On Jentzsch sets of functions analytic in the unit disc

Alexandre Eremenko

Sometime 1993

Dear Lee:

Let f be an analytic function in the unit disc $U = \{z : |z| < 1\}$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |a_n|^{1/n} \to 1.$$

Denote by

$$s_n(z) = \sum_{k=0}^{n} a_k z^k, \quad n = 1, 2, \ldots$$

the partial sums of the Maclaurin series of f. Following L. Rubel we denote by $J(f)$ the set of all limit points of zeros of s_n. Then the classical theorem of Jentzsch says that $J(f)$ contains the unit circle. The structure of $J(f)$ in U is clear: it coinsides with the set of zeros of f. Some interesting results about the part of $J(f)$ outside the unit circle were recently obtained by Rubel's student Qian [Q].

Here we will prove the following theorem which partially answers a question of Rubel.

Theorem 1 *Let h be a function holomorphic in $D(R) := \{z : |z| < R\}$, $1 < R \leq \infty$. Then $J(f) \cap \{z : 1 < |z| < R\} = J(f - h) \cap \{z : 1 < |z| < R\}$.*

The idea of the proof actually goes back to H. Delange's thesis [D], see also [B, p. 386].

Proof. Put

$$u_n = \frac{1}{n} \log |s_n| = \int_{\mathbf{C}} \log |z - \zeta| d\mu_n + \epsilon_n,$$

where $\epsilon_n = (1/n)\log|a_n| \to 0$. Here μ_n are probability measures in $\mathbf{C}$ charging every zero of s_n with mass $1/n$. So in particular $\mu_n(D(1/2)) \to 0$ as $n \to \infty$. Evidently $\{u_n(0)\}$ is bounded from above, so

$$\int_{\mathbf{C}} \log|\zeta| d\mu_n \leq C. \tag{1}$$

Let μ be the weak limit of some subsequence of μ_n in $\bar{\mathbf{C}}$. We have $\mu(D(1/2)) = 0$ and this together with (1) implies that $\mu(\infty) = 0$, and so μ is a probability measure on $\mathbf{C}$. Now we have

$$u_n(z) \to u(z) = \int_{\mathbf{C}} \log|z - \zeta| d\mu,$$

where the convergence takes place in L^1_{loc}. Evidently, $u(z) = 0,\ |z| < 1$, so $\mu(U) = 0$ and

$$\int_{\mathbf{C}} \log|\zeta| d\mu = 0.$$

Thus μ is supported by the unit circle. It follows that u is harmonic in $\mathbf{C}\backslash\bar{U}$ and $u(z) = \log|z| + o(1),\ z \to \infty$. These properties of u imply that $u(z) = \log^+|z|$. Now it follows that μ is the uniform distribution on the unit circle, which constitutes and improvement of the Jentzsch theorem due to Szegö. (The limit $\mu_n \to \mu$ actually exists because the above argument is applicable to *any* limit measure μ.

To continue the proof we need the following result about convergence of potentials [A]. For arbitrary set $E \subset \mathbf{C}$ define the 1-content $m_1(E)$ in the follwing way. Consider all coverings of E by countable systems of discs. For each such covering take the sum of radii of these disks. Then $m_1(E)$ is the infimum of such sums taken over all coverings.

Lemma 1 *Let $\mu_n \to \mu$ weakly. Denote by u_n and u the corresponding potentials. Then*

$$m_1(\{z : |u_n(z) - u(z)| > \delta\} \to 0$$

for every $\delta > 0$.

Fix arbitrary small $\epsilon > 0$ and put $\delta = (1/2)\log(1+\epsilon)$. Choose a natural N so large that for $n > N$ we have

$$m_1(\{z : |u_n(z) - u(z)| > \delta\} < \epsilon \tag{2}$$

and

$$\frac{1}{n}\log|h_n(z)| < \frac{\delta}{2}, \quad |z| < R-\epsilon, \tag{3}$$

where h_n is the n-th partial sum of h.

Let G_n be the union of those components of the set $\{z : 1 < |z| < R,\ u_n(z) < \delta\}$ which intersect the set $\{z : 1+2\epsilon < |z| < R-2\epsilon\}$. We claim that G_n does not intersect $D(1+\epsilon)$ for $n > N$.

Indeed if a component K of G_n intersects $D(1+\epsilon)$ then it intersects both boundary components of the annulus $\{z : 1+\epsilon < |z| < 1+2\epsilon\}$. On the other hand we have $u(z) > 2\delta,\ |z| > 1+\epsilon$ and $u_n(z) < \delta,\ z \in K$. So $|u(z) - u_n(z)| > \delta,\ z \in K \cap \{z : 1+\epsilon < |z| < 1+2\epsilon\}$ and this contradicts (2).

We proved that $G_n \subset \{z : |z| > 1+\epsilon\}$. Similarly we prove that $G_n \subset D(R-\epsilon)$.

Now $u(z) = \log^+|z| > 2\delta,\ z \in G_n$ and $u_n(z) < \delta,\ z \in G_n$, from which follows that $|u(z) - u_n(z)| > \delta,\ z \in G_n$. Thus every component of G_n has diameter less then ϵ in view of (2).

All zeros of $s_n - h_n$ in $\{z : 1+2\epsilon < |z| < R-\epsilon\}$ are contained in G_n by (3). And every component of G_n contains a zero of s_n by the Minimum principle. On the other hand, every zero of s_n in $\{z : 1+2\epsilon < |z| < R-\epsilon\}$ is contained in some component of G_n. And $s_n - h_n$ has a zero in this component by Rouché's theorem.

This shows that the distance between the zero set of s_n and the zero set of $s_n - h_n$ in $\{z : 1+2\epsilon < |z| < R-\epsilon\}$ is at most ϵ, which proves the Theorem.

References

[A] V. S. Azarin, On the asymptotic behavior of subharmonic functions of finite order, Math. USSR Sbornik, 36 (1980), 135-153.

[B] A. Beurling, Collected Works, vol. 1, Birkhäuser, Boston, Basel, Berlin, 1989.

[D] H. Delange, Sur la convergence des séries de polynomes de la forme $\sum a_n P_n(z)$ et sur certaines suites de polynomes, Ann. École Norm. Supèr., 56 (1939), 173-275.

[Q] Qian. Thesis. University of Illinois at Urbana– Champaign.)

Department of Mathematics, Purdue University West Lafayette IN 47907
e-mail: eremenko@math.purdue.edu