Univalent functions of fast growth with gap power series

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Abstract

We construct univalent functions in the unit disc, whose coefficient sequences (a_n) have arbitrarily long intervals of zeros, and at the same time arbitrarily long intervals where $|a_n| > n\epsilon_n$ holds, (ϵ_n) being an an arbitrary prescribed sequence of positive numbers tending to zero. Furthermore we show that the initial interval of coefficients of such a function can be prescribed to be any interior point of a coefficient region.

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We consider the class **S** of normalized univalent functions

$$f(z) = z + a_2 z^2 + \dots$$

in the unit disc **U**. It is known [2, p.15] that, when $f \in \mathbf{S}$, there always exists a limit

$$\alpha(f) := \lim_{n\to\infty} |a_n|/n \in [0,1].$$

Thus if there is an infinite subsequence of zero coefficients, we have $\alpha(f) = 0$. We will show in this paper that the last conclusion is best possible: the sequence (a_n) may have infinitely many long gaps and simultaneously long intervals where $|a_n| > n\epsilon_n$. Here (ϵ_n) is any prescribed sequence such that $(\epsilon_n) \to 0$.

For any power series of the form $f(z) = z + a_2 z^2 + \dots$ we put

$$\sigma_n(f) := (a_2, \dots, a_n) \in \mathbf{C}^{n-1}.$$

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Then the n-th *coefficient region* $\mathbf{V}_n \in \mathbf{C}^{n-1}$ is defined by

$$\mathbf{V}_n := \{\sigma_n(f) : f \in \mathbf{S}\}.$$

Theorem 1. *Suppose that* $a \in \mathrm{int}\mathbf{V}_N,\ (\epsilon_n) \to 0,\ \epsilon_n > 0$, *and let an infinite set* E *of disjoint intervals of integers be given. Then there exists* f *in* $\mathbf{S}$,

$$f(z) = z + a_2 z^2 + \dots$$

with the following properties:
(i) $\sigma_N(f) = a$,
(ii) $a_n = 0,\ n \in I$ *for infinitely many intervals* I *in* E, *and*
(iii) $|a_n| > n\epsilon_n,\ n \in I$ *for infinitely many intervals* I *in* E.
If $a \in \mathrm{int}\mathbf{V}_N \cap \mathbf{R}^{N-1}$ *then the function* f *with the properties* (i)-(iii) *can be chosen with real coefficients.*

Remark. If the coefficients a_n are real then (iii) can be strengthened to $a_n > n\epsilon_n,\ n \in I$. We don't know whether there is a function f in S with *all non-negative* coefficients and properties (ii) and (iii).

The main ingredient of our proof is the following

Proposition 1. *Let* f *in* S *be a polynomial univalent in some neighborhood of the closed unit disc* $\overline{\mathbf{U}}$, *and* N *an integer. Then there exists* F *in* S, *such that* $\alpha(F) > 0$,

$$F(z) = f(z) + O(z^{N+1}), \quad \textit{as} \quad z \to 0 \tag{1}$$

and the image $F(\mathbf{U})$ *is not dense in* $\mathbf{C}$. *If* f *is real*[1] *then* F *is also real.*

Proof. The proofs for the cases of real f and general f are very similar. We first consider the case of real f and then indicate the changes necessary for the general case.

It follows from our assumptions that $\Gamma := f(\partial\mathbf{U})$ is an analytic Jordan curve symmetric with respect to $\mathbf{R}$. Furthermore, $\Gamma \cap \mathbf{R} = \{f(-1), f(1)\}$ and $f(-1) < 0 < f(1)$ because $f(z) > 0$ for $0 < z < 1$. It is easy to see that there is a positive Δ with the following property: for $\delta \in (0, \Delta)$ the half-strip

$$\Pi(\delta) := \{z : |\Im z| < \delta, \Re z > 0\}$$

intersects Γ in a single simple arc γ_δ containing $f(1)$. We write

$$M := \max\{|z| : z \in \Gamma\}, \quad A := \{z : |z| > M + 1, |\arg z| < \pi\}$$

[1] i. e. has real coefficients

and

$$D_\delta := f(\mathbf{U}) \cup \Pi(\delta) \cup A, \quad \text{for} \quad \delta \in (0, \Delta). \tag{2}$$

We also define $D_0 := f(\mathbf{U})$. The region D_δ is simply connected. We consider the conformal homeomorphism $f_\delta : \mathbf{U} \to D_\delta$, normalized by $f_\delta(0) = 0$ and $f'_\delta(0) > 0$. Notice that f_δ is real because D_δ as well as our normalization are symmetric. By the Caratheodory Convergence Theorem [1, p.78]

$$f_\delta \to f \quad \text{as} \quad \delta \to 0+ \tag{3}$$

uniformly on compact subsets of $\mathbf{U}$.

We put $K_\delta := f_\delta^{-1}(D_\delta \backslash D_0)$. Then we have $K_\delta \to \{1\}$ as $\delta \to 0+$. This follows for example from [2, Lemma 7.1, p. 198]. Thus we may assume that our Δ is so small that

$$K_\delta \subset \mathbf{U}^+ := \{z \in \mathbf{U} : \Re z > 0\} \quad \text{for} \quad \delta \in (0, \Delta). \tag{4}$$

Thus f_δ maps the left half of the unit circle, $\{z \in \partial\mathbf{U} : \Re z < 0\}$, into the analytic curve Γ. Thus f_δ has an analytic continuation to a region

$$G_\epsilon := \mathbf{U} \cup \mathbf{U}_{-\epsilon}, \quad \text{where} \quad \mathbf{U}_{-\epsilon} := \{z - \epsilon : z \in \mathbf{U}\}. \tag{5}$$

Here $0 < \epsilon < 1/2$ and ϵ depends of f and Δ but is independent of δ.

As $D_0 \subset D_\delta$ when $\delta \in (0, \Delta)$ the function

$$g_\delta := f_\delta^{-1} \circ f : \mathbf{U} \to \mathbf{U} \tag{6}$$

is well defined, real, univalent and in view of (3) we have

$$g_\delta \to \mathrm{id} \quad \text{as} \quad \delta \to 0+. \tag{7}$$

Now we need the following lemma, where we use for convenience a modified notation: for a power series of the form $\psi(z) = a_1 z + a_2 z^2 + \ldots$ we denote by $\sigma_N^*(\psi)$ the vector $(a_1, \ldots, a_N)$ of the first N coefficients.

Lemma 1. *Given a positive integer N and a positive ϵ there exists a neighborhood of the origin V in R^N and a map $\psi : \mathbf{U} \times V \to \mathbf{C}$ with the following properties:*
(i) *for every λ in V $\psi_\lambda := \psi(., \lambda) : \mathbf{U} \to \mathbf{C}$ is univalent, real and satisfies $\psi_\lambda(0) = 0$ and*

$$\mathbf{U}^+ \subset \psi_\lambda(\mathbf{U}) \subset G_\epsilon, \tag{8}$$

where $\mathbf{U}^+$ is defined in (4) and G_ϵ in (5);
(ii) *the image of the map $b : V \to \mathbf{R}^N$ defined by $b(\lambda) = \sigma_N^*(\psi_\lambda)$ is a neighborhood of the point $(1, 0, \ldots, 0)$ in R^N.*

Proof. We define $\phi(t) = \exp\{-(1-t^2)^{-1}\}$ for $t \in (-1,1)$ and $\phi(t) = 0$ for real t, $|t| \geq 1$. Then ϕ is an infinitely differentiable function. Next we fix N arbitrary points t_n, $2\pi/3 < t_0 < t_1 < \ldots < t_{N-1} < \pi$, and define functions $\phi_j(t) := M\phi(M(t-t_j))$, $t \in \mathbf{R}$, where M is a positive parameter to be specified now. Notice that $\phi_j(t) \to c\delta(t-t_j)$ as $M \to \infty$ in the sense of distributions, where $c = \int \phi(t)dt$ is a positive absolute constant. We claim that M can be chosen in such a way that

$$J_M := \det\left(\int_0^\pi \phi_n(t)\cos mt\,dt\right)_{m,n=0}^{N-1} \neq 0. \tag{9}$$

To prove our claim we note that

$$M\int_0^\pi \phi(M(t-t_j))\cos mt\,dt \to c\cos mt_j, \quad \text{as} \quad M \to \infty$$

for $0 \leq j \leq N-1$ and for every integer m. It follows that, as $M \to \infty$, J_M has the limit

$$J_\infty = c^N \det\left(\cos mt_n\right)_{m,n=0}^{N-1} = c_1 \prod_{n<m}(\cos t_m - \cos t_n) \neq 0,$$

where $c_1 = c^N 2^{N(N-1)/2}$. In fact $\cos mt_n = T_m(\cos t_n)$ where

$$T_m(z) = 2^{m-1}z^m + \ldots$$

is the Tchebychev polynomial of degree m. Thus $c^{-N}J_\infty$ is a polynomial of degree $N(N-1)/2$ in the $\cos t_j$. Comparing coefficients of $\cos t_1 \cos^2 t_2 \ldots \cos^{N-1} t_{N-1}$ we obtain our result.

This shows that if M is large enough then $J_M \neq 0$, and we fix such a value of M (M depends on N from the assumptions of Lemma 1, and on the choice of (t_n)). We also assume that M is so large that all the functions ϕ_j are equal to 0 for $|t| \leq 2\pi/3$.

Now for every λ in R^N we define

$$\phi^\lambda(t) := \sum_{j=0}^{N-1} \lambda_j(\phi_j(t) + \phi_j(-t)), \tag{10}$$

so that ϕ^λ is an even smooth function equal to 0 for $|t| \leq 2\pi/3$. Further we write

$$\psi_\lambda(z) := z\exp\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\phi^\lambda(t)\,dt. \tag{11}$$

This function ψ_λ is analytic in $\mathbf{U}$. It is real because ϕ^λ is even. If we write

$$\psi_\lambda = z\exp(u_\lambda + iv_\lambda),$$

then u_λ and v_λ are conjugate harmonic functions in $\mathbf{U}$. The function u_λ extends continuously to $\partial\mathbf{U}$ and $u_\lambda(e^{it}) = 2\pi\phi^\lambda(t)$ is a smooth function of t. It follows that v_λ also extends continuously to $\partial\mathbf{U}$ and $t \mapsto v_\lambda(e^{it})$ is also a smooth function. Both u_λ and v_λ depend linearly on λ. Thus there exists a neighborhood V' of the origin in $\mathbf{R}^N$ such that

$$\left|\frac{d}{dt}v_\lambda(e^{it})\right| < 1, \quad \text{when} \quad |t| \leq \pi \quad \text{and} \quad \lambda \in V'.$$

It follows that $\arg\psi_\lambda(e^{it}) = t + v_\lambda(e^{it})$ is strictly monotone, and its increment as $t \in [-\pi, \pi]$ is equal to 2π. This implies that ψ_λ is univalent (and in fact starlike) for $\lambda \in V'$ [1, p.41]

The restriction of ϕ^λ to $[-2\pi/3, 2\pi/3]$ is zero, so that $u_\lambda(e^{it}) = 0$ and thus $|\psi_\lambda(e^{it})| = 1$ for $|t| < 2\pi/3$. In addition, $\psi_\lambda \to \mathrm{id}$ as $\lambda \to O$, where O is the origin in R^N, uniformly in $\mathbf{U}$. Thus there is a neighborhood V of the origin such that $V \subset V'$, and (8) holds for $\lambda \in V$. This proves statement (i) of Lemma 1.

To prove statement (ii) we set

$$\sum_{n=0}^{\infty} c_n(\lambda) z^n := \log\frac{\psi_\lambda(z)}{z} = u_\lambda(z) + iv_\lambda(z). \tag{12}$$

Then (10) and (11) imply the following expressions for the coefficients c_n:

$$c_0(\lambda) = 2\sum_{n=0}^{N-1} \lambda_n \int_{-\pi}^{\pi} \phi_n(t)dt \tag{13}$$

and

$$c_m(\lambda) = 4\sum_{n=0}^{N-1} \lambda_n \int_{-\pi}^{\pi} \phi_n(t)\cos mt\, dt, \quad 1 \leq m \leq N-1. \tag{14}$$

Thus the vector $c(\lambda) := (c_0(\lambda), \ldots, c_{N-1}(\lambda))$ depends linearly on the vector λ and the determinant of this linear transformation is a positive multiple of the determinant J_M in (9). Our choice of the parameter M guarantees that this linear transformation is non-singular. Thus $c(V)$ contains a neighborhood of the origin in $\mathbf{R}^N$.

To investigate the dependence of the coefficients of ψ_λ on λ we first notice that they are polynomials in the λ_j. This follows from the expression

$$\begin{aligned}\psi_\lambda(z) &= z\left(b_0(\lambda)+b_1(\lambda)z+\ldots+b_{N-1}(\lambda)z^{N-1}+\ldots\right)\\ &= z\exp(c_0(\lambda)+c_1(\lambda)z+\ldots+c_{N-1}(\lambda)z^{N-1}+\ldots).\end{aligned} \tag{15}$$

Differentiation gives

$$\frac{\partial b_m}{\partial c_n}=\left.\frac{\partial}{\partial c_n}\right|_{c=O}(m!)^{-1}\left.\frac{\partial^m\psi_\lambda}{(\partial z)^m}\right|_{z=0}=(m!)^{-1}\left.\frac{\partial^m}{(\partial z)^m}\right|_{z=0}\left.\frac{\partial\psi_\lambda}{\partial c_n}\right|_{c=O}=\delta_{m,n}, \tag{16}$$

where $\delta_{m,m}=1$ and $\delta_{m,n}=0$ if $m\neq n$. Thus the map $\lambda\mapsto c(\lambda)\mapsto b(\lambda)$ is analytic and has non-zero Jacobian at the origin and thus by the Inverse Function Theorem $b(V)$ contains a neighborhood of the point $(1,0,\ldots,0)$. This proves (ii) in Lemma 1.

□

We apply Lemma 1 with N given in Proposition 1 and ϵ defined in (5) to obtain a neighborhood $V'=b(V)$ of the point $(1,0,\ldots,0)\in\mathbf{R}^N$. Next, using (7) we choose δ in $(0,\Delta)$ and so small that $\sigma_N^*(g_\delta)\in V'$. This means that we can choose λ in V such that

$$b(\lambda)=\sigma_N^*(\psi_\lambda)=\sigma_N^*(g_\delta), \tag{17}$$

which is the same as $\psi_\lambda(z)=g_\delta(z)+O(z^{N+1}),\ z\to 0$.

Now we put

$$F=f_\delta\circ\psi_\lambda \tag{18}$$

and it remains to verify that this F satisfies the requirements of Proposition 1.

First of all, F is univalent in $\mathbf{U}$ because ψ_λ is univalent in $\mathbf{U}$, satisfies (8), and f_δ is univalent in G_ϵ defined in (5).

Second, the definitions (6) and (18) together with (17) imply

$$\sigma_N^*(F)=\sigma_N^*(f)$$

which is the same as (1).

Finally we have to show that $\alpha(F)>0$. It is easy to see from the explicit description of the image domain $D_\delta=f_\delta(\mathbf{U})$ in (2) that f_δ has a double pole at the point 1, and is bounded in $\mathbf{U}$ outside a neighborhood of the point 1. Using the standard notation

$$M(r,h)=\max\{|h(z)|:|z|\leq r\}$$

we conclude that

$$M(r, f_\delta) \sim c(1-r)^{-2} \quad \text{as} \quad r \to 1- \tag{19}$$

with a positive constant c. Now, in view of (8) ψ_λ maps an open arc of the unit circle containing the point 1 into a similar arc. So by the Symmetry Principle ψ_λ has an analytic continuation to the point 1. Thus

$$1 - \psi_\lambda(z) = \psi'_\lambda(1)(1-z) + O((1-z)^2), \quad \text{as} \quad z \to 1, \quad \text{and} \quad \psi'_\lambda(1) > 0.$$

Together with (19) and (18) this implies that

$$M(r, F) \sim c_1(1-r)^{-2} \quad \text{as} \quad r \to 1- \tag{20}$$

where $c_1 > 0$. According to [2, Theorem 1.12] this implies that $\alpha(F) > 0$.

It remains to consider the case in Proposition 1 when f is not real. The proof in this case has to be modified in two places.

1. Construction of the region D_δ in the beginning of the proof of Proposition 1. $\Gamma = f(\partial \mathbf{U})$ is a Jordan curve, so the point $f(1)$ in $\partial\Gamma$ is accessible from the unbounded component of $\mathbf{C}\backslash\Gamma$. Consider a simple curve γ which does not intersect Γ and ∂A, except at the endpoints: one endpoint is $f(1) \in \Gamma$ another is the point $M+1$. A δ-neighborhood of this curve γ will play the role of $\Pi(\delta)$ in (2).

2. We need the following version of Lemma 1.

Lemma 1′. *Given a positive integer N and $\epsilon > 0$, there exists a neighborhood of the origin V in $\mathbf{R}^{2N-1}$ and a map $\psi : \mathbf{U} \times V \to \mathbf{C}$ with the following properties:*
(i) *for every λ in V $\psi_\lambda(., \lambda) : \mathbf{U} \to \mathbf{C}$ is univalent, satisfies* (8), *$\psi_\lambda(0) = 0$, and $\psi'_\lambda(0)$ is real.*
(ii) *the image of the map $a : V \to \mathbf{R} \times \mathbf{C}^{N-1}$, defined by $a(\lambda) = \sigma^*_N(\psi_\lambda)$ is a neighborhood of the point $(1, 0 \ldots, 0)$ in $\mathbf{R} \times \mathbf{C}^{N-1}$.*

Proof. We fix $2N-1$ points t_j, $2\pi/3 < t_1 < \ldots < t_{2N-1} < \pi$. The following determinant plays a role similar to that of J_∞ in the proof of Lemma 1.

$$\begin{vmatrix} 1 & \ldots & 1 \\ e^{it_1} & \ldots & e^{it_{2N-1}} \\ e^{-it_1} & \ldots & e^{-it_{2N-1}} \\ \ldots & \ldots & \ldots \\ e^{(N-1)it_1} & \ldots & e^{(N-1)it_{2N-1}} \\ e^{-(N-1)it_1} & \ldots & e^{-(N-1)it_{2N-1}} \end{vmatrix} \neq 0. \tag{21}$$

We can see this by multiplying the j-th column by $e^{(N-1)it_j}$ for $1 \le j \le 2N-1$, the result being proportional to a Vandermonde determinant.

For every λ in $\mathbf{R}^{2N-1}$ we put

$$\phi^\lambda(t) := \sum_{j=1}^{2N-1} \lambda_j \phi_j(t),$$

where the ϕ_j are defined exactly as in the proof of Lemma 1. The proof of the statement (i) is the same as the proof of (i) in Lemma 1 (The fact that $\psi'_\lambda(0)$ is real is evident from the explicit expression (11).

To prove (ii) we define coefficients $c_j(\lambda)$ by (12). The expressions (13) and (14) have now to be replaced by

$$c_0(\lambda) = \sum_{n=1}^{2N-1} \lambda_n \int_{-\pi}^{\pi} \phi_n(t)dt \tag{22}$$

and

$$c_m(\lambda) = 2\sum_{n=1}^{2N-1} \lambda_n \int_{-\pi}^{\pi} \phi_n(t)e^{imt}dt, \quad 1 \le m \le N-1. \tag{23}$$

If λ is real this is equivalent to

$$\overline{c_m(\lambda)} = 2\sum_{n=1}^{2N-1} \lambda_n \int_{-\pi}^{\pi} \phi_n(t)e^{-imt}dt, \quad 1 \le m \le N-1. \tag{24}$$

Now we consider (22), (23) and (24) as a system of linear equations with respect to λ. For every given vector $(c_0, \ldots, c_{N-1})$ in $\mathbf{R} \times \mathbf{C}^{n-1}$ this system has a unique solution (if M in the definition of ϕ_j is large enough: see the beginning of the proof of Lemma 1) because the determinant (21) is different from zero. But this unique solution is in fact real, because the system is symmetric under complex conjugation.

Thus the real linear map $\mathbf{R}^{2N-1} \to \mathbf{R} \times \mathbf{C}^{N-1}$, $\lambda \mapsto c(\lambda)$ is non-degenerate. The rest of the proof repeats literally the concluding argument in the proof of Lemma 1. This proves Lemma $1'$. $\square$

The rest of the proof of Proposition 1 remains unchanged. $\square$

To prove Theorem 1 we need two more lemmas

Lemma 2.[3, p.9] *The following statements are equivalent:*
(i) $a \in \mathrm{int}\mathbf{V}_N$,

(ii) *there exists* f *univalent in* $D(R) := \{z : |z| < R\}$ *where* $R > 1$, *and* $\sigma_N(f) = a$,
(iii) *there exists* f *in* $\mathbf{S}$ *whose image* $f(\mathbf{U})$ *is not dense in* $\mathbf{C}$ *and* $\sigma_N(f) = a$.

Lemma 3. *If* $a \in \operatorname{int}\mathbf{V}_N$ *then there exists a polynomial* p *univalent in a neighborhood of* $\overline{\mathbf{U}}$ *with* $\sigma_N(p) = a$. *If* $a \in \operatorname{int}\mathbf{V}_N \cap \mathbf{R}^{N-1}$ *then* p *can be chosen real.*

Proof. According to Lemma 2 (ii) there exists f in $\mathbf{S}$ univalent in a disc $D(R) := \{z : |z| < R\}$ where $R > 1$ and $\sigma_N(f) = a$. The proof of this fact given in [3, p.9] actually shows that if a is real than f can be chosen real. Put $R' = (1 + R)/2$ and consider the Jordan regions $D := f(\mathbf{U})$ and $D' = f(D(R'))$. We have $\overline{D} \subset D'$. Let p_n be the n-th partial sum of the Taylor series of f, where $n > N$ and n is chosen large enough to satisfy the following two conditions. Firstly, $\overline{p_n(\mathbf{U})} \subset D'$. Such an n exists because $p_n \to f$ uniformly in $\mathbf{U}$. Secondly, $|f(z) - p_n(z)| < \operatorname{dist}(\partial D', p_n(\mathbf{U}))$ for $z \in \overline{\mathbf{U}}$. When n satisfies these two conditions, the Argument Principle implies that p_n is univalent in $\mathbf{U}$. We take $p = p_n$. $\square$

Proof of Theorem 1. We construct inductively a sequence of polynomials (f_n), $f_n \in S$ with $\deg f_n = d_n$ such that $\sigma_{d_n}(f_n) = \sigma_{d_n}(f_{n+1})$, $n \in \mathbf{N}$.

Let f_1 be the polynomial p constructed in Lemma 3 using the vector a, prescribed in Theorem 1. Assume that f_m has been already constructed where $m \geq 1$, $d_m = \deg f_m$, and let us construct f_{m+1}.

Let d'_m be the smallest integer with the property that $d'_m > d_m$ and the interval (d_m, d'_m) contains an interval I from the set E. We apply Proposition 1 to f_m with $N = d'_m$ and obtain a function $g_m := F$ with $\alpha(g_m) > 0$, initial interval of coefficients matching the whole sequence of coefficients of the polynomial f_m and a gap occurring on the interval I in (d_m, d'_m). Let (a_n) be the sequence of coefficients of g_m. Let k_m be an integer with the properties $k_m > d'_m$ and $|a_n| > n\epsilon_n$ for $n > k_m$. Let d''_m be the smallest integer with the properties $d''_m > k_m$ and the interval (k_m, d''_m) contains an interval from the set E. The image of g_m is not dense, so by Lemma 2 (iii) any initial interval of coefficients of g_m is an interior point of the corresponding coefficient region. Using Lemma 3 we can find a polynomial f_{m+1} univalent in $\overline{\mathbf{U}}$, which has the same initial interval of length d''_m of the Taylor series as f_m.

The sequence (f_m) is convergent uniformly on compact subsets of $\mathbf{U}$ because $\mathbf{S}$ is complete. The limit is the function f in $\mathbf{S}$ which has all the required properties in view of construction of its sequence of coefficients.

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