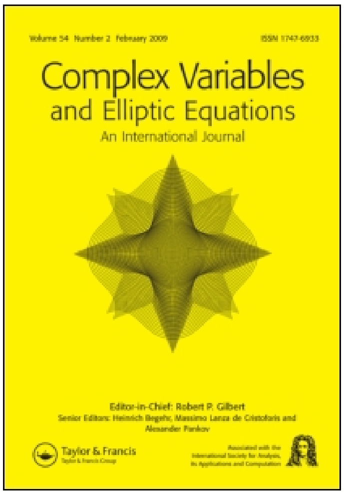

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Some Asymptotic Properties of Meromorphic Functions*

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In §1 of this paper a connection between the order of a meromorphic function and its deficiency in the sense of R. Nevanlinna [1] is studied. For this purpose a class of meromorphic functions with certain extremal property of definciencies is singled out, which permits to obtain as simple corollaries generalizations of several known theorems.

In §2 a connection between critical points of a Riemann surface and the order of growth of the associated meromorphic function is studied. A generalization of the known theorem of Denjoy-Carleman-Ahlfors is obtained. This generalization consists in adding some subclass of indirect critical points into consideration, in addition to the direct ones.

There is no direct connection between critical points and deficient values, first because it is well known that a critical point may have deficiency zero, and second, since there exist examples of deficient values to which no critical points correspond [2,3]. Nevertheless there is an inner connection between the questions considered in §1 and §2. Let us explain this with an example.

A known theorem of Wiman (see, for example [4, p. 301]) states that for entire functions of order $\rho < \frac{1}{2}$ there exists a sequence of circles

*Russian original: Uchenye Zapiski Lvovskogo Gosudarstvennogo Universiteta, 38, 7, 1956, 54-74. Translated by A. Eremenko

$|z| = r_n$, $n \to \infty$, on which the function uniformly tends to ∞. There are several ways to generalize this theorem. First, an entire function has at ∞ a deficient (even Picard) value $\delta(\infty) = 1$. By relaxing this condition we obtain in §1 the following proposition: for a meromorphic function $f(z)$ of order $\rho < \frac{1}{2}$, having $\delta(a) > 1 - \cos \pi\rho$, there exists a sequence of circles $|z| = r_n$, $r_n \to \infty$, on which $f(z)$ uniformly tends to a. The condition on deficiency cannot be relaxed. On the other hand the Riemann surface of an entire function has a direct critical point of a special kind over ∞. By generalizing the Wiman theorem in this direction one can introduce a class of critical points — K-points (the precise definition is in §2), for which the following statement is still true: if the Riemann surface of a meromorphic function of order $\rho < \frac{1}{2}$ has a K-point over a then there exists a sequence of circles $|z| = r_n$, $r_n \to \infty$, on which $f(z)$ uniformly tends to a.

So §1 and §2 give two different very natural approaches to the same problem.[1]

§2

The known theorem of Denjoy-Carleman-Ahlfors (see [1, p. 258–262]) states: *a meromorphic function $w = f(z)$ mapping the plane $z \neq \infty$ onto a Riemann surface F having n direct critical points, has order of growth at least $n/2$, normal type. Here if $n = 1$ the surface is assumed to be decomposable* (this as a rule will not be specially mentioned in what follows); that is F contains a closed Jordan curve C such that over the regions into which it divides the w-plane lie at least two connected infinitely-sheeted pieces of F [10]. In a series of papers [11,12,13,14,15] (I am familiar with their contents only by reviews) Y. Tumura and M. Tsuji singled out from the class of indirect critical points a subclass for which it turns out that the Denjoy-Carleman-Ahlfors theorem is still true, but to establish whether a given critical point belongs to this subclass is possible only by constructing a conformal mapping of the whole Riemann surface on the finite z-plane. In this section a class of critical points is introduced (K-points) which is wider than the class of direct critical points, and for which the Denjoy-Carleman-Ahlfors Theorem remains true. In addition,

[1] §1 is omitted in this translation. Its contents are widely known to specialists and it is contained in [19], [17] and [18] with substantial simplifications.

to classify a critical point as a K-point one has to consider only a neighborhood of this point, which often permits to formulate the criterion in pure geometric terms.

Let a critical point $\mathfrak{U}$ lie over a. Let us map its ε-neighborhood (that is the connected part of the Riemann surface containing $\mathfrak{U}$ and lying over $|w-a|<\varepsilon$ (or $|w|>1/\varepsilon$ if $a=\infty$) onto some region D bounded by Jordan arcs in ζ-plane with a function $w=\psi(\zeta)$. Let us denote by $G_D(\zeta, a_\nu)$ the Green function of the region D with the pole at a_ν which is an a-point of the function $\psi(\zeta)$. Then using the Maximum Principle and Harnack's Theorem it is easy to obtain the following equality

$$\ln\left|\frac{\varepsilon}{\psi(\zeta)-a}\right| = \sum_{\nu=1}^{\infty} G_D(\zeta, a_\nu) + h_\varepsilon(\zeta),$$

where $h_\varepsilon(\zeta)$ is a non-negative harmonic function, and independently from the choice of $\varphi(\zeta)$ we either have $h_\varepsilon(\zeta)\equiv 0$ or $h_\varepsilon(\zeta)>0$, $\zeta\in D$; $\lim_{\zeta\to\zeta'} h_\varepsilon(\zeta)=0$, ζ' being any boundary point of D corresponding to an interior point of F.

DEFINITION If for every small enough ε-neighborhood of the critical point $\mathfrak{U}$ we have $h_\varepsilon(\zeta)>0$ then $\mathfrak{U}$ is called a K-point.

Evidently the membership of $\mathfrak{U}$ in the class of K-points is determined by the properties of the Riemann surface only in sufficiently small neighborhoods of $\mathfrak{U}$. Finding K-points is faciliated by the following two theorems.

THEOREM 4. *If an ε-neighborhood of $\mathfrak{U}$ contains an ε'-neighborhood of $\mathfrak{U}$ and $h_{\varepsilon'}(\zeta)>0$, then $h_\varepsilon(\zeta)>0$.*

Proof Without loss of generality one may assume that $a=\infty$. If $w=\psi(\zeta)$ maps the ε-neighborhood on a region D_ε, then the same function maps the ε'-neighborhood onto some $D_{\varepsilon'}\subset D_\varepsilon$. Assume that $h_\varepsilon(\zeta)\equiv 0$, that is

$$\ln|\varepsilon\psi(\zeta)| = \sum_{\nu=1}^{\infty} G_\varepsilon(\zeta, p_\nu) \qquad (G_\varepsilon\equiv G_{D_\varepsilon}),$$

$$\ln|\varepsilon'\psi(\zeta)| = \ln|\varepsilon\psi(\zeta)| + \ln\frac{\varepsilon'}{\varepsilon} = \sum_{\nu=1}^{\infty} G_\varepsilon(\zeta, p_\nu) + \ln\frac{\varepsilon'}{\varepsilon}.$$

On the part of the boundary of $D_{\varepsilon'}$ which is inside D_ε one has $\ln|\varepsilon'\psi(\zeta)| = 0$ and

$$\sum_{\nu=1}^{\infty} G_\varepsilon(\zeta, p_\nu) = -\ln\frac{\varepsilon'}{\varepsilon} = \ln\frac{\varepsilon}{\varepsilon'}, \tag{1}$$

$$\sum_{\nu=1}^{\infty} G_{\varepsilon'}(\zeta, p_\nu) \le \ln|\varepsilon'\psi(\zeta)| = \sum_{\nu=1}^{\infty} G_\varepsilon(\zeta, p_\nu) + \ln\frac{\varepsilon'}{\varepsilon}, \tag{2}$$

where $G_{\varepsilon'}(\zeta, p_\nu) \equiv 0$, if $p_\nu \notin D_{\varepsilon'}$. On the other hand,

$$\sum_{\nu=1}^{n} G_{\varepsilon'}(\zeta, p_\nu) \ge \sum_{\nu=1}^{n} G_\varepsilon(\zeta, p_\nu) - \ln\frac{\varepsilon}{\varepsilon'}, \quad \zeta \in D_{\varepsilon'},$$

because on the whole boundary of $D_{\varepsilon'}$ the left side is $\equiv 0$ and the right side is ≤ 0 by (1), and in addition the left and right sides have the same poles and the same singular parts at these poles. Passing to the limit when $n \to \infty$ we obtain

$$\sum_{\nu=1}^{\infty} G_{\varepsilon'}(\zeta, p_\nu) \ge \sum_{\nu=1}^{\infty} G_\varepsilon(\zeta, p_\nu) + \ln\frac{\varepsilon'}{\varepsilon},$$

and comparing with (2) we obtain

$$\sum_{\nu=1}^{\infty} G_{\varepsilon'}(\zeta, p_\nu) = \sum_{\nu=1}^{\infty} G_\varepsilon(\zeta, p_\nu) + \ln\frac{\varepsilon'}{\varepsilon} = \ln|\varepsilon'\psi(\zeta)|.$$

Consequently $h_{\varepsilon'}(\zeta) \equiv 0$ which contradicts the assumption.

THEOREM 5. *If for some ε-neighborhood lying over $|w - a| < \varepsilon$ we have $h_\varepsilon(\zeta) > 0$, then this neighborhood contains at least one K-point lying over a.*

Proof Set $a = \infty$, $\ln|\varepsilon\psi(\zeta)| = \sum G_\varepsilon(\zeta, p_\nu) + h_\varepsilon(\zeta)$, $h_\varepsilon(\zeta) > 0$. By the extended Maximum Modulus Principle there has to exist a point ζ' on the boundary of D_ε such that for some sequence $\zeta_k \to \zeta'$, $\zeta_k \in D_\varepsilon$ we have $h_\varepsilon(\zeta_k) \to \infty$. Let to the points from the ε-neighborhood which lie over $|w - a| < \varepsilon/2$ correspond an open set $D'_{\varepsilon/2} \subset D_\varepsilon$. As $\lim_{\zeta_k \to \zeta'} |\psi(\zeta_k)| = \infty$, there exists a connected component of $D'_{\varepsilon/2}$ containing a point ζ_k such that $h_\varepsilon(\zeta_k) > \ln 2$. To this component of $D'_{\varepsilon/2}$

corresponds an $\varepsilon/2$-neighborhood of the critical point. We have

$$\begin{aligned} h_{\varepsilon/2}(\zeta) &= \ln\left|\frac{\varepsilon}{2}\psi(\zeta)\right| - \sum G_{\varepsilon/2}(\zeta, p_\nu) \\ &= \ln|\varepsilon\psi(\zeta)| - \sum G_{\varepsilon/2}(\zeta, p_\nu) - \ln 2 \\ &\geq \ln|\varepsilon\psi(\zeta)| - \sum G_{\varepsilon}(\zeta, p_\nu) - \ln 2 = h_\varepsilon(\zeta) - \ln 2. \end{aligned}$$

But $h_{\varepsilon/2}(\zeta_k) \geq h_\varepsilon(\zeta_k) - \ln 2 > 0$ thus $h_{\varepsilon/2}(\zeta) > 0$. We can apply the same argument to the $\varepsilon/2$-neighborhood. Thus we obtain a sequence of nested $\varepsilon 2^{-m}$-neighborhoods which defines a critical point $\mathfrak{U}$. As $h_{\varepsilon 2^{-n}}(\zeta) > 0$, $n = 1, 2, \ldots$, by Theorem 4 $\mathfrak{U}$ is a K-point.

THEOREM 6. *A meromorphic function $w = f(z)$ which maps the plane $z \neq \infty$ onto the Riemann surface F having n K-points, has the order of growth at least $n/2$, normal type. (We assume in the case $n = 1$ that the surface is decomposable.)*

Proof Choose so small disjoint ε_ν-neighborhoods of K-points $\mathfrak{U}_\nu$, $\nu = 1, \ldots, n$ that their preimages Δ_ν in z-plane do not contain $z = 0$ and there are curves connecting $z = 0$ to each region Δ_ν without entering the the other regions. Let γ_ν be the component of the boundary of Δ_ν which separates Δ_ν from $z = 0$. It divides z-plane into two unbounded regions, the one which contains Δ_ν we denote by $\bar{\Delta}_\nu$. Fix a point $z_\nu \in \Delta_\nu$. Let Θ_r^ν and $\bar{\Theta}_r^\nu$ be the arcs of the circle $|z| = r$, $r > |z_\nu|$ which separate the part of Δ_ν, respectively $\bar{\Delta}_\nu$, containing z_ν, from ∞, and let $\Theta_\nu(r)$ and $\bar{\Theta}_\nu(r)$ be the radian measures of these arcs. Denote by D_r ($\bar{D}_r$) the component of the intersection of Δ_ν ($\bar{\Delta}_\nu$) with the disk $|z| < r$ containing z_ν. Using the Maximum Principle we obtain

$$0 < h_{\varepsilon_\nu}(z_\nu) \leq \max_{|z|=r} h_{\varepsilon_\nu}(z)\omega(z_\nu, \Theta_r^{(\nu)}, D_r), \tag{3}$$

where $\omega(z_\nu, \Theta_r^{(\nu)}, D_r)$ is the harmonic measure of $\Theta_r^{(\nu)}$ with respect to D_r measured at the point z_ν; for constructing $h_{\varepsilon_\nu}(z); D_{\varepsilon_\nu} \equiv \Delta_\nu$ and $\psi(z) \equiv f(z)$ were taken. Set $F_\nu(z) = (\varepsilon_\nu/(f(z) - a_\nu))$; since $T(r, F_\nu) = T(r, f) + O(1)$, it is enough to estimate $T(r, F_\nu)$. Let us denote $M(r, F_\nu) = \max_{|z|=r} |F_\nu(z)|$. Then

$$\max_{|z|=r} h_\varepsilon(z) \leq \max_{|z|=r, z\in\Delta_\nu} \ln\left|\frac{\varepsilon_\nu}{f(z) - a_\nu}\right| \leq \ln^+ M(r, F_\nu)$$

and from (3) we obtain

$$0 < C < \ln^{+} M(r, F_\nu)\omega(z_\nu, \Theta_r^{(\nu)}, D_r). \tag{4}$$

From this, using the Principle of Region Extension and the inequality

$$\frac{1}{r}\int_0^r \ln^{+} M(t, F_\nu)\,dt < C(k)T(kr, F_\nu) \tag{5}$$

(see [6], p. 25) and integrating (4), we obtain that

$$\begin{aligned} 0 < \frac{C}{2} &\le \frac{1}{2r}\int_r^{2r} \ln^{+} M(t, F_\nu)\omega(z_\nu, \Theta_t^{(\nu)}, D_t)\,dt \\ &\le \omega(z_\nu, \Theta_r^{(\nu)}, D_r)\frac{1}{2r}\int_0^{2r} \ln^{+} M(t, F_\nu)\,dt \\ &\le C_1\omega(z_\nu, \Theta_r^{(\nu)}, D_r)T(2r, F_\nu), \end{aligned}$$

$$T(2r, F_\nu) \ge \frac{K}{\omega(z_\nu, \Theta_r^{(\nu)}, D_r)}, \quad K > 0. \tag{6}$$

Now we estimate from above $\omega(z_\nu, \Theta_r^{(\nu)}, D_r) \le \omega(z_\nu, \bar{\Theta}_r^{(\nu)}, \bar{D}_r)$. Then estimating $\omega(z_\nu, \bar{\Theta}_r^{(\nu)}, \bar{D}_r)$ in exactly the same way as in the proof of the Denjoy-Carleman-Ahlfors theorem (see [1], sect. 258-261), we get

$$\ln T(2r) > \pi \int_{r_0}^{r} \frac{dr}{r\bar{\Theta}_\nu(r)} + \text{ const}, \tag{7}$$

from which, using the inequality between harmonic and arithmetic means we obtain

$$\ln T(r) \ge \frac{n}{2}\ln r + \text{ const}, \quad T(r) \ge \text{ const } \cdot r^{n/2} > 0. \tag{8}$$

In this proof the requirement that for $n = 1$ the surface should be decomposable (for $n > 1$ this is automatic), was used in an essential way when constructing γ_ν. Nevertheless this condition can be relaxed, though by doing this one loses the possibility to verify the condition directly by inspecting the Riemann surface F. Without loss of generality we can assume that the only K-point of F lies over ∞. Then *if*

$$\min_{|z|=r} |f(z)| < M < \infty, \quad 0 < r < \infty, \quad \text{then} \quad T(r, f) > \text{ const } \sqrt{r}.$$

In view of Theorem 6 it is enough to consider the case when F is not decomposable. Let Δ be the preimage in z-plane of an ε-neighborhood of $\mathfrak{U}$, $\varepsilon < M^{-1}$. Then every circle $|z| = r$, $r_0 < r < \infty$ intersects the boundary of Δ. Though in the case of non-decomposable surface one cannot construct γ dividing the plane into two unbounded regions, we still have by the Carleman-Milloux theorem (see [1], sect. 82) that $\omega(z_1, \Theta_r, D_r) \leq \omega(|z_1|, |z| = r, K_r)$, where K_r is the disk $|z| < r$ from which the radius $(-r, 0]$ is removed. The harmonic measure $\omega(|z_1|, |z| = r, K_r)$ is easy to estimate: $\omega(|z_1|, |z| = r, K_r) < Cr^{-1/2}$, where C depends only on $|z_1|$. Together with (6) this proves the statement. From here one deduces the following generalization of Wiman's theorem (see the Introduction).

THEOREM 7. *If the Riemann surface, onto which a meromorphic function $w = f(z)$ of order $\rho < \frac{1}{2}$ maps the plane $z \neq \infty$, has a K-point over a then there exists a sequence of circles $|z| = r_n \to \infty$ on which $f(z)$ tends uniformly to a.*

Let us give several theorems which permit to decide whether a given critical point is a K-point.

THEOREM 8. *If $\mathfrak{U}$ is a direct critical point then it is a K-point.*

Indeed, in this case $h_\varepsilon(\zeta) = \ln\left|\dfrac{\varepsilon}{\psi(\zeta) - a}\right| > 0.$

THEOREM 9. *If an ε-neighborhood of $\mathfrak{U}$ is simply connected then for $\mathfrak{U}$ to be a K-point it is necessary and sufficient that the function $\varphi(\zeta) = (\psi(\zeta) - a)/\varepsilon$, where $w = \psi(\zeta)$ is a conformal map of $|\zeta| < 1$ onto the ε-neighborhood, be not a Blaschke product; that is*

$$\varphi(\zeta) \not\equiv B(\zeta) = e^{i\Theta} \prod_{\nu=1}^{\infty} \frac{|a_\nu|}{a_\nu} \cdot \frac{a_\nu - \zeta}{1 - \bar{a}_\nu \zeta}.$$

Indeed, for $\mathfrak{U}$ not to be a K-point it is necessary and sufficient that

$$\ln\left|\frac{1}{\varphi(\zeta)}\right| = \ln\left|\frac{\varepsilon}{\psi(\zeta) - a}\right| = \sum_{\nu=1}^{\infty} G(\zeta, a_\nu),$$

where $G(\zeta, a_\nu)$ is the Green function for the unit disk; that is

$$G(\zeta, a_\nu) = \ln\left|\frac{1 - \bar{a}_\nu \zeta}{a_\nu - \zeta}\right|,$$

but this is equivalent (see [1], sect. 150) to $\varphi(\zeta) \equiv B(\zeta)$.

In typical cases (for example for all entire functions of finite order and $a \neq \infty$) for ε small enough, ε-neighborhoods of $\mathfrak{U}$ are simply connected and bounded by a single Jordan arc. Then the question of whether $\mathfrak{U}$ is a K-point is reduced to finding out if $\varphi(\zeta)$ is a Blaschke product with zeros tending to a single point. With the additional assumption that *all zeros of* $\varphi(\zeta)$ *are real* we will give a necessary and sufficient condition in geometric terms for given simply connected Riemann surface S over the unit disk (one can pass from $|w-a| < \varepsilon$ to $|w| < 1$ by a linear transformation) that the function $w = \varphi(\zeta)$ mapping this surface onto $|\zeta| < 1$ not be a Blaschke product.

Let $w = \varphi(\zeta)$ be a function with zeros at $a_1 < a_2 < \cdots \to 1$ of orders $m-1, m_2, \ldots$ respectively, which maps $|\zeta| < 1$ into $|w| < 1$ such that $\lim_{\rho\to 1} |\varphi(\rho e^{i\theta})| = 1$, $\theta \neq 0$. Then (see [1], sect. 160)

$$w = \varphi(\zeta) = e^{\delta\frac{\zeta+1}{\zeta-1}} B(\zeta) = e^{\delta\frac{\zeta+1}{\zeta-1}} \prod_{\nu=1}^{\infty} \left(\frac{a_\nu - \zeta}{1 - a_\nu \zeta}\right)^{m_\nu},$$

where

$$\sum_{\nu=1}^{\infty} m_\nu (1 - a_\nu) < \infty, \quad \delta \geq 0.$$

Let us map $|\zeta| < 1$ onto $\Re z > 0$ with $z = (1+\zeta)/(1-\zeta)$. Then

$$w = \varphi(\zeta(z)) = \Phi(z) = e^{-\delta z} \prod_{\nu=1}^{\infty} \left(\frac{r_\nu - z}{r_\nu + z}\right)^{m_\nu},$$

where

$$r_\nu = \frac{1+a_\nu}{1-a_\nu} \quad \text{and} \quad \sum_{\nu=1}^{\infty} \frac{m_\nu}{r_\nu} < \infty.$$

After continuation of this function across $\Re z = 0$ we obtain $w = \Phi(z)$- a function meromorphic in the entire finite z-plane.

Let us show that all algebraic branch points of S lie over the real axis; it is enough to show that all zeros of $\Phi'(z)$ in $\Re z > 0$ are real. Indeed if a zero of $\Phi'(z)$ is not a zero of $\Phi(z)$ then it is a zero of Φ'/Φ. Now

$$\frac{\Phi'(z)}{\Phi(z)} = -\delta - \sum_{\nu=1}^{\infty} \frac{2 r_\nu m_\nu}{r_\nu^2 - z^2}. \tag{9}$$

Let $z = x + iy$. Then

$$\Im \frac{\Phi'(z)}{\Phi(z)} = -4xy \sum_{\nu=1}^{\infty} \frac{r_\nu m_\nu}{(r_\nu^2 + y - x^2)^2 + 4x^2y^2},$$

but this expression can be zero only if $y = 0$.

Now let us show that the arc in S which projects to the real axis and which connects in S the two adjacent branch points not lying over zero, itself passes over zero. In addition we show that all branch points not lying over zero have order one.

For this it is enough to show that zeros of $\Phi'(z)$ interlace with zeros of $\Phi(z)$ and that in the case that zeros of $\Phi'(z)$ do not coincide with zeros of $\Phi(z)$ they have multiplicity 1. Indeed,

$$\frac{d}{dx}\left\{\frac{\Phi'(x)}{\Phi(x)}\right\} = -\sum_{\nu=1}^{\infty} \frac{4r_\nu m_\nu x}{(r_\nu^2 - x^2)^2} < 0 \quad \text{for } x > 0.$$

Consequently, in each interval $(r_\nu, r_{\nu+1})$ the function $\Phi'(x)/\Phi(x)$ decreases monotonically changing its sign at the same time, so it has exactly one simple zero on this interval. This proves the statement.

We conclude that the surface S corresponding to the function $\varphi(\zeta)$ has the following structure.

Take the sequence consisting of disks $U = \{w : |w| < 1\}$ with cuts from -1 to $b'_\nu < 0$ and from $b''_\nu > 0$ to $+1$. Denote the ν-th slit disk by S_ν. Let S'_ν be m_ν-sheeted Riemann surface, which is the image of S_ν under the mapping $w \mapsto \delta_\nu w^{m_\nu}$ where $\delta_\nu = -1$ if $(-1)^{m_1+m_2+\dots+m_{\nu-1}} < (-1)^{m_\nu}$, and $\delta_\nu = 1$ otherwise, $\nu \geq 2$. Thus S'_ν is the part of the Riemann surface of the function w^{1/m_ν} which lies over U, has a branch point of order $(m_\nu - 1)$ over $w = 0$ and two cuts I_ν^- and I_ν^+ (only one cut I_1^- over $(-1)^{m_1}R_+$ when $\nu = 1$). The cuts connect certain points over $\delta_\nu(b'_\nu)^{m_\nu}$ and $\delta_\nu(b''_\nu)^{m_\nu}$ each to the closest point on the boundary of S'_ν. We will assume that b'_ν and b''_ν are chosen in such a way that the cut $I_{2\nu}^-$ on $S'_{2\nu}$ has the same projection as the cut $I_{2\nu-1}^-$. Similarly the cut $I_{2\nu}^+$ has the same projection as the cut $I_{2\nu+1}^+$. Now we paste together $S'_{2\nu}$ with $S'_{2\nu-1}$ by identifying the opposite edges of the cuts $I_{2\nu}^-$ and $I_{2\nu-1}^-$. Similarly we paste together $S'_{2\nu}$ and $S'_{2\nu+1}$ by identifying the opposite edges of $I_{2\nu}^+$ and $I_{2\nu+1}^+$, $\nu \geq 1$. As the result we obtain the Riemann surface S spread over U and having infinitely many branch points of first order over $(U\backslash\{0\})\cap R$ and possibly some branch points over 0. O. Teichmüller [2] proved that

$\varphi(\zeta)$ maps $|\zeta| < 1$ on such a Riemann surface S. In particular, if $m_\nu = 1$ for all ν then the Riemann surface S has no branch points over 0; we have $S'_\nu = S_\nu$ for all ν, the cuts I_ν^- and I_ν^+ project to $(-1, b'_\nu]$ and $[b''_\nu, 1)$ respectively. In addition we have $b'_{2\nu} = b'_{2\nu-1}$ and $b''_{2\nu} = b''_{2\nu+1}$.

The shortest asymptotic path on S will be a broken line passing through these branch points; it will pass through all branch points over 0. As this broken line divides S into two symmetric parts, so in turn the function $\varphi^*(\zeta)$ which maps $|\zeta| < 1$ onto S has only real zeros and these zeros converge to one point; consequently $\varphi^*(\zeta)$ belongs to the class of functions $\varphi(\zeta)$ we have been considering.

Enumerate all branch points of S as they occur on sheets of S counting the n-fold ones n times. Let their moduli be $|b_i|$. Whether S will be a neighborhood of a K-point depends only on position of algebraic branch points of the surface S. Let us prove the following theorem.

THEOREM 10. *For S to be a neighborhood of a K-point it is necessary and sufficient that*

$$\sum_{\nu=1}^{\infty} \frac{1}{\inf_{i\geq\nu}\left\{\ln\frac{1}{|b_i|}\right\}} < \infty^2 \tag{10}$$

Proof Set $\mu(r, \Phi) = \inf_{x\geq r}\{\ln|(1)/(\Phi(x))|\}$. This function $\mu(r, \Phi)$ is evidently non-decreasing and

$$\mu(r_\nu, \Phi) = \inf_{i\geq\nu}\left\{\ln\frac{1}{|b_i|}\right\}.$$

Sufficiency. Assume that S is not a neighborhood of a K-point. Then $w = \varphi(\zeta) = B(\zeta)$ and $\Phi(z) = \prod_{\nu=1}^{\infty}((r_\nu - z)/(r_\nu + z))^{m_\nu}$. From (5) follows that

$$\begin{aligned} T(2kr, \Phi) &> C(k)\frac{1}{2r}\int_0^{2r} \ln\left|\frac{1}{\Phi(x)}\right| dx \\ &> C(k)\frac{1}{2r}\int_r^{2r} \ln\left|\frac{1}{\Phi(x)}\right| dx \geq C(k)\frac{1}{2}\mu(r, \Phi). \end{aligned}$$

[2] This will prove that for S to be a Riemann surface of the Blaschke product with real zeros converging to one point it is necessary and sufficient that the series (10) is divergent — this statement has an independent interest.

But $\lim_{r\to\infty} T(r,\Phi)/r = 0$ (see [1], sect. 190); thus

$$\lim_{r\to\infty} \frac{\mu(r,\Phi)}{r} = 0. \tag{11}$$

Now we apply the Carleman formula (see [16], ch IV, §14) to the function $\Phi(z)$ regular in $\Re z > 0$:

$$\sum_{r_\nu \le R} m_\nu \left(\frac{1}{r_\nu} - \frac{r_\nu}{R^2}\right) = \frac{1}{\pi R}\int_{-\pi/2}^{\pi/2} \ln|\Phi(Re^{i\theta})|\cos\theta\, d\theta + \frac{1}{2\pi}\int_0^R \left(\frac{1}{t^2} - \frac{1}{R^2}\right)$$

$$\times \ln|\Phi(it)\Phi(-it)|\, dt - \frac{1}{2}\Re\Phi'(0).$$

But $|\Phi(it)| = |\Phi(-it)| = 1$ and

$$\Re\Phi'(0) = (\Phi'(0))/(\Phi(0)) = -2\sum_{\nu=1}^{\infty}(m_\nu)/(r_\nu) \qquad \text{(see (9))}.$$

Hence

$$\sum_{r_\nu \le R} m_\nu \left(\frac{1}{r_\nu} - \frac{r_\nu}{R^2}\right) = \frac{1}{\pi R}\int_{-\pi/2}^{\pi/2} \ln|\Phi(Re^{i\theta})|\cos\theta\, d\theta + \sum_{\nu=1}^{\infty}\frac{m_\nu}{r_\nu}$$

and

$$\sum_{r_\nu \le R} \frac{r_\nu m_\nu}{R} + \sum_{r_\nu > R} \frac{R m_\nu}{r_\nu} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \ln\left|\frac{1}{\Phi(Re^{i\theta})}\right|\cos\theta\, d\theta. \tag{12}$$

Let us denote by D_R the intersection of the regions $\{0 < \arg z < \pi/2\}$ and $\{|z| > R\}$. For $x > R$ we have $\ln|\Phi(x)| \le -\mu(R,\Phi)$, and everywhere in D_R we have $\ln|\Phi(z)| \le 0$. So by the Two Constants Theorem (see [1], sect. 36) for $z \in D_R$ we have $\ln|\Phi(z)| < -\omega(z; x > R; D_R)\mu(R,\Phi)$. Let $\lambda = \min \omega(z; x > R; D_R)$ with z on the arc $|z| = 2R$, $0 \le \arg z \le \pi/4$. Evidently $\lambda > 0$ does not depend on R. Then on the arc $|z| = 2R$, $-\pi/4 < \arg z < \pi/4$ we have

$$\ln|\Phi(z)| < -\lambda\mu(R,\Phi). \tag{13}$$

From (12) and (13) follows that

$$\sum_{r_\nu \le 2R} \frac{r_\nu m_\nu}{2R} + \sum_{r_\nu > 2R} \frac{2R m_\nu}{r_\nu} > \frac{1}{\pi}\int_{-\pi/4}^{\pi/4} \ln\left|\frac{1}{\Phi(2Re^{i\theta})}\right|\cos\theta\, d\theta$$

$$> \frac{\sqrt{2}\lambda}{4}\mu(R,\Phi) = C\mu(R,\Phi).$$

From this using simple transformations which do not decrease the left hand side of the inequality we obtain

$$\frac{1}{2}\sum_{r_\nu \le R} \frac{r_\nu m_\nu}{R} + 2\sum_{r_\nu > R} \frac{R m_\nu}{r_\nu} > C\mu(R, \Phi),$$

$$\frac{1}{2} n(R, 0, \Phi) + 2\sum_{r_\nu > R} \frac{R m_\nu}{r_\nu} > C\mu(R, \Phi). \tag{14}$$

In view of (11) there exists such a sequence $R_i \to \infty$ that $(\mu(R_i))/(R_i) > (\mu(R))/(R)$ for all $R > R_i$ and thus $(\mu(R_i))/(\mu(r_\nu)) > (R_i)/(r_\nu)$, $r_\nu > R_i$. From (14) follows that

$$\frac{1}{2} n(R_i, 0, \Phi) + 2\sum_{r_\nu > R_i} m_\nu \frac{\mu(R_i)}{\mu(r_\nu)} > C\mu(R_i), \quad C > 0,$$

$$\frac{1}{2}\frac{n(R_i, 0, \Phi)}{\mu(R_i)} + 2\sum_{r_\nu > R_i}^{\infty} \frac{m_\nu}{\mu(r_\nu)} > C > 0. \tag{15}$$

The condition (10) implies that $\sum_{\nu=1}^{\infty}(m_\nu)/(\mu(r_\nu)) < \infty$ and $\lim_{i\to\infty} \sum_{r_\nu > R_i}(m_\nu)/(\mu(r_\nu)) = 0$ since $\sum_{r_\nu > R_i}(m_\nu)/(\mu(r_\nu))$ is the tail of a convergent series. On the other hand because $1/\mu(r_\nu)$ is monotone decreasing and the series is convergent it follows that $\nu/\mu(r_\nu) \to 0$, that is

$$\frac{n(r_\nu, 0, \Phi)}{\mu(r_\nu)} \to 0 \quad \text{and} \quad \frac{n(R_i, 0, \Phi)}{\mu(R_i)} \to 0,$$

since $\mu(R_i) \ge \mu(r_{\nu_i})$ and $n(R_i, 0, \Phi) = n(r_{\nu_i}, 0, \Phi)$ where r_{ν_i} is the maximal value of $r_\nu \le R_i$. The left side of (15) tends to 0 as $i \to \infty$ which contradicts (15). Our assumption was incorrect.

Necessity. Let S be a neighborhood of a K-point. Then $w = \Phi(z) = e^{-\delta z}\prod_{\nu=1}^{\infty}((r_\nu - z)/(r_\nu + z))^{m_\nu}$, where $\delta > 0$ and $\sum_{\nu=1}^{\infty}(m_\nu)/(r_\nu) < \infty$. But $\mu(r, \Phi) > \delta r$ and $\sum_{\nu=1}^{\infty}(1)/(\mu(r_\nu, \Phi)) < \infty$ which is equivalent to (10).

Theorem 10 is proved completely.

The meromorphic function $w = \Phi(z)$ which we used as an auxiliary one nicely illustrates what was proved above. After symmetric continuation of the Riemann surface S across $|w| = 1$ we obtain a complete surface $\tilde{S}$. The function $w = \Phi(z)$ mapping $|z| < \infty$ onto this surface, as follows from what has been said above, has at most order one, mean type. A study of the class of the surfaces $\tilde{S}$ was made by O. Teichmüller [2] who raised the problem of giving a geometric characterization of surfaces $\tilde{S}$ corresponding to the extremal case when $w = \Phi(z)$ has order one, mean type. The following theorem, which is a simple corollary of Theorems 9 and 10, gives an exhaustive answer to Teichmüller's question.

THEOREM 11. *A meromorphic function mapping the finite z-plane onto $\tilde{S}$ will have order one, mean type, if and only if both critical points of $\tilde{S}$ over 0 and over ∞ are K-points. If they are not K-points then the order of the function may range from zero order to order one, minimal type, inclusively. An increase of the order of growth to order one, mean type, is caused by fast and uniform rapprochement of algebraic branch points, of which the exact measure is given by*:

$$\sum_{\nu=1}^{\infty} \frac{1}{\inf_{i\geq\nu}\left\{\ln\frac{1}{|b_i|}\right\}} < \infty.$$

Thus for certain classes of Riemann surfaces, for the mapping functions to be of order $\rho = n/2$, mean type, the condition of presence of n K-points is not only sufficient but also necessary.

If an ε-neighborhood of the point $\mathfrak{U}$ is not simply connected then its mapping onto a region D_ε is often hard to perform, so the following sufficient condition is useful. By adding to the ε-neighborhood any adjacent connected finitely-sheeted pieces of the Riemann surface F, which is the image of $z \neq \infty$ under a meromorphic function $w = f(z)$, we obtain a simply connected $\tilde{\varepsilon}$-neighborhood $\mathfrak{U}$; its preimage $\tilde{\Delta}$ in the z-plane is obtained if to the region Δ we will add all its holes. Let us map the disk $|\zeta| < 1$ onto the $\tilde{\varepsilon}$-neighborhood of $\mathfrak{U}$ with the help of the function $w = \tilde{\psi}(\zeta)$. Put $\tilde{\varphi}(\zeta) = (\tilde{\psi}(\zeta) - a)/\varepsilon$.

THEOREM 12. *For $\mathfrak{U}$ to be a K-point each of the following conditions is sufficient*:

a) $\limsup_{r\to 1} m(r, 1/(\tilde{\varphi})) > 0$, *if $\tilde{\varphi}(\zeta)$ is a function of bounded characteristic*;

b) $\lim_{r\to 1} N(r, 0, \tilde{\varphi}) < \infty$, *if $\tilde{\varphi}(\zeta)$ is a function of unbounded characteristic.*

Proof In both cases $\lim_{r\to 1} N(r, 0, \tilde{\varphi}) = N(1, 0, \tilde{\varphi}) < \infty$. It follows that

$$\sum_{\nu=1}^{\infty} \tilde{G}(\zeta, a_\nu) < \infty, \quad \zeta \neq a_\nu, \tag{16}$$

where $\tilde{G}(\zeta, a_\nu)$ is the Green function for $|\zeta| < 1$ having pole at zeros a_ν of the function $\tilde{\varphi}(\zeta)$. Indeed, $\sum \tilde{G}(0, a_\nu) = N(1, 0, \tilde{\varphi}) < \infty$,[3] and from Harnack's theorem follows convergence in (16) in $|\zeta| < 1$. Let us denote by D_ε the preimage of an ε-neighborhood of $\mathfrak{U}$ in $|\zeta| < 1$ and by $G(\zeta, a_\nu)$ the Green function of D_ε with the pole at the zero a_ν of $\tilde{\varphi}(\zeta)$ if $a_\nu \in D_\varepsilon$, and $G(\zeta, a_\nu) \equiv 0$ if $a_\nu \notin D_\varepsilon$. Assume that $h_\varepsilon(\zeta) \equiv 0$. Then

$$\ln^+ \left| \frac{1}{\tilde{\varphi}(\zeta)} \right| \equiv \sum_{\nu=1}^{\infty} G(\zeta, a_\nu) \leq \sum_{\nu=1}^{\infty} \tilde{G}(\zeta, a_\nu), \quad \zeta \in D_\varepsilon.$$

At preimages of $|w - a| < \varepsilon$ which contain $a_\nu \in D_\varepsilon$ we always have $h(\zeta) \equiv 0$. Thus

$$\begin{aligned} m\left(r, \frac{1}{\tilde{\varphi}(\zeta)}\right) &= \frac{1}{2\pi} \int_0^{2\pi} \ln^+ \left| \frac{1}{\tilde{\varphi}(re^{i\theta})} \right| d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \left\{ \sum_{\nu=1}^{\infty} \tilde{G}(re^{i\theta}, a_\nu) \right\} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \ln^+ \left| \frac{1}{B(re^{i\theta})} \right| d\theta \to 0 \end{aligned}$$

as $r \to 1$ (see [1], sect. 165), where $B(\zeta)$ is a Blaschke product taken over the zeros of $\tilde{\varphi}(\zeta)$[4]. But this contradicts the assumption of the theorem (in case b)) because then $\tilde{\varphi}(\zeta)$ would be a function of bounded characteristic. Consequently $h_\varepsilon(\zeta) > 0$ and $\mathfrak{U}$ is a K-point.

If for some $\tilde{\varepsilon}$-neighborhood of $\mathfrak{U}$ the function $\tilde{\varphi}(\zeta)$ has unbounded characteristic and $\lim_{r \to 1} N(r, 0, \tilde{\varphi}) = \infty$ (let us call it a K'-point in this case), though one cannot conclude that $\mathfrak{U}$ is a K-point; one can still improve the estimate of the order of $f(z)$.

THEOREM 13. *A meromorphic function $w = f(z)$ mapping the plane $z \neq \infty$ onto the Riemann surface F having $n \geq 0$ K-points and at least one K'-point, has order of growth at least $(n+1)/2$.*

Proof (Notations as above). Consider an $\tilde{\varepsilon}$-neighborhood of a K'-point. By definition $\lim_{r \to 1} N(r, 0, \tilde{\varphi}) = \infty$. Then (compare the proof of Theorem 12) $\sum \tilde{G}(\zeta, a_\nu) = \infty$. So $\sum \tilde{G}_{\tilde{\Delta}}(z, a_\nu) = \infty$, where $\tilde{G}_{\tilde{\Delta}}$ is the

[3] Without loss of generality we may assume that $\zeta = 0$ is not a zero of $\tilde{\varphi}(\zeta)$.

[4] It can be constructed because $N(1, 0, \tilde{\varphi}) < \infty$ which is equivalent to $\sum (1 - |a_\nu|) < \infty$.

Green function of $\bar{\Delta}$. Then all the more $\sum_{\nu=1}^{\infty} G_{\bar{\Delta}}(z, a_\nu) = \infty$. Let us map conformally the region $\bar{\Delta}$ bounded by one Jordan curve γ onto the halfplane $\Im\eta > 0$, $\eta = \rho e^{it}$, $\eta(\infty) = \infty$. Then $\sum_{\nu=1}^{\infty} G(\eta(z), \alpha_\nu) = \infty$, where $\alpha_\nu = \eta(a_\nu)$. But $G(\eta, \alpha_\nu) = \ln|(\eta - \bar{\alpha}_\nu)/(\eta - \alpha_\nu)|$. Thus

$$\prod_{\nu=1}^{\infty} \left| \frac{1 - \frac{\eta}{\bar{\alpha}_\nu}}{1 - \frac{\eta}{\alpha_\nu}} \right| = \infty, \quad \Im\eta > 0,$$

from which ([1], sect. 176, 190) follows that

$$\limsup_{\rho\to\infty} \frac{\ln n_\eta(\rho, \infty)}{\ln \rho} \geq 1, \tag{17}$$

where $n_\eta(\rho, \infty)$ is the number of poles of Green functions in the half-disk $\Im\eta > 0$, $|\eta| \leq \rho$. By the Ahlfors Distortion Theorem ([1], sect. 78, 79),

$$|\eta(z)| \geq K \exp\left(\pi \int_{r_0}^{|z|} \frac{dr}{r\bar{\Theta}(r)}\right),$$

and so

$$n_\eta\left(K \exp\left(\pi \int_{r_0}^{r} \frac{dr}{r\bar{\Theta}(r)}\right), \infty\right) \leq n_z(r, \infty) \leq N_z(er, \infty) \leq T(er, f).$$

In view of (17) for some sequence $r_i \to \infty$ we have

$$\ln n_\eta\left(K \exp\left(\pi \int_{r_0}^{r_i} \frac{dr}{r\bar{\Theta}(r)}\right), \infty\right) \geq (1 - \varepsilon_i)\pi \int_{r_0}^{r_i} \frac{dr}{r\bar{\Theta}(r)}, \quad \varepsilon_i \to 0,$$

$$\ln T(er, f) \geq (1 - \varepsilon_i')\pi \int_{r_0}^{r_i} \frac{dr}{r\bar{\Theta}(r)}.$$

Combined with (7) this gives

$$\ln T(r_i) \geq (1 - \varepsilon'_i)\frac{n+1}{2} \ln r_i,$$

$$\varepsilon'_i \to 0, \quad \limsup_{r\to\infty} \frac{\ln T(r)}{\ln r} \geq \frac{n+1}{2}.$$

So if for some $\bar{\varepsilon}$-neighborhood the function $\bar{\varphi}(\zeta)$ has unbounded characteristic, the estimate for the order of $w = f(z)$ can be improved by 1/2 in any case, independently of whether $N(1, 0, \bar{\varphi}) < \infty$ or $N(1, 0, \bar{\varphi}) = \infty$; in the case when there are several such neighborhoods the first case permits to obtain a more precise estimate.

Remark 1. Theorem 6 says that $\liminf_{r\to\infty} r^{-n/2}T(r, f) > 0$ while in Theorem 13 only $\limsup_{r\to\infty} \ln T(r, f)/\ln r$ is estimated.

Remark 2. By Theorem 5 in every ε-neighborhood where $h_\varepsilon(\zeta) > 0$ there exists at least one K-point. On the other hand, if for some $\tilde{\varepsilon}$-neighborhood the function $\tilde{\varphi}(\zeta)$ has unbounded characteristic then, as can be easily shown by an example, in this neighborhood there may be no critical points at all. Nevertheless the entire proof of Theorem 13 is still applicable, so in its formulation a K'-point may be replaced by K'-neighborhood.

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References

[1] Nevanlinna, R. (1953). Eindeutige Analitische Funktionen, Springer, Berlin.
[2] Teichmüller, O. (1939). Vermutungen und Sätze über die Wertverteilung gebrochener Funktionen endlicher Ordnung, Deutsche Math., **4**, 163-190.
[3] Mme L. Schwartz (1941). Exemple d'une fonction méromorphe ayant des valeurs déficientes non asymptotiques, C. R. Acad. Sci., Paris, **212**, 382-384.
[4] Titchmarsh, E. (1939). The Theory of Functions, Oxford.
[5] Valiron, G. (1950). Sur les valeurs déficientes des fonctions méromorphes d'ordre nul, C.R. Acad. Sci., Paris, **230**, 40-42.
[6] Nevanlinna, R. (1929). Le théorème de Picard-Borel et la théorie des fonctions méromorphes, Paris.
[7] Ullrich, E. (1929). Über die Ableitung einer meromorpher Funktion, Sitzungsber., Preuss. Akad. Wiss., Phys-math. Kl., **27**, 592-608.
[8] Shah, S. (1944). A note on meromorphic functions, Math. Student, **12**, 67-70.
[9] Arima, K. (1952). On maximum modulus of integral functions, J. Math. Soc. japan, **4**, 62-66.
[10] Ullrich, E. (1936). Flächenbau und Wachstumsordnung bei gebrochenen Funktionen, Jahrsber. deutch. Math.-Verein., **46**, 232-274.
[11] Tumura, Y. (1941). Sur les théorèmes de M. Valiron et les singularités transcendentes indirectement critiques, Proc. Imp. Acad. Tokyo, **17**, 65-69.
[12] Tumura, Y. (1942). Sur les singularités non directement critiques, Proc. Imp. Acad. Tokyo, **18**, 606-611.
[13] Tumura, Y. (1941). Sur la distribution dès valeurs, Proc. Imp. Acad. Tokyo, **18**, 612-616.
[14] Tumura, Y. (1943). Recherches sur la distribution des valeurs des fonctions analytiques, Jap. J. Math., **18**, 797-876.
[15] Tsuji, M. (1942). On the behavior of an inverse function of a meromorphic function at its transcendental singular point III, Proc. Imp. Acad. Tokyo, **18**, 132-139.
[16] Chebotarev, N. G. and Meiman, N. N. (1949). Routh-Hurwitz Problem for polynomials and entire functions, Proc. Steklov Math. Inst. Acad. Sci., **26**, (Russian).

[17][5] Hayman, W. (1964). Meromorphic Functions, Oxford, Clarendon Press.
[18][5] Gol'dberg, A. A. and Ostrovskii, I. V. (1970). Distribution of values of meromorphic functions, Moscow, Nauka, (Russian)
[19][5] Gol'dberg, A. A. (1957). Inequality for logarithmically convex functions, Doklady Akademii nauk Ukrainskoi SSR, 3, 227-230, (Russian, Ukrainian).

[5] Added by the translator