Nevanlinna functions with real zeros

A. Eremenko* and S. Merenkov†

March 5 2005

Abstract

We show that for every non-negative integer d, there exist differential equations $w'' + Pw = 0$, where P is a polynomial of degree d, such that some non-trivial solution w has all roots real.

AMS Subject Class.: 34M05, 30D30, 30C99.

1 Introduction

Consider a differential equation

$$w'' + Pw = 0, \tag{1}$$

where P is a polynomial of the independent variable. Every solution w of this equation is an entire function. We are interested in solutions w all of whose roots are real. If (1) has two linearly independent solutions with this property then $\deg P = 0$, see [7, 8]. Here we study equations (1) that have at least one solution with all roots real.

The question of describing equations (1) with this property was proposed by S. Hellerstein and J. Rossi in [3, Probl. 2.71]. According to [6], up to trivial changes of variables, only the following four examples were known until recently.

- $\deg P = 0$. If k is real, all solutions of $w'' + k^2 w = 0$ are trigonometric functions.

*Supported by NSF grants DMS-0100512 and DMS-0244547.

†Supported by NSF grants DMS-0400636 and DMS-0244547.

- $\deg P = 1$. The Airy equation $w'' - zw = 0$, has a solution $\mathrm{Ai}(z)$ whose roots lie on the negative ray.

- $\deg P$ is even and $w = p \exp q$, where p and q are polynomials, and all roots of p are real. In this case, the set of roots of w is evidently finite. For example, if $P(z) = 1 - z^2 + 2n$, where n is a positive integer, then the equation (1) has solutions $w = H_n(z) \exp(-z^2/2)$, where H_n are Hermite's polynomials whose roots are all real.

- $P(z) = az^4 + bz^2 - c$. Gundersen [6] proved that for every $a > 0$ and $b \geq 0$ one can find an infinite set of real numbers c, such that some solution of (1) has infinitely many roots, almost all of them real. When $b = 0$ this result, with *all* roots real, was earlier obtained by Titchmarsh [16, p. 172].

Here and in what follows "almost all" means "all except finitely many". Recently Kwang C. Shin [13] proved a similar result for a degree 3 polynomial:

- For every real a and $b \leq 0$ there exist an infinite set of positive numbers c such that the equation $w'' + (z^3 + az^2 + bz - c)w = 0$ has a solution with infinitely many roots, almost all of them real.

On the other hand, Gundersen [7] proved the following theorems:

Theorem A *If $d = \deg P \equiv 2 \pmod 4$, and w is a solution of (1), with almost all roots real, then w has only finitely many roots.*

Theorem B *If (1) possesses a solution w with infinitely many real roots, then P is a real polynomial, and w is proportional to a real function.*

We also mention a result of Rossi and Wang [12] that if (1) has a solution with infinitely many roots, all of them real, then at least half of all roots of P are non-real. In view of Theorem B we restrict from now on to the case of real polynomials P in (1). Our results are:

Theorem 1 *For every d, there exist w satisfying (1) with $\deg P = d$ and such that all roots of w are real. For every positive d divisible by 4, there exist w with infinitely many roots, all of them real, as well as w with any prescribed finite number of roots, all of them real.*

Theorem 2 *Let w be a solution of the equation (1) whose all roots are real, and $d = \deg P$. Then:*

(a) For $d \equiv 0 \pmod 4$ the set of roots of w is either finite or unbounded from above and below (as a subset of the real axis).
(b) For odd d there are infinitely many roots of w all of which lie on a ray.

Theorem 2 can be generalized to the case that almost all roots of w are real.

In comparison with the existence results of Titchmarsh, Gundersen and Shin mentioned above, our Theorem 1 gives more precise information on the roots w: they are all real. On the other hand we can tell less about the polynomial P.

Our proofs are based on a geometric characterization of meromorphic functions of the form $f = w_1/w_2$, where w_1 and w_2 are linearly independent solutions of (1), due to F. and R. Nevanlinna [9, 10], which will be explained in the next section.

2 A class of meromorphic functions

We associate with (1) another differential equation

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = 2P. \tag{2}$$

The expression in the left hand side of (2) is called the *Schwarzian derivative* of f. The following well-known fact is proved by simple formal computation.

Proposition 1 *The relation $f = w_1/w_2$ gives a bijective correspondence between solutions f of (2) and classes of proportionality of pairs (w_1, w_2) of linearly independent solutions of (1).*

Thus when P is a polynomial, all solutions of (2) are meromorphic in the complex plane, and they are all obtained from each other by post-composition with a fractional-linear transformation. We call solutions of equations (2) with polynomial right hand side *Nevanlinna functions*. Equation (2) has a real solution if and only if P is real.

It is easy to see that meromorphic functions f satisfying (2) are local homeomorphisms. In other words, $f'(z) \neq 0$ and all poles are simple.

F. and R. Nevanlinna gave a topological characterization of all meromorphic functions f which may occur as solutions of (2). To formulate their result we recall several definitions.

A *surface* is a connected Hausdorff topological manifold of dimension 2 with countable base.

A continuous map $\pi : X \to Y$ of surfaces is called *topologically holomorphic* if it is open and discrete. According to a theorem of Stoïlov [15] this is equivalent to the following property. For every $x \in X$, there is a positive integer k and complex local coordinates z and w in neighborhoods of x and $\pi(x)$, such that $z(x) = 0$ and the map π has the form $w = z^k$ in these coordinates. The integer $k = k(x)$ is called the *local degree* of π at the point x. So π is a local homeomorphism if and only if $n = 1$ for every $x \in X$.

A pair (X, π) where X is a surface and $\pi : X \to \overline{\mathbf{C}}$ a topologically holomorphic map is called *a surface spread over the sphere* (Überlagerungsfläche in German). Two such pairs (X_1, π_1) and (X_2, π_2) are considered equivalent if there is a homeomorphism $h : X_1 \to X_2$ such that $\pi_2 = \pi_1 \circ h$. So, strictly speaking, a surface spread over the sphere is an equivalence class of such pairs.

If $f : D(R) \to \overline{\mathbf{C}}$ is a meromorphic function in some disc $D(R) = \{z : |z| < R\}$, $R \leq \infty$ then $(D(R), f)$ defines a surface spread over the sphere. We will call the equivalence class of $(D(R), f)$ the *surface associated with* f. It is the same as the Riemann surface of f^{-1}, as it is defined in [1, p. 288], completed with algebraic branch points as in [1, p. 300].

In the opposite direction, suppose that (X, π) is a surface spread over the sphere. Then there exists a unique conformal structure on X which makes π holomorphic. If X is open and simply connected, the Uniformization Theorem says that there exists a conformal homeomorphism $\phi : D(R) \to X$, where $R = 1$ or ∞. This ϕ is defined up to a conformal automorphism of $D(R)$. The function $f = \pi \circ \phi$ is meromorphic in $D(R)$, and (X, π) is (a representative of) the surface associated with f.

If $R = \infty$ we say that (X, π) is of *parabolic type*.

We consider surfaces spread over the sphere (X, π) where X is open and simply connected[1], π a local homeomorphism, and subject to additional conditions below.

Suppose that for some finite set $A \subset \overline{\mathbf{C}}$ the restriction

$$\pi : X \backslash \pi^{-1}(A) \to \overline{\mathbf{C}} \backslash A \quad \text{is a covering map.} \tag{3}$$

Fix an open topological disc $D \subset \overline{\mathbf{C}}$ containing exactly one point a of the set

[1]That is homeomorphic to the plane.

A. If V is a connected component of $\pi^{-1}(D\backslash\{a\})$ then the restriction

$$\pi : V \to D\backslash\{a\} \tag{4}$$

is a covering of a ring, and its degree k does not depend on the choice of the disc D. The following cases are possible [11]:

a) $k = \infty$. Then (4) is a universal covering, V is simply connected and its boundary consists of a single simple curve in X tending to "infinity" in both directions. In this case we say that V defines a *logarithmic singularity* over a. Notice that the number of logarithmic singularities over a is independent of the choice of D.

b) If $k < \infty$ and there exists a point $x \in X$ such that $\widetilde{V} = V \cup \{x\}$ is an open topological disc, then $\pi : \widetilde{V} \to D$ is a ramified covering and has local degree k at x. As we assume that π is a local homeomorphism, only $k = 1$ is possible, so $\pi : \widetilde{V} \to D$ is a homeomorphism.

c) If $k < \infty$ but there is no point $x \in X$ such that $V \cup \{x\}$ is an open disc, then we can add to X such an "ideal point" and define the topology on $\widetilde{X} = X \cup \{x\}$ so that it remains a surface. Then $\widetilde{X}$ is a sphere, as a one-point compactification of an open simply connected surface, and our local homeomorphism extends to a topologically holomorphic map between spheres whose local degree equals one everywhere except possibly one point. It easily follows that the local degree equals one everywhere, the extended map is a homeomorphism. This implies that our original map π is an embedding.

So in any case the degree of the map (4) can be only 1 or ∞.

Definition We say that (X, π) is an *N-surface* if X is open and simply connected, π is a local homeomorphism, condition (3) is satisfied, and there are only finitely many logarithmic singularities.

Unless π is an embedding, as in case c), the number of logarithmic singularities is at least two, because a sphere minus one point is simply connected, so every covering over such surface is a homeomorphism. All cases with two logarithmic singularities can be reduced by a fractional-linear transformation of $\overline{\mathbf{C}}$ to the case $\exp : \mathbf{C} \to \overline{\mathbf{C}}$.

The name N-surface is chosen in honor of F. and R. Nevanlinna. The complete official name of this object would be "An open simply connected surface spread over the sphere without algebraic branch points and with finitely many logarithmic singularities".

Theorem C (i) *Every N-surface is of parabolic type, that is its associated*

functions are meromorphic in the plane $\mathbf{C}$.
(ii) *If an N-surface* (X, π) *has* $n \geq 2$ *logarithmic singularities then the associated meromorphic functions* $f = \pi \circ \phi$ *satisfy a differential equation* (2) *in which* $\deg P = n - 2$.
(iii) *For every polynomial* $P \neq 0$, *every solution* f *of* (2) *is a meromorphic function in the plane whose associated surface is an N-surface with* $n = \deg P + 2$ *logarithmic singularities. If* $P = 0$ *there are no logarithmic singularities and* f *is fractional-linear.*

Two meromorphic functions f_1 and f_2 are called equivalent if $f_1(z) = f_2(az + b)$ with $a \neq 0$. Theorem C establishes a bijective correspondence between equivalence classes of Nevanlinna functions and N-surfaces.

In this paper we use only statements (i) and (ii) of Theorem C. The connection between N-surfaces and differential equations was apparently discovered by F. Nevanlinna who proved (iii) in [9]. Statements (i) and (ii) were proved for the first time by R. Nevanlinna in [10]. Then Ahlfors [2] gave an alternative proof based on completely different ideas. A modern version of this second proof uses quasiconformal mappings [5]. This modern proof is reproduced in [4]. All these authors were primarily interested in the theory of meromorphic functions, and used differential equations as a tool. Apparently, the only application of Theorem C to differential equations is due to Sibuya [14] who deduced from it the existence of equations (1) with prescribed Stokes multipliers.

In view of Theorem C, to obtain our results, we only need to single out those N-surfaces that are associated with real meromorphic functions with real roots.

3 Speiser graphs

We recall a classical tool for explicit construction and visualization of N-surfaces. It actually applies to all surfaces spread over the sphere that satisfy (3). First we suppose that a surface spread over the sphere (X, π) satisfying (3) is given, and $A = \{a_1, \ldots, a_q\}$ is the set in (3). We call elements of A *base points*. Consider a *base curve* that is an oriented Jordan curve Γ passing through $a_1, \ldots, a_q$. Choosing a base curve defines a cyclic order on A, and we assume that the enumeration is consistent with this cyclic order and interpret the subscripts as remainders modulo q.

The base curve Γ divides the Riemann sphere $\overline{\mathbf{C}}$ into two regions which we denote $D^{\times}$ and D°, so that when Γ is traced according to its orientation, the region $D^{\times}$ is on the left. We choose points $\times \in D^{\times}$ and $\circ \in D^{\circ}$, and connect these two points by q disjoint simple arcs L_j so that each L_j intersects Γ at exactly one point, and this point belongs to the arc $(a_j, a_{j+1}) \subset \Gamma$. We obtain an *embedded graph* $L \subset \overline{\mathbf{C}}$ having two vertices $\times$ and $\circ$ and q edges L_j. This embedded graph defines a cell decomposition of the sphere, whose 2-cells (faces) are components of the complement of L, 1-cells (edges) are the open arcs L_j and 0-cells (vertices) are the points $\times$ and $\circ$. Each face contains exactly one base point.

The preimage of this cell decomposition under π is a cell decomposition of X, because as we saw in the previous section, each component of the preimage of a cell is a cell of the same dimension (0, 1 or 2). The 1-skeleton $S = \pi^{-1}(L) \subset X$ is a connected properly embedded graph in X. As S completely defines the cell decomposition, we will permit ourselves to follow the tradition and speak of this graph instead of the cell decomposition, and use such expressions as "faces of S" meaning the faces of the cell decomposition.

We label vertices of S by $\times$ and $\circ$, according to their images under π, and similarly label the faces by the base points a_j. Our labeled graph S (or more precisely, the labeled cell decomposition) has the following properties:

1. Every edge connects a $\times$-vertex to a $\circ$ vertex.

2. Every vertex belongs to the boundaries of exactly q faces having all q different labels.

3. The face labels have cyclic order $a_1, \ldots, a_q$ anticlockwise around each $\times$-vertex, and the opposite cyclic order around each $\circ$-vertex.

The labeled graph S is called the *Speiser graph* or the *line complex* of the surface spread over the sphere (X, π).

A face of S is called *bounded* if its boundary consists of finitely many edges and vertices. It follows from property 1 that the numbers of edges and vertices on the boundary of a bounded face are equal and even. If a is a base point, all solutions of the equation $\pi(x) = a$ belong to the bounded faces labeled by a, and each such face contains exactly one solution. If k is the local degree of π at this point x then the face is a $2k$-gon, that is bounded by $2k$ edges and $2k$ vertices. As we consider local homeomorphisms is this paper, $k = 1$ for every bounded face. Existence of 2-gonal faces leads to *multiple edges* of the Speiser graph.

Suppose now that X is a surface and a labeled cell decomposition of X with 1-skeleton S is given such that 1, 2 and 3 hold. If we choose a set $A \subset \overline{\mathbf{C}}$ of q points and a curve $\Gamma \subset \overline{\mathbf{C}}$ passing through the points of the set A, and define L as above, then there exists a topologically holomorphic map π such that $S = \pi^{-1}(L)$. This map π is unique up to pre-composition with a homeomorphism of X. A verification of this statement is contained in [5]. We recall the construction.

The labels of faces define labels of edges: an edge is labeled by j if it belongs to the common boundary of faces with labels a_j and a_{j+1}. This defines a map of the 1-skeleton of S to the 1-skeleton L of the cell decomposition of the sphere: each edge of S labeled by j is mapped onto L_j homeomorphically, and such that the orientation is consistent with the vertex labeling. It is easy to see that this map is a covering $S \to L$. The boundary of each face covers a topological circle formed by two adjacent edges and two vertices of L. Such a map extends to a ramified covering between faces, ramified only over the base points (unramified for 2-gonal or unbounded faces, which is the case for all Speiser graphs we consider).

For a given cyclically ordered set A and a surface X, the correspondence between topologically holomorphic maps π and Speiser graphs is not canonical: it depends on the choice of the base curve Γ.[2]

It is easy to single out those Speiser graphs that correspond to N-surfaces: the ambient surface X is open and simply connected, and two additional properties hold:

4. Each face has either two or infinitely many boundary edges.

5. The set of unbounded faces is finite.

Property 4 corresponds to the assumption that π is a local homeomorphism, and property 5 follows from the fact that unbounded faces correspond to "logarithmic singularities" that is to the components V in (4) where the covering has infinite degree.

So we have

Proposition 2 *Let (X, π) be an N-surface, and a a base point. Then each solution of the equation $\pi(x) = a$ is contained in a face which is a 2-gon, and is labeled by a. Each such face contains exactly one solution of this equation.*

[2] It is the isotopy class of Γ with fixed set A that matters.

For each Speiser graph S corresponding to an N-surface, we define a new graph $T(S)$ with the same vertices: two vertices are connected by a single edge in T if they are connected by at least one edge in S. Thus T is obtained from S by replacing all edges connecting a pair of vertices with a single edge. Property 4 of S implies that T is a tree. Faces of T are exactly the unbounded faces of S. We assume that vertices and unbounded faces inherit their labels from S.

The tree T is properly embedded in X. Each vertex has at least two and at most q adjacent edges in T and the cyclic order of face labels around the $\times$-vertices of T is the same as in S.

Examples of trees T can be seen in Figures 1–5. The right half of each figure represents a base curve with base points marked on it. The left half is the tree T with the labeling of unbounded faces.

Suppose that S has $n > 2$ unbounded faces. Then the tree $T(S)$ has n maximal (by inclusion) infinite "branches" consisting of vertices of degree 2, of the form

$$-\circ-\times-\circ-\times-\circ-\ldots \quad \text{or} \quad -\times-\circ-\times-\circ-\times-\ldots.$$

Such branches are called *logarithmic ends*. A tree $T(S)$ is the union of its logarithmic ends and a finite subtree.

For example, the trees in Figures 1 and 2 have 14 logarithmic ends; each of them is the union of these 14 ends and a finite graph with 11 vertices. The tree in Fig. 3 is the union of 11 logarithmic ends and a finite subgraph with 9 vertices. The tree in Fig. 4 is the union of 12 logarithmic ends and a finite subgraph with 10 vertices, and the tree in Fig 5 is the union of 13 logarithmic ends and a finite subgraph with 11 vertices.

We excluded in our definition of logarithmic ends those graphs S that contain only two unbounded faces. There is only one tree, T(S) corresponding to such graphs. This tree is homeomorphic to a line, and corresponds to a function of the form $L \circ \exp(az + b)$, where L is fractional linear.

4 Symmetric Speiser graphs

A *symmetric* surface spread over the sphere is defined as a triple (X, π, s), where $s : X \to X$ is a homeomorphism such that $s \circ s = \text{id}$ and $\pi \circ s(x) = \overline{\pi(x)}$, and the bar denotes complex conjugation. It is clear that such s is an anticonformal homeomorphism of X. If (X, π) is of parabolic type, and

$\phi : \mathbf{C} \to X$ a conformal homeomorphism then $\phi^{-1} \circ s \circ \phi$ is an anticonformal involution of the complex plane. Each such involution is conjugate to $z \mapsto \overline{z}$ by a conformal automorphism of $\mathbf{C}$. So for a symmetric surface spread over the sphere there exists a uniformizing map ϕ with the property $\phi(\overline{z}) = s \circ \phi(z)$, $z \in \mathbf{C}$. The set of fixed points of s is called the *axis* (of symmetry); it is the image of the real line under ϕ.

If f is a real function meromorphic in $\mathbf{C}$ then its associated surface has a natural involution which makes it a symmetric surface spread over the sphere. In the opposite direction, to a symmetric surface of parabolic type spread over the sphere a real meromorphic function is associated.

It is clear that the set $A = \{a_1, \ldots, a_q\}$ of base points of a symmetric N-surface is invariant under complex conjugation. Suppose for a moment that at most two of the points $a_1, \ldots, a_q$ are real. Then there exists a base curve Γ passing through $a_1, \ldots, a_q$ which is symmetric with respect to complex conjugation. Choosing the $\times$ and $\circ$ points on the real axis we can perform the construction of the Speiser graph symmetrically. The resulting graph S and the tree $T(S)$ will be *symmetric* in the following sense. The involution s will send each vertex to a vertex with the same label, each edge to an edge and each face to a face with complex conjugate label.

In the general case, that more than two base points are allowed on the real line, one has to modify a little the definition of the Speiser graph. Let a be a real base point of a symmetric N-surface (X, π, s). Consider an open (round) disc $D \subset \overline{\mathbf{C}}$ centered at a and not containing other base points. Let V be a component of $\pi^{-1}(D)$ which defines a logarithmic singularity (see Section 2). Then one of the following: either V is invariant under s, or $s(V) = V'$ where V' is another component of $\pi^{-1}(D)$, disjoint from V. In the latter case, V is disjoint from the symmetry axis. In the former case, we will call the logarithmic singularity *real*.

We claim that there are at most two real logarithmic singularities. Indeed, for a real logarithmic singularity, the intersection of V with the symmetry axis consists of a "ray", and there cannot be more than two disjoint "rays" on the symmetry axis.

The symmetry axis divides X into two "halfplanes", and each non-real logarithmic singularity (more precisely, its defining region V) belongs to one of these "halfplanes". Thus the non-real logarithmic singularities are split into two classes, say C_+ and C_- according to the "halfplane" they belong, and the regions V and V' always belong to different classes.

The real logarithmic singularities lie over at most two base points.

Let a be a real base point such that there are no real logarithmic singularity over a. Consider a homeomorphism η_+ of the Riemann sphere which is the identity outside D and sends the point a to the point $a + i\epsilon$, where $\epsilon > 0$ is so small that $a + i\epsilon \in D$. Let $\eta_-(z) = \overline{\eta_+(\overline{z})}$. We deform our map π in the following way:

$$\pi^*(x) = \begin{cases} \eta_+ \circ \pi(x), \; x \in V, & V \in C_+, \\ \eta_- \circ \pi(x), \; x \in V, & V \in C_-, \\ \pi(x) & \text{otherwise.} \end{cases}$$

Evidently, the new N-surface is symmetric. Let E be the set projections of real logarithmic singularities, card $E \leq 2$. Performing the deformation described above for all real base points except two of them, a' and a'', such that $E \subset \{a', a''\}$ we obtain a new symmetric N-surface which has the property that only two base points are real. So a symmetric base curve can be chosen and a symmetric Speiser graph constructed. The Speiser graph of this deformed surface does not depend on ϵ as soon as ϵ is positive and small enough, and we call it a *symmetric Speiser graph of* (X, π). The number of base points of (X, π^*) is larger than that of (X, π); to preserve properties 2 and 3, we can use two different labels, say a^+ and a^- for a base point a as above. The faces of S over $a + i\epsilon$ are labeled by a^+, those over $a - i\epsilon$ by a^-.

Symmetric Speiser graphs have all the properties 1–5 listed above, and in addition, they are preserved by an orientation-reversing involution of the ambient surface. This includes the action of s on labels if we consider a^+ and a^- as complex conjugate.

Given a symmetric Speiser graph one can construct a symmetric surface spread over the sphere corresponding to this Speiser graph. First we replace all labels a^+ by $a + i\epsilon$ and a^- by $a - i\epsilon$, then choose a symmetric base curve passing through the new base points, points $\times$ and $\circ$ on the real axis and a symmetric graph L, and perform all construction preserving symmetry. Then we apply the inverse of the deformation described above to place the base points in their original position.

We will need two simple properties of symmetric Speiser graphs S and trees $T(S)$.

A. If a logarithmic end of T intersects the axis then it is contained in the axis.

Indeed, suppose first that two vertices v_1 and v_2 of the same logarithmic end belong to the axis. If there are vertices of this logarithmic end between

v_1 and v_2 that do not belong to the axis then the vertices between v_1 and v_2 together with their symmetric vertices and v_1 and v_2 will contain a cycle in T which is impossible. If all vertices between v_1 and v_2 belong to the axis then the whole end has to belong to the axis, because all vertices of the end have degree 2, and T is symmetric.

If exactly one vertex of a logarithmic end belongs to the axis then the whole Speiser graph contains only vertices of degree 2, so there are only two unbounded faces, the situation we excluded in the definition of a logarithmic end.

B. Every (open) edge either belongs to the axis or is disjoint from it.

Indeed, the endpoints of an edge intersecting the axis would be interchanged by the involution but this is impossible because they are differently labeled.

In the following Proposition 3 we consider a symmetric N-surface satisfying the following

Assumptions *The number of logarithmic ends is at least* 3*, zero is a base point, there is at most one real logarithmic singularity not lying over zero, and the symmetric Speiser graph does not have labels* 0^+ *or* 0^-*.*

Comments. The assumption that there are at least three logarithmic ends excludes only the trivial cases when $P = \text{const}$. The assumption that 0 is a base point does not restrict generality because an extra base point can be always added. The third assumption excludes the cases when the number of real roots is finite (if there are two non-zero real logarithmic singularities, then f has non-zero limits along the real axis as $x \to +\infty$ and $x \to -\infty$, so the set of real roots is finite). If this third assumption is satisfied, we can always construct the symmetric Speiser graph in such a way that it does not contain labels 0^+ and 0^-.

Proposition 3. *Let $S \subset \mathbf{C}$ be a symmetric Speiser graph corresponding to a symmetric N-surface, f an associated real Nevanlinna function, and the above Assumptions are satisfied. Then all roots of f are real if and only if S has the following property: each vertex belongs either to the axis of symmetry, or to the boundary of an unbounded face labeled by* 0*.*

Proof. Suppose that all roots of f are real. This means that all preimages of 0 lie on the axis of symmetry. By Proposition 2, these preimages are in bijective correspondence with 2-gonal faces F labeled by zero. We have for such faces $F \cap s(F) \neq \emptyset$, and thus by the symmetry of the graph, $F = s(F)$. It follows that both vertices on the boundary of F belong to the axis.

Every vertex belongs to the boundary of some face labeled by 0 by property 2 of the Speiser graphs (Section 3). If this face is bounded we conclude from the above that the vertex lies on the symmetry axis. This proves necessity of the condition of Proposition 3.

Now we suppose that each vertex belongs either to the symmetry axis or to the boundary of an unbounded face labeled by zero. As every vertex belongs to the boundary of *only one* face labeled by 0, we conclude that every 2-gonal face labeled by zero has one boundary vertex on the axis, and thus its other boundary vertex also belongs to the axis. We conclude that this face is symmetric, and thus the π-preimage of 0 contained in this face belongs to the axis.

Proof of Theorem 2. The cases $P = 0$ and $d = 0$ are trivial, so we assume that $d \geq 1$. If there are two real logarithmic singularities over non-zero points then d is even and f has finitely many roots, so there is nothing to prove. Thus we suppose from now on that the Assumptions stated above are satisfied.

If the number $n = \deg P + 2$ of logarithmic ends is even, then either none or two of them belong to the axis of symmetry.

If none of the logarithmic ends belongs to the axis then f has finitely many roots.

Now suppose that there are two logarithmic ends on the axis of symmetry. Consider one of them. Let a and b be the labels of the two unbounded faces adjacent to it. Then $a \neq b$ and $a = \overline{b}$, so neither a nor b can be 0. This implies that all vertices on this logarithmic end belong to the boundaries of 2-gonal faces whose labels are 0, and thus we obtain an infinite sequence of real roots. As there are two logarithmic ends on the axis of symmetry, the sequence of roots is unbounded from above and below. This proves (a).

That for $d = 4k, k \geq 1$ both cases actually occur is demonstrated by Figures 1 and 2.

If n is odd then exactly one logarithmic end belongs to the axis of symmetry. The other end of the symmetry axis is contained in an infinite face and bisects it (by symmetry). It follows that π has a limit on this other end of the symmetry axis. If the limit is non-zero, (b) immediately follows. If the limit is zero, we notice that this end of the axis belongs to a neighborhood V of a real logarithmic singularity over 0, and again statement (b) follows.

Proof of Theorem 1. It is enough to display a Speiser graph S for each case. For simplicity we only show in Figures 1–5 the trees $T(S)$ on the left of each picture and the base curves with base points on the right. It can be easily seen that each of our trees has unique extension to a symmetric Speiser graph S that has the property described in Proposition 3.

Remarks. Suppose that all base points of a symmetric N-surface are real. Then we can construct another kind of Speiser graph which we call *anti-symmetric*, without using the perturbation procedure described above. Namely, take the real axis as the base curve, and choose $\times$ and $\circ$ at the points $\pm i$. The corresponding Speiser graph is preserved by the involution, except that the vertex labels are now interchanged. It is easy to see that in such anti-symmetric graph there can be no vertices on the axis (the values of our function at vertices are $\pm i$ and the function is real), and thus *exactly one edge* of $T(S)$ intersects the axis, because $T(S)$ is connected and symmetric with respect to the axis. We conclude that

Any real Nevanlinna function with at least three logarithmic singularities and only real asymptotic values can have at most one real zero.

We thank the referee for many valuable remarks and suggestions.

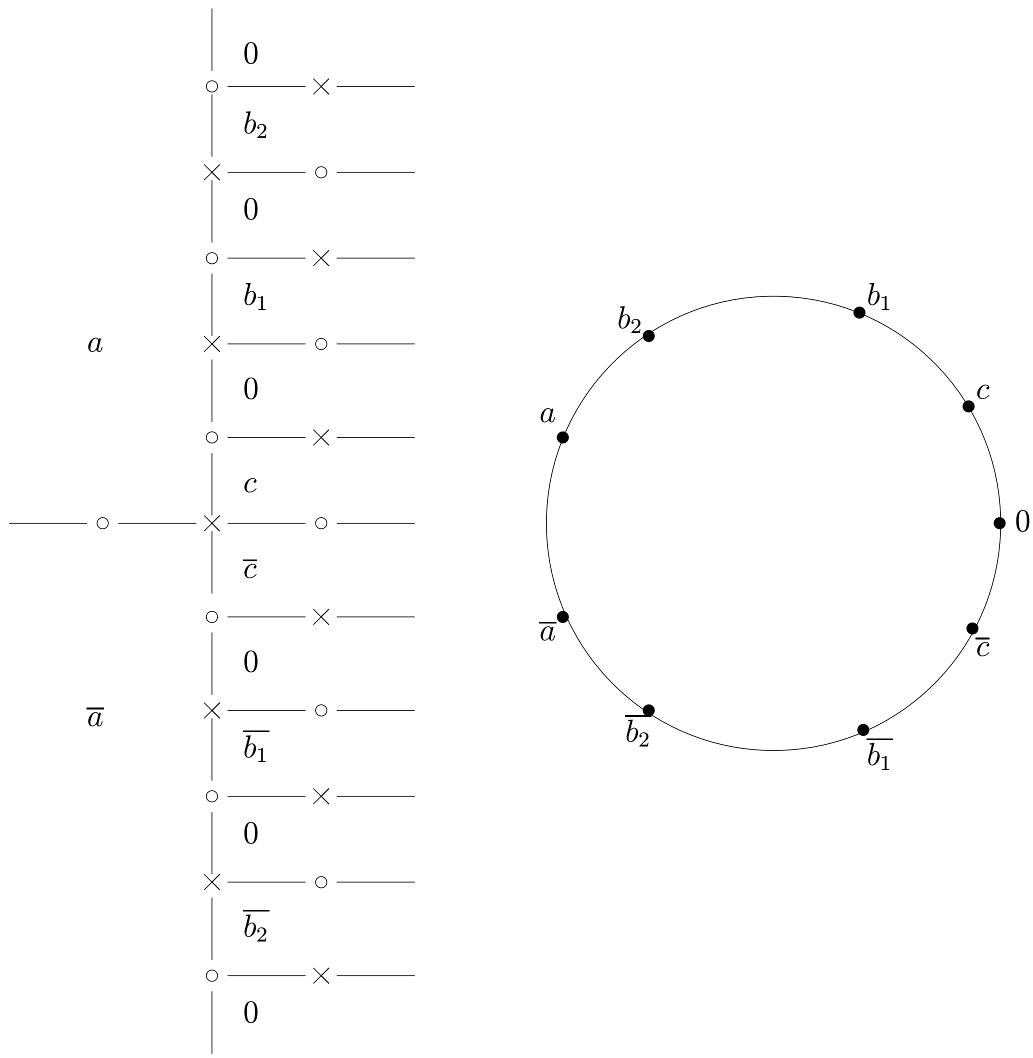

Figure 1: $d \equiv 0 \pmod 4$, the sequence of roots infinite in both directions. The tree $T(S)$ in this picture has 14 logarithmic ends, and $d = 12$.

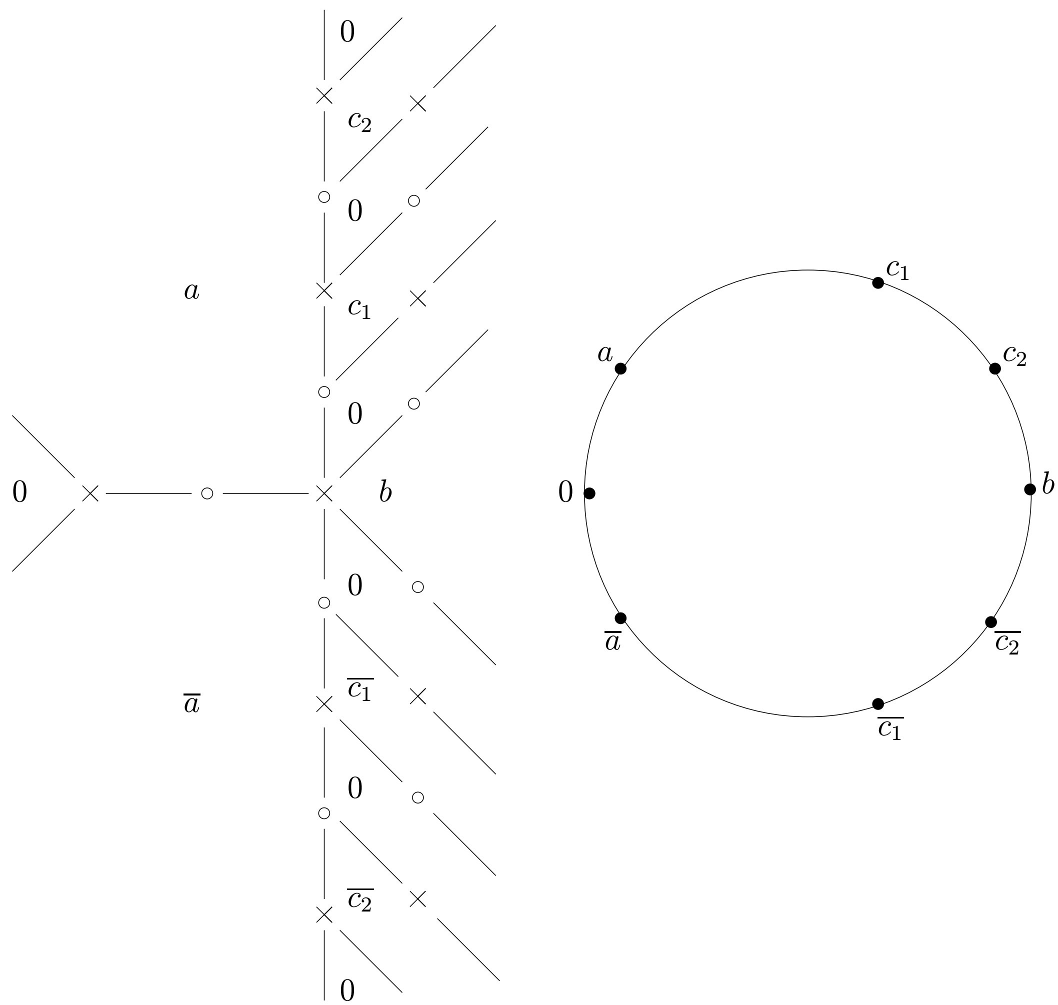

Figure 2: $d \equiv 0 \pmod 4$, the sequence of roots is finite.

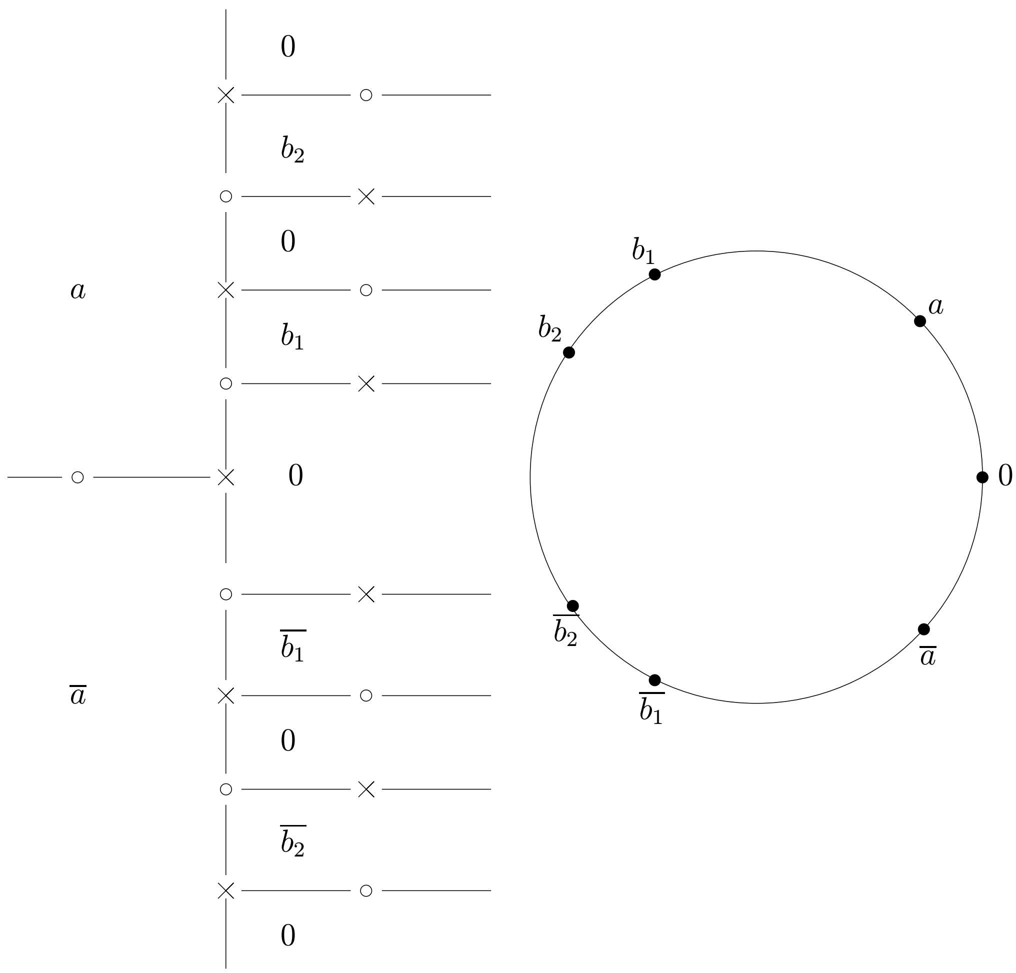

Figure 3: $d \equiv 1 \pmod 4$, the sequence of roots is infinite in one direction. In this picture, $d = 9$.

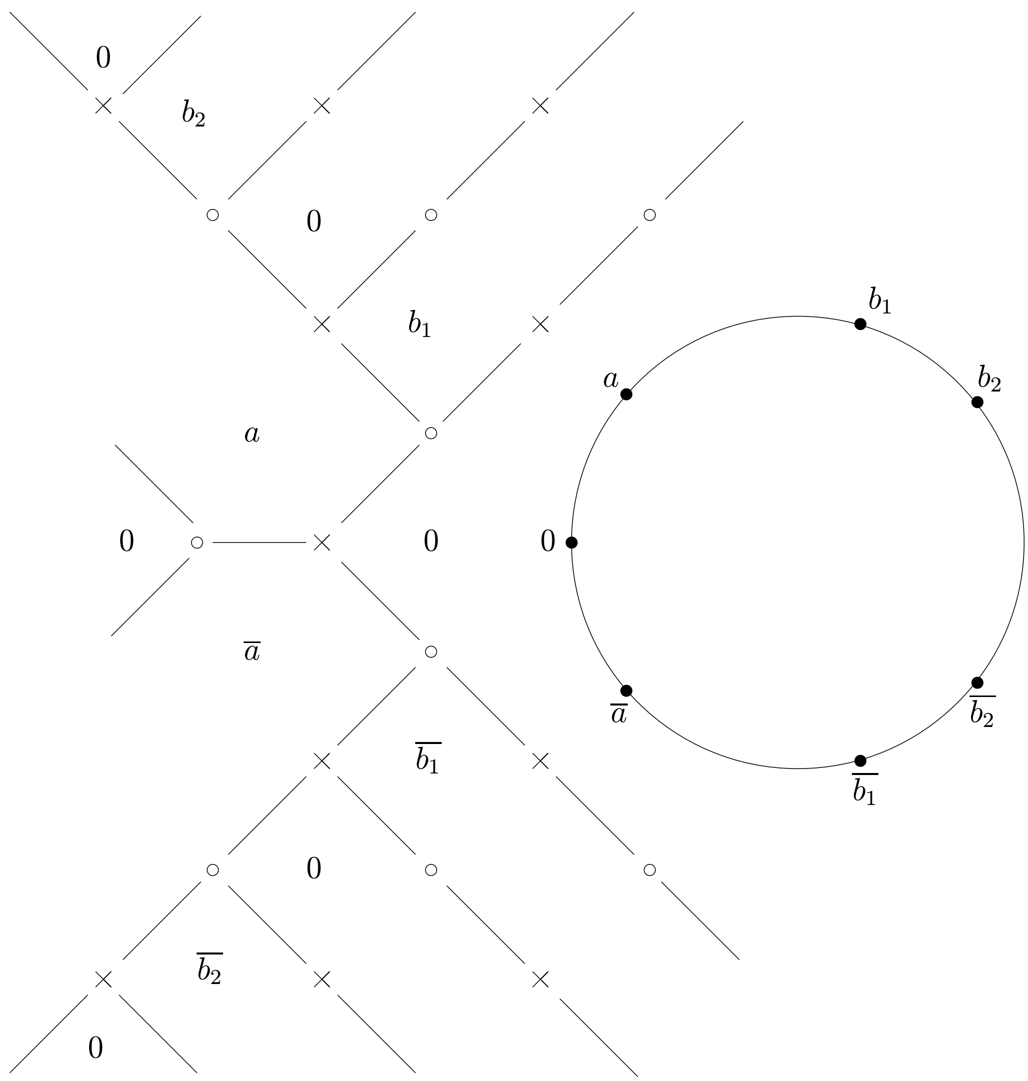

Figure 4: $d \equiv 2 \pmod 4$, the sequence of roots is finite. In this picture, $d = 10$.

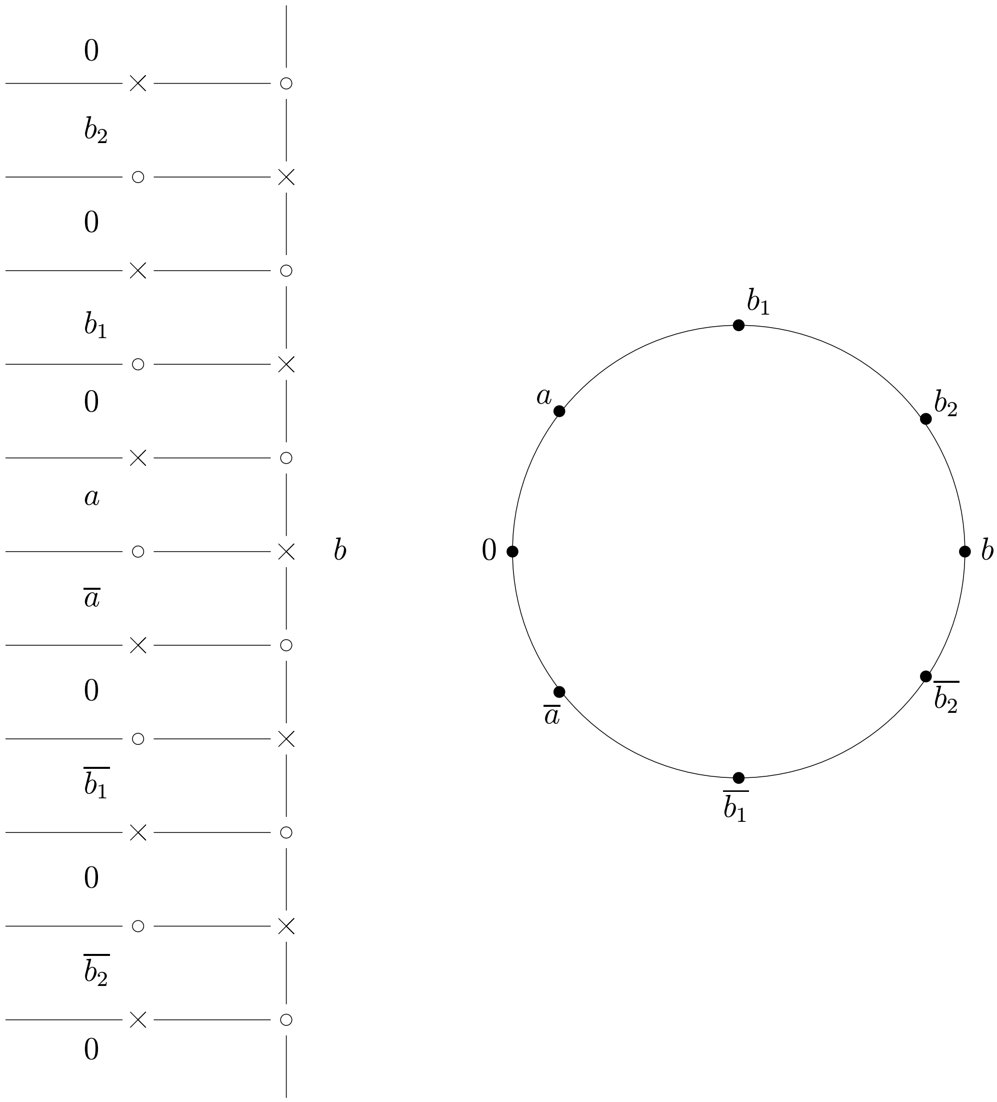

Figure 5: $d \equiv 3 \pmod 4$, the sequence of roots is infinite in one direction. In this picture, $d = 11$.

References

[1] L. Ahlfors, Complex Analysis, McGraw-Hill, NY, 1979.

[2] L. Ahlfors, Über eine in der neueren Wertverteilungstheorie betrachtete Klasse transcendenter Funktionen, Acta math., (1932) 375–406.

[3] D.A. Brannan and W.K. Hayman, Research problems in complex analysis, Bull. London Math. Soc., 21 (1989) 1–35.

[4] D. Drasin and A. Weitsman, Meromorphic functions with large sums of deficiencies, Adv. Math., 15 (1975) 93–126.

[5] A.A. Gol'dberg and I.V. Ostrovskii, Distribution of values of meromorphic functions, Moscow, Nauka, 1970 (Russian).

[6] G. Gundersen, Solutions of $f'' + P(z)f = 0$ that have almost all real zeros, Ann. Acad. Sci. Fenn. Math. 26 (2001) 483–488.

[7] G. Gundersen, On the real zeros of solutions of $f'' + A(z)f = 0$, where $A(z)$ is entire, Ann. Acad. Sci. Fenn. Math. 11 (1986) 275–294.

[8] S. Hellerstein and J. Rossi, Zeros of meromorphic solutions of second order linear differential equations, Math. Z., 192 (1986) 603–612.

[9] F. Nevanlinna, Über eine Klasse meromorpher Funktionen, 7 Congress Math. Scand., Oslo, 1929.

[10] R. Nevanlinna, Über Riemannsche Flächen mit endlich vielen Windungspunkten, Acta math., 58 (1932), 292–373.

[11] R. Nevanlinna, Analytic functions, Springer, NY, 1953.

[12] J. Rossi and S. Wang, The radial oscillation of solutions to ODE's in the complex domain, Proc. Edinburgh Math. Soc., (2) 39 (1996) 473–483.

[13] K. Shin, New polynomials P for which $f'' + P(z)f = 0$ has a solution with almost all real zeros. Ann. Acad. Sci. Fenn. Math. 27 (2002), no. 2, 491–498.

[14] Y. Sibuya, Global theory of a second order linear ordinary differential equation with a polynomial coefficient, North-Holland, Amsterdam, 1975.

[15] S. Stoïlov, Leçons sur les principes topologiques de la théorie des fonctions analytiques, Paris, Gauthier-Villars, 1956.

[16] E. Titchmarsh, Eigenfunction expansions associated with second-order differential equations, vol. 1, second edition, Oxford, Oxford UP, 1962.

Purdue University
West Lafayette IN
eremenko@math.purdue.edu

University of Michigan
Ann Arbor MI
merenkov@umich.edu