been thoroughly investigated (see, for example, [1], [13], [14]). Using Fatou's result, Azarina [15] gave a complete description of all solutions G of (0.4) with $\deg g = 1$ that are meromorphic on $\mathbf{C}$. Next, assume that $\deg g \geq 2$. Equation (0.4) is encountered in several papers of Fatou and Julia; the long paper [16], which sums up all the preceding results, is especially devoted to this equation. As a rule, the solution G is a very complicated multivalued function about which little of consequence can be said in the general case. The study of single-valued solutions is of interest. Following Julia, we confine ourselves to the single-valued transcendental(2) solutions G having finitely many essentially singular points in $\overline{\mathbf{C}}$. It is easy to see that the set E of these singular points is completely invariant with respect to g, and hence $E \subset E(g)$. With the help of conjugacy the matter can be reduced to one of the following two cases:

a) $g(z) = z^n$, $n \in \mathbf{Z}\backslash\{0, \pm 1\}$, and G is meromorphic in $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$.

b) g is a polynomial, and G is meromorphic in $\mathbf{C}$.

The main result in [16] is that in both cases a) and b) we must have $J(f) = \overline{\mathbf{C}}$, and the function G takes all values in $\overline{\mathbf{C}}$ infinitely many times. Julia stopped with this investigation, since there were no suitable methods for studying the dynamics of functions f with $J(f) = \overline{\mathbf{C}}$. Examples of relations of the form

(2)The author does not know any results about rational solutions.

(0.4) constructed by means of elliptic functions were presented in the same article [16]. At the time these were the only known examples of rational functions with $J(f) = \overline{\mathbf{C}}$.(3)

We find all triples (g, f, G) of functions satisfying (0.4) and the condition a) or b). They are all explicitly expressed in terms of elliptic functions.

§1. Formulation of results

Let us start with some definitions. According to Thurston ([18], [19]), a (two-dimensional) orbifold is defined to be a Riemann surface S together with a function $n: S \to \mathbf{N} \cup \{\infty\}$ equal to 1 outside a discrete set of points. (This is the same as a marked Riemann surface; we prefer the short modern term.) Orbifolds (S_1, n_1) and (S_2, n_2) are regarded as equivalent if there exists a conformal homeomorphism

$$\varphi: S_1 \backslash \{z: n_1(z) = \infty\} \to S_2 \backslash \{z: n_2(z) = \infty\}, \qquad n_2(\varphi(z)) \equiv n_1(z).$$

For example, if $S_1 = \mathbf{C}$ and $n_1 \equiv 1$, while $S_2 = \overline{\mathbf{C}}$, $n_2(\infty) = \infty$, and $n_2(z) = 1$ for $z \neq \infty$, then $(S_1, n_1) = (S_2, n_2)$. If the surface S is compact, then the Euler characteristic χ of the orbifold $\mathscr{O} = (S, n)$ is defined as follows. We triangulate S with the condition that all points z with $n(z) \geq 2$ be vertices. Let Δ be the number of faces, Γ the number of edges, and P the set of vertices in this triangulation. Then

$$\chi(\mathscr{O}) = -\Delta + \Gamma - \sum_{z \in P} \frac{1}{n(z)}.$$

A cover of orbifolds $R: (S_1, n_1) \to (S_2, n_2)$ is defined to be a holomorphic branched cover R

$$S_1 \backslash \{z: n_1(z) = \infty\} \to S_2 \backslash \{z: n_2(z) = \infty\}$$

with the property $\deg_z R \cdot n_1(z) = n_2(R(z))$, $z \in S_1$. Here $\deg_z R$ is the multiplicity of the function R at z. A cover is said to be universal if S_1 is simply connected and $n_1 \equiv 1$. If $f_1: \mathscr{O}_1 \to \mathscr{O}$ and $f_2: \mathscr{O}_2 \to \mathscr{O}$ are universal covers, then there exists a conformal homeomorphism $\varphi: \mathscr{O}_1 \to \mathscr{O}_2$ such that $f_1 = f_2 \circ \varphi$. If $\mathscr{O}_1$ and $\mathscr{O}_2$ are orbifolds with compact surfaces, and $R: \mathscr{O}_1 \to \mathscr{O}_2$ is a finite-sheeted cover, then the Riemann-Hurwitz formula holds:

$$\chi(\mathscr{O}_1) = \deg R \cdot \chi(\mathscr{O}_2).$$

We are interested in orbifolds $\mathscr{O} = (\overline{\mathbf{C}}, n)$ with $\chi(\mathscr{O}) = 0$, i.e.,

$$\sum_{z \in \overline{\mathbf{C}}} \left(1 - \frac{1}{n(z)}\right) = 2.$$

This equation has six solutions:

$$(\infty, \infty), \quad (2, 2, \infty), \tag{1.1}$$

$$(2, 4, 4), \quad (3, 3, 3), \quad (2, 3, 6), \quad (2, 2, 2, 2). \tag{1.2}$$

Associated with each solution except the last is a unique orbifold to within conformal equivalence, and associated with the solution $(2, 2, 2, 2)$ is a family depending on a single complex parameter. Each orbifold in (1.1), (1.2) has a

(3)The first such example is usually attributed to Lattès (1918). However, Boettcher had an analogous example in 1903 [17].

universal cover by the plane $\mathbf{C}$ and has the form $\mathbf{C}/\Gamma$, where Γ is a discontinuous group of orientation-preserving mixings of the plane ([18], [19], [22]). We present an explicit form for universal covering functions F and the generators of the groups Γ ([19], [22]):

1) (∞, ∞); $\exp 2\pi z$; $z \mapsto z + i$;
2) $(2, 2, \infty)$; $\cos 2\pi z$; $z \mapsto z+1$, $z \mapsto -z$;
3) $(2, 4, 4)$; $\wp^2(z, 1, i)$; $z \mapsto z+1$, $z \mapsto iz$;
4) $(3, 3, 3)$; $\wp'(z, 1, \omega)$; $z \mapsto z+1$, $z \mapsto z+\omega$, $z \mapsto \omega^2 z$;
5) $(2, 3, 6)$; $(\wp')^2(z, 1, \omega)$; $z \mapsto z+1$, $z \mapsto z+\omega$, $z \mapsto \omega z$;
6) $(2, 2, 2, 2)$; $\wp(z, 1, \tau)$; $z \mapsto z+1$, $z \mapsto z+\tau$, $z \mapsto -z$.

Here $\wp(z, \omega_1, \omega_2)$ is the Weierstrass elliptic function with periods ω_1 and ω_2, $\omega = e^{\pi i/3}$, and $\operatorname{Im}\tau > 0$. The meromorphic functions, F in 1)–6) admit many interesting characterizations. For example, every meromorphic periodic solution of the Poincaré equation has the form $L_1 \circ F \circ L_2$, where L_1 is a linear fractional transformation, and L_2 is a linear function [27] (see also [14]).

Associated with each of the orbifolds (1.1), (1.2) is a family of rational functions that implement a cover $f\colon \mathscr{O} \to \mathscr{O}$. All such functions f are obtained from the commutative diagram

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\Lambda} & \mathbf{C} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ \mathscr{O} & \xrightarrow{f} & \mathscr{O} \end{array}$$

Here $F\colon \mathbf{C} \to \mathscr{O} = \mathbf{C}/\Gamma$ is the universal cover, and Λ is a conformal homeomorphism with $\Lambda\Gamma \subset \Gamma$. We present a list of admissible Λ for 1)–5) that give functions f with $\deg f \geq 2$ [19]:

1) $z \mapsto nz$, $n \in \mathbf{Z}$, $|n| > 1$;
2) $z \mapsto nz$, $z \mapsto nz + \frac{1}{2}$, $n \in \mathbf{Z}$, $|n| > 1$;
3) $z \mapsto az$, $z \mapsto az + \frac{1}{2}(1+i)$, $a \in \mathbf{Z}[i]$, $|a| \geq 2$;
4) $z \mapsto az$, $a \in \mathbf{Z}[\omega]$, $|a| \geq 3$, $\omega = e^{\pi i/3}$.
5) $z \mapsto az$, $z \mapsto az + \frac{1}{3}(1+\omega)$, $z \mapsto az + i\sqrt{3}/3$, $a \in \mathbf{Z}[\omega]$, $|a| \geq 3$, $\omega = \exp \pi i/3$.

In case 6) the conformal homeomorphisms $\Lambda(z) = nz + \alpha$, $n \in \mathbf{Z}$, $|n| \geq 2$, $2\alpha \in \Gamma$, are admissible for each τ, and for certain special τ there are also other possibilities for Λ (so-called complex multiplications), the complete description of which we omit (see, for example, [19]).

In case 1) $f(z) = z^n$, while in case 2) $f = T_n$ to within conjugacy. The functions f corresponding to cases 3)–6) were the first examples of rational functions with $J(f) = \overline{\mathbf{C}}$. Many other examples in which $J = \overline{\mathbf{C}}$ are now known (see, for example, [10], [20]).

THEOREM 1. *Suppose that f_1 and f_2 are rational functions such that* $\deg f_j \geq 2$ *and $f_1^m \neq f_2^n$, $m, n \in \mathbf{Z}$. If $f_1 \circ f_2 = f_2 \circ f_1$, then there exists an orbifold $\mathscr{O}$ of type* (1.1) *or* (1.2) *such that f_1 and f_2 are covers $\mathscr{O} \to \mathscr{O}$.*

We now consider covers $\mathscr{O}_1 \to \mathscr{O}_2$ of different orbifolds in the list 1)–6). If $\mathscr{O}_k = \mathbf{C}/\Gamma_k$, then a cover $\mathscr{O}_1 \to \mathscr{O}_2$ exists if and only if $\Gamma_1 \subset \Gamma_2$. The degree of the cover is equal to the index of Γ_1 in Γ_2. Therefore, covers $\mathscr{O}_1 \to \mathscr{O}_2$ of infinite degree are possible when $\mathscr{O}_1$ has the form (1.1), and when $\mathscr{O}_2$ has the form (1.2), where the pair $(\mathscr{O}_1, \mathscr{O}_2) = ((2, 2, \infty), (3, 3, 3))$ is excluded. Such covers G can be expressed in terms of the universal covers F_k of the orbifolds $\mathscr{O}_k$: $F_2 = G \circ F_1$.

THEOREM 2. *Suppose that g and f are rational functions,* $\deg g \geq 2$, *and G is a meromorphic function in* $\mathbf{C}$ *or in* $\mathbf{C}^*$ *with an essential singularity at* ∞. *If G satisfies the equation $G \circ g = f \circ G$, then there exist orbifolds $\mathscr{O}_1$ of type* (1.1) *and $\mathscr{O}_2$ of type* (1.2) *such that the commutative diagram*

$$\begin{array}{ccc} \mathscr{O}_1 & \xrightarrow{g} & \mathscr{O}_1 \\ G\downarrow & & \downarrow G \\ \mathscr{O}_2 & \xrightarrow{f} & \mathscr{O}_2 \end{array}$$

consists of covers.

The examples given in [16] correspond to the case when $\mathscr{O}_2$ has type $(2, 2, 2, 2)$.

We remark that if $h: \mathscr{O} \to \mathscr{O}$ is a cover of an orbifold in the list 1)–6) and $h = h_1^m$, $m \in \mathbf{N}$, then $h_1: \mathscr{O} \to \mathscr{O}$ is also a cover. This remark permits us to replace the functions f_i, f, and g by iterates of them in the proofs of Theorems 1 and 2.

§2. Auxiliary results

The measure on the Julia set that describes the asymptotic distribution of the roots of the equation $f^n(z) = a$ as $n \to \infty$ will serve as the main tool in the proofs of Theorems 1 and 2. Unless otherwise specified, all measures are assumed to be locally finite Borel measures.

Let U and V be domains in $\overline{\mathbf{C}}$, and $\psi: U \to V$ a sufficiently nice function (for example, holomorphic on U or a homeomorphism). Then for every measure μ on V the inverse image

$$(\psi^* \mu)(E) = \int_{\psi E} n_\psi(z)\, d\mu, \qquad E \subset U,$$

is defined, where $n_\psi(z)$ is the number of inverse images in U of a point $z \in V$, with multiplicity taken into account. In particular, a linear operator $A_f = f^*/\deg f$ acting on the set of probability measures in $\overline{\mathbf{C}}$ is defined for a rational function f. There exists a unique probability measure μ_f with the properties

$$A_f \mu_f = \mu_f, \qquad \mu_f(E(f)) = 0. \tag{2.1}$$

This measure does not have discrete components; its support coincides with the Julia set. The measure μ_f is called the equilibrium measure of f. The existence and uniqueness of μ_f for each rational function f was first proved by Lyubich [21].(4) Later a number of other proofs appeared (see, for example, [23]). In the present article μ_f is used mainly for studying the dynamics of functions f with $J(f) = \overline{\mathbf{C}}$. This is precisely the case when the methods of

(4)It was also shown in [21] that μ_f is the unique maximal entropy measure for the function f.

Fatou and Julia are not applicable. We show that the measures μ_f corresponding to rational functions f and f_j in equations (0.1) and (0.4) have a very specific property. But first we investigate this property by itself.

A smooth nonsingular vector field in a domain $V \subset \mathbf{R}^2$ is a smooth function $w: V \to \mathbf{R}^2\backslash\{0\}$. Associated with w is a local phase flow $g_t: V \to \mathbf{R}^2$—a solution of the Cauchy problem

$$\frac{d}{dt}g_t = w(g_t), \qquad g_0 = \mathrm{id},$$

which is defined for sufficiently small $t \in \mathbf{R}$. If $\varphi: U \to V$ is a diffeomorphism, then the inverse image $\varphi^* w$ of the vector field w is defined as

$$(\varphi^* w)(z) = (\varphi'(z))^{-1}(w(\varphi(z))), \tag{2.2}$$

where φ' is the derivative of φ (a linear mapping of the tangent space). The straightening theorem (see [24], Chapter 2, §7) asserts that an arbitrary smooth nonsingular vector field can locally be turned into a constant vector field by means of a diffeomorphism.

Definition. A measure μ on $\mathbf{R}^2$ is said to be lamellar at a point $z_0 \in \operatorname{supp}\mu$ if in some neighborhood of this point there exists a smooth nonsingular vector field w such that its local phase flow preserves μ. If $w(z_0) = a$, then μ is said to be lamellar in the direction a.

With the help of the straightening theorem we obtain an equivalent definition: there exists a diffeomorphism ψ of some neighborhood U of z_0 onto a domain $V \subset \mathbf{R}^2$ such that the measure $\nu = (\varphi^{-1})^*\mu$ is invariant with respect to shifts in the direction of the x-axis. The last condition is equivalent to ν being the product of dx and some measure $d\nu_1(y)$ (x and y are the Cartesian coordinates in $\mathbf{R}^2$).

If μ is lamellar at the point z_0, then it is also lamellar at all points close to z_0. The property of being lamellar is preserved under the action of the operator ψ^* if ψ is a diffeomorphism.

Proposition 1. *Suppose that the measure μ is lamellar at a point z_0 in the two directions a and b, and the vectors a and b are linearly independent* (*over* $\mathbf{R}$). *Then in some neighborhood of z_0 the measure μ is absolutely continuous and has a smooth nonvanishing density.*

Proof. Denote by $B_k(z_0)$ the family of neighborhoods E of z_0 with the property $\max\{|z_0 - \zeta|: \zeta \in \partial E\} \le k\min\{|z_0 - \zeta|: \zeta \in \partial E\}$, $k > 1$. Let $\mu'(z_0) = \lim \mu(E)/|E|$, $\operatorname{diam} E \to 0$, $E \in B_k(z_0)$, where $|\cdot|$ is Lebesgue measure. According to a theorem of Lebesgue (see [25], Chapter IV), the derivative μ' exists almost everywhere for each $k > 1$. We show that μ' exists everywhere in some neighborhood of z_0 and is a smooth function there.

Let $g_t(z)$ and $h_t(z)$ be local phase flows preserving μ, with the corresponding vector fields linearly independent at z_0. It is easy to see that there exists a neighborhood V of z_0 with the property that for each pair of points $z_1, z_2 \in V$ the phase curves $g_t(z_1)$ and $h_t(z_2)$ intersect at a unique point $z_3 \in V$. Therefore, there exist r and s such that $h_s \circ g_r(z_1) = z_2$. The diffeomorphism $\varphi = h_s \circ g_r$ preserves the measure μ and maps the family $B_k(z_1)$ into a family $B_K(z_2)$ with some K that can be chosen independently of $z_1, z_2 \in V$.

We have $|\varphi(E)| \sim c|E|$ for $E \in B_k(z_1)$, $\operatorname{diam} E \to 0$, where c is the Jacobian of φ. Thus, if $\mu'(z_1)$ exists, then $\mu'(z_2)$ exists, and $\mu'(z_2) = \mu'(z_1)/c$. Obviously, c depends smoothly on z_1 and z_2.

We have shown that μ' exists and is a smooth function. It remains to show that μ is absolutely continuous (then its density is necessarily equal to μ'). Let $\varepsilon > 0$ be arbitrary, and take a set K, $\overline{K} \subset V$, $|K| < \varepsilon$. It is necessary to estimate $\mu(K)$. Let U be an open set containing K, with $|U| < 2\varepsilon$, and choose $M > \max\{\mu'(z) : z \in \overline{U}\}$. For each $z \in K$ we consider a disk $O(z) \subset U$ about z such that $\mu(O(z)) \leq M|O(z)|$. According to the theorem of Besicovitch (see, for example, [26], Chapter 1, Theorem 1.1), there is an at most countable covering $\{O_j\}$ of K by these disks such that each point of the plane belongs to at most six disks. Then

$$\mu(K) \leq \sum_k \mu(O_j) \leq M \sum_j |O_j| \leq 6M|U| \leq 12M\varepsilon,$$

which is what was to be proved.

Absolutely continuous measures with smooth nonvanishing densities will simply be called smooth measures below.

We consider the behavior of lamellar measures under holomorphic mappings of the plane. Let us now identify $\mathbf{R}^2$ with $\mathbf{C}$, and vector fields with smooth functions $V \to \mathbf{C}^*$. If φ is a holomorphic function, then formula (2.2) is preserved, with φ' the complex derivative.

Suppose that U and V are neighborhoods of zero, $f: U \to V$ is a holomorphic mapping with $f(z) = az^k + O(z^{k+1})$, $z \to 0$, μ is a measure on V, and $\nu = f^*\mu$ is the inverse image of μ.

PROPOSITION 2. *If $k \geq 3$ and ν is lamellar at zero, then ν is smooth.*

PROOF. It suffices to confine ourselves to the case when $f(z) = z^k$. Then $\nu(E) = \nu(\varepsilon_k E)$, where $E \subset U$ is an arbitrary Borel set, and $\varepsilon_k = \exp 2\pi i/k$. Therefore, if ν is lamellar at zero in the direction a, then it is also lamellar in the direction $\varepsilon_k a$. If $k \geq 3$, then the vectors a and ε_k are linearly independent over $\mathbf{R}$, and an application of Proposition 1 completes the proof.

PROPOSITION 3. *If $k \geq 2$, then the measures μ and ν cannot be simultaneously lamellar at zero.*

PROOF. If ν is a smooth measure, then its image μ is a smooth measure in a deleted neighborhood of zero, and its density has a singularity at zero. This contradicts the condition that μ be lamellar.

Thus, it suffices in view of Proposition 2 to consider the case when $k = 2$. It can be assumed that $f(z) = z^2$. Suppose that both measures μ and ν are lamellar. Denote by w and u the corresponding vector fields. We have

$$u(z) = u(0)(1 + o(1)), \quad w(z) = w(0)(1 + o(1)), \qquad z \to 0.$$

At the point $z \in U$ the measure ν must be lamellar in the direction of the inverse image $f^*w(f(z))$ of the vector $w(f(z))$. We have

$$f^*w(f(z)) = \frac{w(0)}{2z}(1 + o(1)), \qquad z \to 0.$$

It is easy to see that there is a sectorial domain A of angle $3\pi/4$ with vertex at zero in which the vectors $u(z)$ and $f^*w(f(z))$ are linearly independent over $\mathbf{R}$. By Proposition 1, ν is smooth in A. Therefore, μ is smooth in $f(A)$, which is a sectorial domain of angle $3\pi/2$. But μ is lamellar at 0. Obviously, the images of the angle $f(A)$ under the action of the local phase flow determined by the vector field w fill a complete neighborhood of zero. Therefore, μ is smooth in a neighborhood of zero. But then ν is also smooth, and this is impossible, as we saw at the beginning of the proof.

PROPOSITION 4. *If μ satisfies the condition $g_t^*\mu = e^{at}\mu$ in a neighborhood of z_0, where g_t is the local phase flow and $a > 0$, then μ is lamellar at z_0.*

PROOF. By the straightening theorem, it suffices to confine ourselves to the case when $g_t(x, y) = (x + t, y)$. In this case we have $\mu(E + t) = e^{at}\mu(E)$. Let

$$\nu(E) = \int_E e^{-ax}\, d\mu \qquad (z = x + iy).$$

It is easy to see that the measure ν is invariant with respect to the flow g_t. Therefore, $d\nu = (dx) \times d\lambda(y)$ and $d\mu = e^{ax}dx\, d\lambda(y)$. We verify that μ is invariant with respect to the local phase flow

$$(x, y) \mapsto (\varphi_t(x), y), \qquad \varphi_t(x) = \frac{1}{a}\log(e^{ax} + t),$$

which satisfies the differential equation

$$\frac{d}{dt}\varphi_t = \frac{1}{a}e^{-a\varphi_t}.$$

Indeed,

$$e^{a\varphi_t(x)}\left(\frac{d}{dx}\varphi_t(x)\right) dx\, d\lambda(y) = e^{ax}\, dx\, d\lambda(y).$$

Thus, μ is lamellar.

§3. Commuting functions

We proceed to a proof of Theorem 1. Let $f_1 \circ f_2 = f_2 \circ f_1$. If $f_1^n z = z$, then $f_1^n \circ f_2 z = f_2 \circ f_1^n z = f_2 z$, i.e., f_2 maps the finite set of roots of the equation $f_1^n z = z$ into itself. Therefore, f_1 and f_2 have infinitely many common periodic points. All but finitely many of them are repelling. Replacing f_1 and f_2 by certain of their iterates (again denoted by f_1 and f_2), we have the existence of a common repelling fixed point z_0. Let $\lambda_j = f_j'(z_0)$, $\Lambda_j\colon z \mapsto \lambda_j z$, $j = 1, 2$. We consider the Poincaré function:

$$F \circ \Lambda_1 = f_1 \circ F, \quad F(0) = z_0, \quad F'(0) = 1. \tag{3.1}$$

Let $\varphi = f_2 \circ F \circ \Lambda_2^{-1}$. We have $\varphi(0) = z_0$, $\varphi'(0) = 1$, and

$$\begin{aligned} f_1 \circ \varphi &= f_1 \circ f_2 \circ F \circ \Lambda_2^{-1} = f_2 \circ f_1 \circ F \circ \Lambda_2^{-1} \\ &= f_2 \circ F \circ \Lambda_1 \circ \Lambda_2^{-1} = f_2 \circ F \circ \Lambda_2^{-1} \circ \Lambda_1 = \varphi \circ \Lambda_1, \end{aligned}$$

i.e., φ satisfies (3.1) as F. In view of the uniqueness of the normalized Poincaré function $\varphi = F$, i.e.,

$$F \circ \Lambda_2 = f_2 \circ F. \tag{3.2}$$

Thus, F is a common Poincaré function for f_1 and f_2.

Let us now consider the operators A_{f_1} and A_{f_2} defined at the beginning of §2. Obviously, $A_{f_1}A_{f_2} = A_{f_2 \circ f_1} = A_{f_1 \circ f_2} = A_{f_2}A_{f_1}$. Therefore, the unique fixed points of these operators coincide: $\mu_{f_1} = \mu_{f_2} = \mu$. In particular, the Julia sets coincide: $J(f_1) = J(f_2) = J$.

Let $\nu = F^*\mu$. Equations (3.1) and (3.2) give $\Lambda_j^* F^* = F^* f_j^*$, $j = 1, 2$. Since $f_j^*\mu = \deg f_j \cdot \mu$, we get

$$(\deg f_j)^{-1}\Lambda_j^*\nu = \nu, \qquad j = 1, 2. \tag{3.3}$$

Observe now that in view of (3.1) and (3.2) it follows from $f_1^m \neq f_2^n$ that $\Lambda_1^m \neq \Lambda_2^n$ for all $m, n \in \mathbf{N}$. Therefore, it is possible to choose m_k and n_k tending to ∞ such that $\Lambda_1^{m_k}\Lambda_2^{-n_k} \to 1$. Using (3.3), we get

$$\frac{(\deg f_2)^{n_k}}{(\deg f_1)^{m_k}}\Lambda_1^{*m_k}\Lambda_2^{*-n_k} \to \mathrm{id},$$

consequently, the group generated by the transformations $(\deg f_j)^{-1}\Lambda_j^*$, $j = 1, 2$ is nondiscrete, and its closure contains a one-parameter subgroup $\Gamma = \{B_t : t \in \mathbf{R}^+\}$, $B_t\nu(E) = t^\rho \nu(t^a E)$, $a \in \mathbf{C}^*$, $\rho \geq 0$. The measure ν is invariant with respect to Γ, namely,

$$B_t\nu = \nu, \qquad t \in \mathbf{R}^+; \tag{3.4}$$

therefore, by Proposition 4, it is lamellar everywhere in $\mathbf{C}^*$ (the vector field has a singularity at zero). If $I = \operatorname{supp}\nu = F^{-1}J$, then in view of (3.4)

$$t^a I = I \quad \text{for all } t > 0. \tag{3.5}$$

Two cases must now be considered.

First case. $J = \overline{\mathbf{C}}$. Then $I = \mathbf{C}$. If a point $z \in \overline{\mathbf{C}}$ has a simple (not multiple) inverse image $\zeta \in F^{-1}(z)$, $\zeta \neq 0$, then μ is lamellar at z. This follows from the fact that $\nu = F^*\mu$ is lamellar at ξ, and F is a diffeomorphism in a neighborhood of ζ. Next, by Proposition 3, all the F-inverse images of z in $\mathbf{C}^*$ are simple. Now suppose that z does not have simple inverse images in $\mathbf{C}^*$. If ν is not smooth at all points of $F^{-1}(z)\backslash\{0\}$, then all these points have multiplicity 2 in view of Proposition 2. Suppose now that $\zeta \in F^{-1}(z)$, and ν is smooth at ζ. Then μ is smooth in a deleted neighborhood of z. Consequently, ν is smooth in deleted neighborhoods of all points in $F^{-1}(z)\backslash\{0\}$. Since ν is also lamellar in $\mathbf{C}^*$, we get that it is smooth at all points in $F^{-1}(z)\backslash\{0\}$. If one of these points has multiplicity k, then μ has the following form in a neighborhood of z: $p(z')\,dx\,dy$, $z' = x + iy$, where $p(z') \sim c|z - z'|^{2(1-k)}$ as $z' \to z$. Therefore, all points $\zeta \in F^{-1}(z)\backslash\{0\}$ hav ethe same multiplicity.

Thus, to every point $z \in \overline{\mathbf{C}}$ there corresponds a positive integer $n(z)$ such that all points $\zeta \in F^{-1}(z)\backslash\{0\}$ have multiplicity $n(z)$. Note that $F|_{\mathbf{C}^*}$ takes all values in $\overline{\mathbf{C}}$. If this were not so, then f_j would have a nonempty exceptional set, which is impossible because $J = \overline{\mathbf{C}}$.

We consider the orbifold $\mathscr{O} = (\overline{\mathbf{C}}, n)$ and show that the $f_j : \mathscr{O} \to \mathscr{O}$ are covers. Take an arbitrary point $z \in \overline{\mathbf{C}}$ and let $\zeta \in F^{-1}(z)\backslash\{0\}$. Define $z_j = f_j(z)$ and $\zeta_j = \Lambda_j\zeta$. Then $F(\zeta_j) = z_j$ by virtue of (3.1) and (3.2). It follows from the same equations that $\deg_z f_j \cdot \deg_\zeta F = \deg_{\zeta_j} F$, i.e., $\deg_z f_j \cdot n(z) = n(z_j)$, $j = 1, 2$, which is what was required.

By the Riemann-Hurwitz formula $\deg f_1 \cdot \chi(\mathscr{O}) = \chi(\mathscr{O})$, which implies that $\chi(\mathscr{O}) = 0$. This proves Theorem 1 in the first case.

As mentioned in the Introduction, the second case was analyzed by Fatou and Julia. However, the use of lamellar measures essentially simplifies the proof here also.

SECOND CASE. J is nowhere dense. Then $I = \operatorname{supp} \nu$ is also nowhere dense. It follows from (3.5) that I is either a union of logarithmic spirals (in particular, rays) starting from zero and disjoint in $\mathbf{C}^*$, or a union of the point 0 itself and circles about zero. In any case an arbitrary point $\zeta \in I \backslash \{0\}$ has a neighborhood V such that $I \cap V$ is diffeomorphic to the product of an open interval and some closed nowhere dense subset of a closed interval.

We show that I is a line or ray. Note that $z_0 = F(0) \notin E(f_j)$. This follows from the description of the exceptional set $E(f)$ in the Introduction and the fact that z_0 is a repelling fixed point. Therefore, the Poincaré function F takes the value z_0 infinitely many times. Take a point $\zeta \in F^{-1}(z_0)$, $\zeta \neq 0$. Suppose that W is a sufficiently small neighborhood of z_0 and let U_1 and U_2 be the components of $F^{-1}W$ containing the points 0 and ζ, respectively. The neighborhoods are chosen so that the restriction $F|_{U_1}$ is univalent (recall that $F'(0) = 1$), while the restriction $F|_{U_2}$ does not have critical points other than perhaps the point ζ. Obviously, $\zeta \in I$, because $F(\zeta) = z_0 \in J$. The component of $I \cap U_2$ containing ζ is a simple analytic curve. Therefore, the component of $J \cap W$ containing z_0 is also a simple analytic curve (if ζ is a critical point, then this curve ends at z_0 but has a definite tangent there). Since $F \colon U_1 \to V$ is a univalent conformal mapping, the component of $U_1 \cap I$ containing 0 is a simple analytic curve that possibly ends at zero but has a tangent there. By the above description of the structure of I, this is possible only if I is a line or ray.

For simplicity we reduce both cases to a single case. Let $F_1(\zeta) = F(c\zeta^m)$, where $m = 1$ if I is a line and $m = 2$ if I is a ray, and the number c is chosen so that the set $I_1 = F_1^{-1}(J)$ is the real line $\mathbf{R}$. We have

$$F_1(\rho_j \zeta) = f_j \circ F_1(\zeta), \qquad \rho_j^m = \lambda_j. \tag{3.6}$$

Let $\nu_1 = F_1^* \mu$, $\operatorname{supp} \nu_1 = \mathbf{R}$. It follows from (3.5) that the number a in (3.4) and (3.5) is real. Without loss of generality we assume that $a > 0$. Then it follows from (3.4) that $\nu_1(tE) = t^\sigma \nu_1(E)$ for all $t \in \mathbf{R}^+$, all Borel sets $E \subset \mathbf{R}$, and some $\sigma > 0$. Therefore, the measure ν_1 has the form

$$c|x|^{\sigma - 1}\, dx. \tag{3.7}$$

We show that $\sigma = 1$. Let us first see that J is a simple analytic curve (closed or not). Suppose that $z \in J$, $\zeta \in F_1^{-1}(z)$, V is a neighborhood of z, and $U \subset F_1^{-1}(V)$ is a neighborhood of ζ. Choose V small enough that F_1 does not have critical points in $U \backslash \zeta$. Moreover, we assume, without loss of generality, that U is convex. Then $U \cap \mathbf{R}$ is an open interval. If ζ is not a critical point, then $V \cap J$ is a simple open arc. If ζ is a critical point, then $\deg_\zeta F_1 = 2$ and $V \cap J$ is a half-open arc with endpoint at z. This implies that J is a simple analytic curve. If J is a closed curve, then F_1 does not have critical points in $\mathbf{R}$. If J is a nonclosed curve with endpoints z_1 and z_2, then the set of critical points of F_1 on $\mathbf{R}$ coincides with $F_1^{-1}(\{z_1, z_2\})$ and all these critical points have multiplicity 2. Suppose now that $\zeta_1 \in F_1^{-1}(z_0)$, $\zeta_1 \neq 0$ (here $z_0 = F_1(0)$). It follows from the foregoing that $\deg_0 F_1 = \deg_{\zeta_1} F_1$; therefore, there exists a conformal mapping φ of a neighborhood of zero onto a neighborhood of ζ_1 such that $F_1(\zeta) = F_1(\varphi(\zeta))$, $\varphi(0) = \zeta_1$. It follows from the definition of ν_1 that φ preserves ν_1. Therefore, the density of ν_1 with respect to the measure dx cannot vanish nor be infinite at 0, because it does not vanish nor equal ∞ at the point ζ_1. This proves that $\sigma = 1$ in (3.6), and the measure ν_1 is proportional to dx.

Let ζ_1 and ζ_2 be arbitrary distinct inverse images of an arbitrary point $z \in J$. As above, there exists a germ of a conformal mapping φ such that $\varphi(\zeta_1) = \zeta_2$ and $F_1(\zeta) = F_1(\varphi(\zeta))$ in a neighborhood of ζ_1. This mapping φ preserves the real axis and Lebesgue measure on it. Therefore, $\varphi(\zeta) = \pm\zeta + T$. This implies that the points glued together by F_1 belong to the orbits of some group Γ consisting of transformations $z \mapsto \pm z + T$, $T \in \mathbf{R}$. In other words, F_1 is a universal cover of one of the orbifolds of type (1.1). Together with (3.6) this finishes the proof.

§4. The Julia equation

We prove Theorem 2. Let $n = \deg g$ and $m = \deg f$. Assume that g is a polynomial. (According to the Introduction, this leaves out only the case $g(z) = z^{-n}$, $n \in \mathbf{N}\backslash\{1\}$, which can, however, be reduced to the case of a polynomial $g^2(z) = z^{n^2}$ by passing to the equation $G \circ g^2 = f^2 \circ G$.) We need the

THEOREM OF BOETTCHER ([3], [9], [10], [17]). *For each polynomial g of degree $n \geq 2$ there exists a neighborhood D of ∞ and a simply connected conformal mapping $B: \Delta \to D$, where $\Delta = \{z \in \overline{\mathbf{C}}: |z| > r\}$, $r > 1$, such that*

$$B(z^n) = g \circ B(z), \qquad z \in \Delta. \tag{4.1}$$

Let $\varphi = G \circ B \circ \exp: H \to \overline{\mathbf{C}}$, where $H = \{z: \operatorname{Re} z > \log r\}$:

$$\begin{array}{ccc|l}
H & \xrightarrow{z \mapsto nz} & H & \\
\exp\downarrow & & \downarrow\exp & \\
\Delta & \xrightarrow{z \mapsto z^n} & \Delta & \\
B\downarrow & & \downarrow B & \varphi \\
D & \xrightarrow{g} & D & \\
G\downarrow & & \downarrow G & \\
\overline{\mathbf{C}} & \xrightarrow{f} & \overline{\mathbf{C}} & \downarrow
\end{array}$$

The function φ is meromorphic on H and satisfies the equation

$$\varphi \circ N = f \circ \varphi, \qquad N: z \mapsto nz, \tag{4.2}$$

which follows from (0.4) and (4.1). Moreover, it is obvious that

$$\varphi \circ T = \varphi, \quad \text{where } T\colon z \mapsto z + 2\pi i. \tag{4.3}$$

We remark that $\varphi\colon H \to \overline{\mathbf{C}}$ is surjective, since G takes all values in $\overline{\mathbf{C}}$ infinitely many times (see the Introduction). Let $\mu = \mu_f$ be the equilibrium measure, and let $\nu = \varphi^*\mu$. Then $\operatorname{supp}\nu = H$, because $\operatorname{supp}\mu = J(f) = \overline{\mathbf{C}}$ in view of the result of Julia given in the introduction. By (4.2), $N^*\varphi^* = \varphi^* f^*$. Considering that $f^*\mu = m\mu$, we get

$$N^*\nu = m\nu, \tag{4.4}$$

and it follows from (4.3) that

$$T^*\nu = \nu. \tag{4.5}$$

It follows from (4.4) and (4.5) that ν is invariant with respect to the transformations $N^{-k}TN^k\colon z \mapsto z + 2\pi i/n^k$, $k \in \mathbf{Z}$. Consequently, it is invariant with respect to the closed group generated by these transformations, i.e.,

$$T_a^*\nu = \nu, \qquad a \in \mathbf{R},$$

where $T_a\colon z \mapsto z + ia$. Thus, ν is lamellar in H.

Repeating word-for-word the arguments in the proof of Theorem 1 (the first case; the role of $F\colon \mathbf{C}^* \to \overline{\mathbf{C}}$ is played by $\varphi\colon H \to \overline{\mathbf{C}}$), we get that there exists an orbifold $\mathscr{O}_2$ of type (1.2) such that $f\colon \mathscr{O}_2 \to \mathscr{O}_2$ is a cover.

Suppose now that F_1 is the Poincaré function for g that corresponds to a fixed point z_0, and $G'(z_0) \neq 0$. (Such a fixed point z_0 can be found because the repelling periodic points are dense in the Julia set; if necessary, replace g and f by certain of their iterates.) We have

$$F_1 \circ \Lambda = g \circ F_1, \quad F_1(0) = z_0, \quad F_1'(0) = 1, \quad \Lambda\colon z \mapsto g'(z_0)\cdot z. \tag{4.6}$$

Let $F = G \circ F_1$. Then

$$F \circ \Lambda = G \circ g \circ F_1 = f \circ G \circ F_1 = f \circ F,$$
$$F(0) = G(z_0), \qquad F'(0) = G'(z_0) \neq 0,$$

i.e., F is proportional to a Poincaré function for f. In particular, F is meromorphic on $\mathbf{C}$. (This was not at once completely obvious, because G is meromorphic only on $\mathbf{C}^*$). We show that $F\colon \mathbf{C} \to \mathscr{O}_2$ is a universal cover. Choose a universal cover $\Phi\colon \mathbf{C} \to \mathscr{O}_2$ with the conditions $F(0) = \Phi(0)$ and $F'(0) = \Phi'(0)$. Then $f \circ \Phi$ is also a universal cover; therefore, there exists a linear function L with the properties $f \circ \Phi = \Phi \circ L$ and $L(0) = 0$. We have $L'(0) = \Lambda'(0)$, since $f' \circ \Phi(0) = f' \circ F(0)$. Consequently, $L = \Lambda$ and $\Phi = F$, because a normalized solution of the Poincaré equation is unique.

Thus, $F\colon \mathbf{C} \to \mathscr{O}_2$ is a universal cover of orbifolds. Suppose that $\mathscr{O}_2 = (\overline{\mathbf{C}}, n_2)$ and $\Sigma_2 = \{z\colon n_2(z) > 1\}$. It follows from the relation

$$F = G \circ F_1 \tag{4.7}$$

that G is a branched cover over $\overline{\mathbf{C}}$ that can be branched only over Σ_2, and all the G-inverse images of a point $e \in \Sigma_2$ have multiplicity dividing $n_2(e)$. Therefore, we can introduce an orbifold $\mathscr{O}_1 = (\mathbf{C}, n_1)$ such that $G\colon \mathscr{O}_1 \to \mathscr{O}_2$

is a cover of orbifolds. In other words, we let

$$n_1(z) = \frac{n_2(G(z))}{\deg_z G}, \qquad z \in \mathbf{C}.$$

If G is meromorphic only in $\mathbf{C}^*$, then we set $n_1(0) = \infty$. Denote by Σ_1 the set of points at which $n_1(z) > 1$. It follows from (4.7) that F_1 is unbranched over $\mathbf{C}\backslash\Sigma_1$ and all the inverse images of a point $z \in \Sigma_1$ have multiplicity $n_1(z)$. It remains to show that $g: \mathscr{O}_1 \to \mathscr{O}_1$ is a cover of orbifolds. This can be done exactly as in the proof of the first case of Theorem 1. Instead of (3.1) and (3.2) we use (4.6). Next, $\deg g \cdot \chi(\mathscr{O}_1) = \chi(\mathscr{O}_1)$ implies that $\chi(\mathscr{O}_1) = 0$. Thus, $\mathscr{O}_1$, regarded as an orbifold $(\overline{\mathbf{C}}, n_1)$ with $n_1(\infty) = \infty$, has type (1.1). The theorem is proved.

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