Topics in Geometric theory of meromorphic functions

A. Eremenko*

September 21, 2024

Abstract

This is a mini-course taught by the author in IMPAN in May 2023. It discussed the following circle of questions: Let X be an open simply connected Riemann surface with some concrete geometric description, and ϕ is a uniformizing function. How properties geometric properties of X are related to properties of ϕ. Classical and modern results are discussed, and some proofs are included.

1. Introduction

According to the Uniformization Theorem, for every simply connected Riemann surface X there exists a conformal homeomorphism $\phi : X_0 \to X$, where X_0 is one of the three standard regions: the Riemann sphere $\overline{\mathbf{C}}$, the complex plane $\mathbf{C}$ or the unit disc $\mathbf{U}$. We say that the *conformal type* of X is *elliptic, parabolic* or *hyperbolic*, respectively. The map ϕ is called the *uniformizing map.* If X is given by some geometric construction, the problem arises to relate properties of ϕ to those of X. This includes the determination of the conformal type of X, which is called the *type problem.*

The case which was studied most is that $X \subset \mathbf{C}$ is a simply connected region, $X \neq \mathbf{C}$. Then X is of hyperbolic type, and ϕ is a univalent function

*Supported by the Simons Foundation Award No. 663281 granted to the Institute of Mathematics of the Polish Academy of Sciences for the years 2021-2023 and the mini-semester Modern holomorphic dynamics and related fields held at the Faculty of Mathematics, Informatics and Mechanics of the University of Warsaw (MIMUW) within the Thematic Research Programme action, a part of the Excellence Initiative – Research University Programme (IDUB) at the University of Warsaw.

in $\mathbf{U}$. An example of the result relating geometric properties of $X \subset \mathbf{C}$ and properties of ϕ is the classical theorem of Caratheodory: *ϕ has a continuous extension to the closed disc if and only if ∂X is locally connected.* This is an example of *exact correspondence* between a class of functions and a class of regions.

Second example is the Koebe 1/4 theorem.

Yet another example is the Ahlfors distortion theorem which relates the growth of a uniformizing map to the geometry of the image domain.

We recall a more general construction of X. By a (topological) *surface* we mean in this text a real manifold of dimension 2 which is Hausdorff, connected, oriented, and has a countable base.

A surface *spread over the sphere* is a pair (X, p), where X is a topological surface and $p : X \to \overline{\mathbf{C}}$ a continuous, open and discrete map. This map p is usually called the *projection.* The natural equivalence relation is $(X, p) \sim (Y, q)$ if there is a homeomorphism $\phi : X \to Y$ with the property $p = q \circ \phi$.

By a *geometric property* of a surface spread over the sphere we mean a property which is invariant when p is replaced by a composition $p \circ \phi$ with an arbitrary homeomorphism of X.

A more general *topological property* is one that remains unchanged when p is replaced by $\psi \circ p \circ \phi$, where $\psi : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ and $\phi : X \to X$ are homeomorphisms.

Examples of topological properties are an omitted value, or completely ramified value, or a direct singularity. An example of a geometric property is the Bloch radius.

According to a theorem of Stoïlov, every continuous open and discrete map p between surfaces locally looks like $z \mapsto z^n$. Those points where $n > 1$ are isolated, they are called *critical points*, or multiple points of multiplicity n. Stoïlov's theorem implies that there is a unique conformal structure on X which makes p holomorphic. If ϕ is a uniformizing map, then $f = p \circ \phi$ is a meromorphic function in one of the three standard regions $\overline{\mathbf{C}}, \mathbf{C}$ or $\mathbf{U}$. The surface (X, p) spread over the sphere is then the "Riemann surface of f^{-1}". We also call it the surface associated to f, or say that f is associated with (X, p).

Every surface admits a topologically holomorphic map to the sphere, this is another result of Stoïlov.

If D is a region on the sphere, a *branch* of p^{-1} in D is a continuous function $\psi : D \to X$ such that $p \circ \psi = \mathrm{id}_D$.

We can define the length of a curve in X as the spherical length[1] of its image under p. Then X becomes a metric space with an *intrinsic metric*, which means that the distance between two points is the infimum of the lengths of curves connecting these points. Similarly, if $p : X \to \mathbf{C}$, and the Euclidean metric in $\mathbf{C}$ is used to measure lengths of curves, we obtain a Riemann surface *spread over the plane*. Theses intrinsic metrics are called *spherical* and *Euclidean* metrics on X.

The intrinsic metric on X is a smooth Riemannian metric of constant curvature on the complement of the critical set of p, and the critical points are the *conic singularities* of the metric. It is easy to show that the intrinsic metric on X determines the projection p up to an isometry of the image sphere or the plane. In what follows, unless otherwise stated, (X, p) denotes a simply connected surface spread over the sphere, equipped with the intrinsic spherical metric.

2. Singularities

There is another useful metric on X, the Mazurkiewicz metric, $\rho(x, y) := \inf\{\text{diam } p(\gamma)\}$ for $x, y \in X$, where diam is the diameter with respect to the spherical metric in $\overline{\mathbf{C}}$, and the infimum is taken over all curves $\gamma \subset X$ connecting x and y. Every non-critical point $x \in X$ has a neighborhood where the Mazurkiewicz metric coincides with the intrinsic one, but in general the Mazurkiewicz metric is smaller. For example, on the surface $(\mathbf{C}, \cos)$ spread over the sphere, the intrinsic distance between 0 and $2\pi k$ is $2\pi k$, while the Mazurkiewicz distance is π.

Mazurkiewicz metric is a convenient tool to define the "singularities of the inverse function".

Let X^* be the completion of X with respect to the Mazurkiewicz metric. Then p has a unique continuous extension to X^*. The elements of the set $Z = X^* \backslash X$ are called *transcendental singularities* of (X, p). We denote by $Z_a = \{z \in Z : f(z) = a\}$ the set of transcendental singularities lying over the point $a \in \overline{\mathbf{C}}$.

The simplest example of a transcendental singularity is a logarithmic branch point: that is a point in Z which is isolated in Z.

The *algebraic singularities* are just the critical points of p, so they belong to X. The images of transcendental and algebraic singularities under p will

[1] We choose the spherical length element to be $2|dz|/(1 + |z|^2)$, so that the curvature of the spherical metric is $+1$.

be called *singular values.* The images of critical points are called *critical values.*

If $A = \overline{f(Z)}$ is the closure of singular values then

$$F : X \backslash f^{-1}(A) \to \overline{\mathbf{C}} \backslash A$$

is a covering map.

There is another way to define the space X^* independent on any metric. Let a be a point in $\overline{\mathbf{C}}$, and $\mathbf{U}(a)$ the set of all connected neighborhoods of a. Consider a map F which to every $U \in \mathbf{U}(a)$ puts into correspondence a connected component of the set $p^{-1}(U)$, so that $U_1 \subset U_2$ implies $F(U_1) \subset F(U_2)$, and all $F(U)$ are non-empty. Two cases are possible:

a) $\bigcap_{U \in \mathbf{U}(a)} F(U) =$ *one* point in X, or

b) $\bigcap_{U \in \mathbf{U}(a)} F(U) = \emptyset$.

In the second case we say that F defines a *transcendental singularity* over a. We add all these transcendental singularities to X, and obtain the set X^*. A base of neighborhoods of a transcendental singularity is defined as the collection of the sets $F(U)$ used in this definition, and the projection p extends by the formula $p(x) = a$ for a transcendental singularity x over a. Then the extended map p is continuous on X^*.

It is easy to show that both definitions are equivalent.

An *asymptotic curve* with an *asymptotic value* $a \in \overline{\mathbf{C}}$ is a curve $\gamma : [0, 1) \to X$ such that $\gamma(t) \to \infty$ as $t \to 1$, where ∞ is the added element of one-point compactification of X, and $p(\gamma(t)) \to a$.

Every asymptotic curve with asymptotic value a defines a transcendental singularity over a, and vise versa. One can define an equivalence relation on the set of asymptotic curves with given asymptotic value so that the classes of equivalence will correspond bijectively to transcendental singularities.

The set of asymptotic values is an analytic set in the sense of Suslin [25].

The following classification of transcendental singularities is due to F. Iversen. A transcendental singularity F over a is called *direct* if for some $U \in \mathbf{U}(a)$ the image $p(F(U))$ does not contain a.

Otherwise the transcendental singularity is called *indirect.* That is for an indirect singularity all $F(U)$ contain a-points of p.

The simplest kind of direct singularities are logarithmic ones. They are defined by the property that $p : F(U) \to U$ is a universal covering of $U \backslash \{a\}$ for some $U \in \mathbf{U}(a)$.

When U is a spherical or Euclidean disk centered at an asymptotic value a, the components of $F(U)$ are usually called *tracts* over a.

Examples. $(\mathbf{C}, \exp)$ has two transcendental singularities: one over 0 and one over ∞. They are both logarithmic branch points.

$(\mathbf{C}, \cos)$ has two logarithmic singularities over ∞ and two critical values over $1, -1$.

$(\mathbf{C}, z \sin z)$ has one direct singularity over ∞, and it is not isolated since it is the limit (with respect to Mazurkiewicz metric) of critical values.

$(\mathbf{C}, (\sin z)/z)$ has two indirect singularities over 0, two logarithmic singularities over ∞, and infinitely many critical values converging to 0.

By the modular function we mean the universal cover $\mathbf{U} \to \mathbf{C}\backslash\{0, 1\}$. it has infinitely many logarithmic singularities over $0, 1, \infty$ and no other singularities.

Peter Seibert [32]–[35] tried to characterize those metric spaces that can occur as the set of singularities of open simply connected surfaces spread over the sphere. He denotes $Z_a = p^{-1}(a) \cap Z$. The following properties are easy to prove: Z and Z_a are complete Hausdorff spaces with countable dense subsets, Z_a are totally disconnected. Besides the topology defined by Mazurkiewicz metric, each of these spaces has a cyclic order structure induced by the natural cyclic order on asymptotic curves. The canonical topology defined by this order is coarser than the metric topology. Seibert defines a continuous order-preserving injection $\phi : Z \to \mathbf{T}$, where $\mathbf{T}$ is the unit circle. However in general such a bijection cannot be a homeomorphism: for example, for the elliptic modular function, the topology of Z is discrete, while the cyclic order is that of the subset of the unit circle consisting of the points whose arguments are commensurable with π. Ullrich constructed an example of a parabolic surface whose set Z has this topology and this order.

Now, if Λ is any closed subset of the unit circle, then there exist surfaces of both hyperbolic and parabolic types whose singular set Z is order-preserving homeomorphic to Λ.

3. Speiser class and line complexes

It is useful to have a way to visualize surfaces spread over the sphere. The simplest way to do this is called the line complex, and it is defined for the class of surfaces (X, p) with the property that all singularities (transcendental and algebraic) lie over a finite set $A \in \overline{\mathbf{C}}$, so that

$$p : X \backslash p^{-1}(A) \to \overline{\mathbf{C}} \backslash A \tag{1}$$

is an (unramified) covering map. The class of such surfaces spread over the sphere is called the Speiser class and is denoted by $\mathbf{S}$.

We also define a wider class of surfaces, which are called *cellular*. They are characterized by the property that all singular values belong to some Jordan curve Γ. We assume that this curve is real analytic (in all known applications it is sufficient to consider only circles). The p-preimage of such a curve γ is called a *net*. This net consists of curves in X which can cross only at the critical points, and never accumulate in X. These curves break X into simply connected regions, and one can draw a net in the plane (when X is open), or on the sphere. Two nets are called equivalent if they correspond by a homeomorphism of X. Equivalence class of nets determines p completely. A net decomposes the plane into simply connected regions (faces) open curves (edges) and points (vertices).

Now we return to the Speiser class. The elements of A are singular values. Consider an oriented Jordan curve Γ which contains all elements of A. It defines a cell decomposition of $\overline{\mathbf{C}}$ with two faces D_x on the left hand side of γ and D_o on the right hand side of Γ. The edges of this cell decomposition are open arcs of Γ between the adjacent points of A, and vertices are points of A. The preimage $p^{-1}(\Gamma)$ is the net of p.

Now we consider the dual cell decomposition of that defined by Γ. It has two vertices which are traditionally denoted by $x \in D_x$ and $o \in D_0$, and $q = |A|$ edges. There is a cyclic order of the points of A consistent with the orientation of Γ, and the corresponding cyclic order on the edges, so we can label the vertices and edges by residues modulo q. Each face of the dual cell decomposition contains exactly one point of A, and we label it by this point. Preimage of this dual cell decomposition is a cell decomposition of X (which we can identify with the plane or with the sphere, depending on whether X is open or closed) and it is completely defined by its 1-skeleton. This one skeleton is called the *line complex* or the Speiser graph. It is an embedded graph, and the faces, edges and vertices of the corresponding cell decomposition are labeled by their p-images. Two Speiser graphs are equivalent if they correspond by a homeomorphism of X respecting the labels.

It is clear that Speiser graphs have these properties:

(i) They are bi-partite, and all vertices have the same degree q

(ii) Their faces are open disks with even or infinite number of vertices on the boundary.

Every properly embedded graph in the plane or in the sphere with these properties is a line complex of some surface spread over the sphere.

Multiple edges between two vertices result in digonal faces. Replacing each bunch of adjacent multiple edges by a single edge, we obtain a *reduced line complex*. It is a bi-partite embedded graph whose vertices have bounded degree, and still satisfies (ii). If the faces of such a graph are labeled as above, the full line complex can be uniquely recovered by adding extra edges in the appropriate places.

The line complex depends not only on p but also on the choice of the base curve Γ. Continuous deformation of this curve with fixed set A does not change the line complex, so the mapping class group of the sphere with q marked points acts on line complexes, and this action can be explicitly described [22].

D. Sullivan asked for which surfaces spread over the sphere, the conformal type is defined by topology. More precisely, let (X, p) be a surface spread of the sphere. We say that it is of stable type, if $(X, \phi \circ p)$ has the same conformal type as (X, p) for every homeomorphism ϕ of the sphere.

This is the case for surfaces of class S, since the singular values can be displaced quasiconformally. This observation is due to Teichmüller, and this was one of the first applications of quasiconformal mappings to the theory of meromorphic functions. This result can be somewhat generalized by assuming that singularities are "uniformly isolated" that is the distances between them (in spherical or Mazurkiewicz metric) are at least $\delta > 0$. On the other hand, it was proved by Volkovyskii [39, Sect. 86-92] that the the parabolic type of the surface surface $(\mathbf{C}, (\sin z)/z)$ is unstable. One can also give examples of parabolic surfaces with non-isolated singularities whose type is stable. So it remains unclear whether there is a reasonable characterization of surfaces of stable type, even in parabolic case.

4. Theorems of Gross and Iversen

After these topological preliminaries, we begin to investigate the geometric conditions for the conformal type. First we notice that the set of asymptotic values by itself does not permit to make any conclusions about the conformal type.

There are both parabolic and hyperbolic surfaces spread over the sphere whose set of asymptotic values is any analytic set $A \subset \overline{\mathbf{C}}$.

Different constructions were given by Heins and by Canton, Drasin and Granados.

In the parabolic case, there is one condition on this set:

Theorem 1. (F. Iversen) *If a meromorphic function f in the plane omits a value a, then a is an asymptotic value.*

Proof. This is easy if f is a rational function, so assume that it is transcendental. Suppose wlog that $a = \infty$, so that f is entire. For any $r > 0$, consider the set $\{z : |f(z)| > r\}$. It is not empty for every $r > 0$ by Liouville's theorem. Let U_r be a component of this set. By Phragmén–Lindelöf theorem, f is unbounded in U_r. This implies that we can choose nested components for all R, that is for $r_1 > r_2$, $U_{r_1} \subset U_{r_2}$. This defines a singularity over ∞, and thus an asymptotic curve.

So the set of asymptotic values of a surface of parabolic type can be large. However in some sense the singularities must be "rare". This is the contents of the following

Theorem 2. (Gross) *Let f be a meromorphic function in the plane, and $f(z_0) = w_0$, $f'(z_0) \neq 0$, and ϕ is a germ of f^{-1} which sends w_0 to z_0. Then the maximal starlike region to which ϕ has an analytic continuation contains a ray from w_0 to ∞ in almost every direction.*

Remarks. Of course, infinity does not play any special role, and was used only for simplicity of formulation. Same applies to the condition $f'(z_0) \neq 0$.

Proof. Suppose for simplicity that $z_0 = w_0 = 0$. Let G be the maximal starlike region to which ϕ has an analytic continuation. For a maximal segment $[0, w) \subset G$, w is either critical or asymptotic value. The set of critical values is countable, so all we need to prove is that the set of directions of segments which end at an asymptotic value have measure 0. Let $G_R = G \cap \{w : |w| < R\}$. It is sufficient to prove the statement for maximal segments whose endpoints are in G_R. The germ ϕ has an analytic continuation to G_R and maps it on some region D in z-plane, $0 \in D$. Any maximal interval in G_R which ends at an asymptotic value corresponds to a curve in D from 0 to ∞. Consider the intersection of a circle $|z| = r$ with D. It consists of countably many arcs γ. The images $f(\gamma)$ of these arcs in D_R separate 0 from the asymptotic values which are the endpoints of the maximal segments in G_R.

So it is sufficient to prove that the total length of these images with respect to the metric $|dw|/|w|$ can be made arbitrarily small by an appropriate choice or r. This length is

$$s := \int_{\ell_r} \left| \frac{f'}{f} \right| r \, d\theta,$$

where $\ell_r = \{z \in D : |z| = r\}$. By Cauchy–Schwarz inequality,

$$s^2 \leq 2\pi r \int_{\ell_r} \left| \frac{f'}{f} \right|^2 r \, d\theta.$$

Dividing by $2\pi r$ and integrating from some $r_0 > 0$ to ∞, we obtain

$$\int_{r_0}^{\infty} \frac{s^2(r)}{2\pi r} dr \leq \int_{r_0}^{\infty} dr \int_{\ell_r} \left| \frac{f'}{f} \right|^2 r \, d\theta.$$

the integral in the RHS is at most the area of G_R from which a neighborhood of 0 is removed, with respect to the metric $|dw|/|w|$, so this area is clearly finite. Therefore, the integral in the LHS must be convergent, which implies $\liminf_{r\to\infty} s(r) = 0$. This completes the proof.

This was the first application of the Length and Area argument in these lectures, and we will see many more.

One important consequence of the Gross Theorem is

Theorem 3. (Iversen) *Let f be a meromorphic function in the plane, ϕ is any holomorphic germ of f^{-1} at a point w_0, and $\gamma : [0,1] \to \overline{\mathbf{C}}$ a curve with $\gamma(0) = w_0$. Then for every $\epsilon > 0$ there is a curve $\gamma_1 : [0,1] \to \overline{\mathbf{C}}$ with $\gamma_1(0) = w_0$ and $\gamma_1(t) - \gamma(t)| \leq \epsilon, t \in [0,1]$ such that ϕ has an analytic continuation along γ_1.*

So the set of singularities of a parabolic surface cannot contain continuum.

There are two interesting unsolved questions related to the Gross' theorem.

Question 1. *Can the estimate of the size of the exceptional set in the Gross theorem be improved?*

It is known that it can have the power of continuum, [39, Sect. 45], but in all known examples it is a much smaller set than zero (logarithmic) capacity. Is it really always of zero capacity?

Question 2. *Let $F(z,w)$ be an entire function of two variables. Consider a germ ϕ of the implicit function satisfying $F(z,\phi(z))=0$. Does the maximal star of analyticity of ϕ contain rays in almost every direction?*

It is due to Julia that germs in Question 2 have the Iversen property. But there is no known analog of Gross's theorem for this case.

5. Theorems of Picard and Bloch and some of their development

Picard's theorem saying that a non-constant meromorphic function cannot omit 3 values can be considered as a sufficient condition of hyperbolic type: if the projection map omits three values than the surface must be of hyperbolic type.

The number of generalizations of Picard's theorem is really enormous, so we can consider here only some lines of development.

A point $a \in \overline{\mathbf{C}}$ is called a *totally ramified value* (of multiplicity $m \geq 2$) of a surface (X,p) spread over the sphere, if all preimages of a under p are multiple, (of multiplicity at least m). We allow $m=\infty$ which means that the value a is omitted.

Nevanlinna's theorem *If a surface spread over the sphere has q totally ramified values of multiplicities m_k, $1 \leq k \leq q$, and*

$$\sum_{k=1}^{q}\left(1-\frac{1}{m_k}\right) > 2, \tag{2}$$

then the surface is hyperbolic. In particular, a parabolic surface has at most four totally ramified values.

The example $(\mathbf{C},\wp)$ shows that a parabolic surface can indeed have 4 totally ramified values. Moreover, one can easily show that the equation

$$\sum_{k=1}^{q}\left(1-\frac{1}{m_k}\right) = 2$$

has 6 solutions up to permutation of the m_k:

$$(\infty,\infty),\ (2,2,\infty),\ (2,4,4),\ (3,3,3),\ (2,4,6),\ (2,2,2,2).$$

and to each of these solutions corresponds an elliptic or trigonometric function.

We give a short proof of this theorem which is due to Robinson, and based on the

Ahlfors Lemma. *Let u be a subharmonic function in the disk $|z| < R$ that satisfies*

$$\Delta u \geq e^{2u}. \tag{3}$$

Then $u(z) \leq v_R$ where

$$v_R(z) := \log \frac{2R}{R^2 - |z|^2}.$$

Remarks. 1. Here $v_R(z)|dz|$ is the line element of the the hyperbolic metric, which is a complete conformal metric of curvature -1 on the disk. Such metric exists and is unique on every hyperbolic Riemann surface.

2. Inequality (3) is in the sense of D'-distributions. In many applications the function u is not smooth. Ahlfors wrote before the appearance of the theory of distributions, so he defines what he calls an "ultrahyperbolic metric" $\rho(z)|dz|$ as follows:

(i) ρ is upper semi-continuous,

(ii) at every z_0 with $\rho(z_0) > 0$ there exists a *supporting metric* ρ_0, of class C^2 in a neighborhood of z_0 such that $\Delta \log \rho_0 \geq \rho_0^2$, and $\rho \geq \rho_0$, while $\rho(z_0) = \rho_0(z_0)$.

The formulation with generalized Laplacian is due to M. Heins.

3. The general framework of metric spaces with length element $\rho(z)|dz|$ where ρ is not smooth is provided by Aleksandrov's theory of "surfaces of bounded curvature", which treats arbitrary metrics with $\rho = e^u$, where u is a difference of subharmonic functions. The integral curvature on such a surface is a Radon measure.

In particular, if $\rho(w) \sim |w|^{\alpha-1}$, where $\alpha > 0$, the metric has a *conic singularity* with angle $2\pi\alpha$. When such a metric is pulled back by a function $w = f(z)$ with critical point of order $m-1$ (that is $f(z) = z^m$ in local coordinates) then the pull back metric has a conic angle $m\alpha$:

$$|w|^{\alpha-1}|dw| = m|z|^{m\alpha-m}|z|^{m-1}|dz| = m|z|^{m\alpha-1}|dz|.$$

To a conic singularity with angle $2\pi\alpha$ corresponds an atom of the curvature mass $2\pi(1-\alpha)$. (Laplacian at a conic singularity has an atom of opposite sign).

For example, the surface of a cube, has total curvature 4π since it is homeomorphic to the sphere. The curvature is zero everywhere except the vertices, and at each vertex we have the total angle $3\pi/2$, so each vertex gives an atom of mass $\pi/2$ and the sum over 8 vertices is 4π.

Proof of Ahlfors Lemma Fix any $r \in (0, R)$. If the set $D = \{z : |z| < r,\ u(z) > v_r(z)\}$ is not empty, then there is a point in this set where $u - v$ has a local maximum. Indeed, since $v_r(z) \to +\infty$ as $|z| \to r$, we have $v(z) = v_r(z)$ on the boundary of each component of D, so a positive upper semi-continuous function must have a global maximum in this component. On the other hand, $u - v$ is subharmonic on D since

$$\Delta(u - v_r) \geq e^{2u} - e^{2v_r} > 0.$$

This contradicts the maximum principle, and proves the lemma.

Corollary. *If a Riemann surface possesses a conformal metric of curvature* $\leq -\delta < 0$*, then it is hyperbolic.*

Proof of Nevanlinna's theorem. Without loss of generality $a_k \in \mathbf{C}$ for all k. Consider the metric on the Riemann sphere with the line element $\lambda(w)|dw|$, where

$$\lambda(w) = \prod_{k=1}^{n} (1 + |w - a_k|^{\epsilon})|w - a_k|^{1/m_k - 1},$$

where $\epsilon > 0$ is so small that

$$n\epsilon - \sum_{k=1}^{n} \left(1 - \frac{1}{m_k}\right) < -2,$$

which is possible to achieve in view of our assumption (2). Using the formula,

$$\Delta \log(1 + |z|^{\epsilon}) = \epsilon^2 |z|^{\epsilon - 2}(1 + |z|^{\epsilon})^{-2}$$

one can be verify directly that the curvature of this metric on the complement of the singularities a_k is $\leq -\delta < 0$ for some δ. When we pull back this metric via f, the angles at all the singularities will be $\geq 2\pi$; in local coordinates at a point of multiplicity m we have

$$|w|^{1/m_k - 1}|dw| \sim |z|^{m/m_k - m + m - 1}|dz| = |z|^{m/m_k - 1}|dz|,$$

and since it is assumed that $m \geq m_k$, we have a metric of curvature $\leq -\delta$. By Ahlfors Lemma, such a metric cannot exist in the plane.

Of curse one could use the hyperbolic metric with conic singularities, but Ahlfors Lemma allows greater flexibility, and permits to construct the metric explicitly.

Ahlfors's Five Islands Theorem. *Suppose that for five Jordan regions with disjoint closures on the sphere, there are no branches of p^{-1} in any of these regions. Then X is of hyperbolic type.*

This theorem was stated for the first time by Bloch [10] (with discs instead of Jordan regions) and proved by Ahlfors in [2], as a corollary from his "Uberlagerungsflachentheorie". It is a recent discovery [6] that actually Theorem 1.2 can be derived from Theorem 1.1 by a simple argument. This derivation is based on the following important principle:

Zalcman's Lemma. *Let F be a family of meromorphic functions in some region D, which is not normal in D. Then there exists a sequence $f_n \in F$, and two sequences $r_n > 0$ and $z_n \in D$ such that there exists a non-constant limit*

$$f(z) = \lim_{n\to\infty} f_n(r_n z + z_n),$$

uniform on compact subsets of $\mathbf{C}$, and moreover,

$$f^{\#}(z) := \frac{|f'(z)|}{1+|f(z)|^2} \leq f^{\#}(0) = 1, \quad z \in \mathbf{C}.$$

Proof. Without loss of generality we may assume that D is the unit disc, functions of the family F are meromorphic in the closure of D, and F is not normal at 0. This means that there exists sequence $f_n \in F$ and $w_n \to 0$ such that $f_n^{\#}(w_n) \to \infty$. Then

$$\max_{z\in D}(1-|z|)f_n^{\#}(z) = (1-|z_n|)f_n^{\#}(z_n) \geq (1-|w_n|)f_n^{\#}(w_n) \to \infty.$$

Thus $r_n = 1/f_n^{\#}(z_n) = o(1-|z_n|)$. We claim that r_n and z_n have the required property. Indeed, putting $g_n(z) = f_n(r_n z + z_n)$, we obtain $g_n^{\#}(0) = 1$ and

$$g_n^{\#}(z) = r_n f_n^{\#}(r_n z + z_n) \leq r_n f_n^{\#}(z_n)\frac{1-|z_n|}{1-|r_n z + z_n|} \leq \frac{1-|z_n|}{1-|z_n|-r_n|z|} \to 1,$$

because $r_n = o(1 - |z_n|)$. So g_n is a normal family. After selecting a subsequence we get $g_n \to f$ for some meromorphic function f, $f^\#(z) \leq 1, z \in \mathbf{C}$ and $f^\#(0) = 1$, which proves all statements of the lemma.

Now we derive Theorem 1.2 by contradiction. Suppose that f is a meromorphic function in the plane, and D_k are five disjoint Jordan regions such that there are no inverse branches of f^{-1} in D_k, $1 \leq k \leq 5$. Choose five points a_k on the sphere and consider quasiconformal homeomorphisms ψ_n of the sphere which map each D_k into $1/n$-neighborhood of a_k. Then the surfaces $(\mathbf{C}, \psi_n \circ f)$ spread over the sphere are all parabolic because the maps $\psi_n \circ f$ are *quasiregular*[2], so there exist homeomorphisms ϕ_n of $\mathbf{C}$ such that $f_n = \psi_n \circ f \circ \phi_n$ are meromorphic functions. These functions have no inverse branches over $1/n$ neighborhoods of a_k. We can use arbitrarily in the choice of ϕ_n to normalize our functions: $f_n(0) = 0$, $f_n'(0) = 1$. If the family $\{f_n\}$ is normal in the whole plane, then the limit functions are non-constant because of the normalization, and it is easy to see that a_k are totally ramified values of these functions, contradicting Theorem 1.1. If $\{f_n\}$ is not a normal family in the plane, we apply Zalcman's lemma to make it normal, and again obtain a contradiction.

Zalcman's lemma shows the importance of study of meromorphic functions with bounded spherical derivative. Very little is known about this class. We mention a theorem of Clunie and Hayman that if an entire function has bounded spherical derivative than it is at most of order one, normal type. (A typical meromorphic function of this class has order two, normal type). This theorem of Clunie and Hayman was subject to several generalizations. Eremenko extended it to holomorphic maps $\mathbf{C} \to \mathbf{P}^n$: *if such a map omits n hyperplanes, and has bounded spherical derivative, then it is of at most normal type, order* 1. Duval and da Costa extended this in the spirit of the second main theorem of Nevanlinna: if f is a meromorphic function with bounded spherical derivative, then

$$T(r, f) \leq N(r, a) + O(r), \quad \text{for every} \quad a \in bC.$$

They also obtained a similar result for the maps $\mathbf{C} \to \mathbf{P}^n$, though it is less precise than expected. Tsukamoto investigated the optimal constant in

[2]Quasiregular maps in dimension 2 are compositions of holomorphic functions with quasiconformal maps. Quasiconformal maps preserve the conformal type of a simply connected Riemann surface.

the following problem: what is the maximal value of $\limsup T(r,f)/r$ for meromorphic functions whose spherical derivative does not exceed 1. The extremal function is supposed to be the Weierstrass function for a hexagonal lattice.

Original application of Ahlfors Lemma was to the estimate of the Bloch constant.

Theorem of Valiron *Let (X,p) be a parabolic surface spread over the plane. Then X contains Euclidean disks of arbitrarily large radius.*

Using Valiron's idea, Bloch stated the more general theorem:

Theorem of Bloch *There is an absolute constant B, such that for every function holomorphic in the unit disk and satisfying $|f'(0)| = 1$ and for every $\epsilon > 0$ there is a branch of f^{-1} defined in some disk of radius $> B - \epsilon$.*

Following Ahlfors, we derive Bloch's theorem from the Ahlfors Lemma, and show that $B \geq \sqrt{3}/4$. Let $R(w)$ be the radius of the largest unramified disk centered at w on the Riemann surface of the inverse function. Consider the metric $\lambda(w)|dw|$ with

$$\lambda(w) = \frac{A}{\sqrt{R(w)}(A^2 - R(w))},$$

where $A^2 > B_f := \sup_w R(w)$. Then one can show that the pull-back $\lambda(f(z))|f'(z)||dz|$ is a metric of curvature $\leq -\delta < 0$ in the unit disk.

Comments. The expression

$$\frac{A}{\sqrt{|w|}(A^2 - |w|)}$$

is the density of the complete metric of curvature -1 in the disk $|w| < A^2$, with conic singularity with angle π at 0. If D is the largest schlicht disk around w_0, then there is a singularity w_1 of f^{-1} on the boundary of this disk, whose distance to w_0 is $R(w_0)$, and

$$\frac{A}{\sqrt{|w - w_1|}(A^2 - |w - w_1|)}|dw|$$

is the supporting metric at w_0 in the sense of Ahlfors.

The problem of finding the exact value of the optimal constant B in Bloch's theorem is still unsolved. Heins showed that $B > \sqrt{3}/4$, and Bonk was the first to obtain an explicit number $> \sqrt{3}/4$. This was only slightly improved since then: the world record is $B \geq \sqrt{3}/4 + 2 \cdot 10^{-4} \approx 0.435$. On the other hand, there is a very plausible conjecture that

$$B = \sqrt{\pi} 2^{1/4} \frac{\Gamma(1/3)}{\Gamma(1/4)} \sqrt{\frac{\Gamma(11/12)}{\Gamma(1/12)}} \approx .4719.$$

The extremal function is the covering of the complement of the hexagonal lattice with simple ramification points over the lattice points.

If we take the five regions in Ahlfors's theorem to be spherical discs of equal radii, Theorem 1.1 implies the following [2]: *Suppose that for some* $\epsilon > 0$ *there are no branches of* p^{-1} *in discs of radii* $\pi/4 - \epsilon$. *Then* X *is of hyperbolic type.* The question arises, what is the best constant for which this result still holds. Let $B(X,p)$ be the supremum of radii of discs where branches of p^{-1} exist, and $B = \inf B(X,p)$, where the infimum is taken over all surfaces of elliptic or parabolic type. Ahlfors's estimate $B \geq \pi/4$ was improved by Pommerenke [30] to $B \geq \pi/3$, and recently the sharp result was obtained in [11]:

$$B = b_0 := \arccos(1/3) \approx 0.39\pi.$$

We have $B(\mathbf{C}, \wp) = B$, where $\wp$ is the Weierstrass function of a hexagonal lattice. It is interesting to notice that $B = b_0$ implies Theorem 1.1 by a simple argument given in [11].

For surfaces (X,p) of elliptic type we have $B(X,p) > b_0$, but it is not known whether the constant b_0 is best possible in this inequality.

We sketch the proof for elliptic surfaces. First one constructs a triangulation T of X into geodesic triangles, so that the vertices of this triangulation coincide with the set of critical points, and the circumscribed radius of each triangle is at most $B(X,p)$. This is always possible to do if $B < \pi/2$ which we can assume. Suppose now that $B(X,p) \leq b_0$. Then the circumscribed radius of each triangle is at most b_0, and an elementary geometric argument shows that the area of each triangle is at most π. Notice that by Gauss formula, $\text{area}(\Delta) = \sum \alpha(\Delta) - \pi$, where $\alpha(\Delta)$ is the sum of the angles of Δ. As $\text{area}(\Delta) \leq \pi$ we conclude that $\text{area}(\Delta) \leq \alpha(\Delta)/2$. If we denote by $\alpha(v)$ the total angle at a vertex, then $\alpha(v) = 4\pi$, assuming that all critical points have multiplicity 2. If d is the degree of our rational function then the total area is

$$4\pi n = \sum_{\Delta \in T} \text{area}(\Delta) \leq \frac{1}{2} \sum_{v} \alpha(v) = 2\pi(2n-2),$$

and this is a contradiction.

The proof of $B \geq b_0$ for parabolic surfaces is more complicated. For our class of surfaces with intrinsic metric, one can define integral curvature [31] as a signed Borel measure on X which is equal to the area on the smooth part of X and has negative atoms at the critical points of p. The assumption that $B(X,p) < b_0$ implies that the atoms of negative curvature are sufficiently dense on the surface, so that on large pieces of X the negative part of the curvature dominates the positive part. Then a bi-Lipschitz modification of the surface is made, which spreads the integral curvature more evenly on the surface, resulting in a surface whose Gaussian curvature is bounded from above by a negative constant, and the Ahlfors–Schwarz lemma implies hyperbolicity. A non-technical exposition of the ideas of this proof is given in the survey [13] which contains some further geometric applications of this technique of spreading the curvature by bi-Lipschitz modifications of a surface.

6. Sufficient conditions of parabolic type

Let $ds = \lambda|dz|$ be a conformal metric in a disk $\{z : |z| < R\}$, where $R \leq \infty$. We assume it is complete. Let W_r be the metric disk centered at the origin, Γ_r the component of ∂W_r which separates 0 from the circle $|z| = R$, and

$$L(r) = \int_{\Gamma_r} ds,$$

the length of Γ_r. Then we have:

$$2\pi \leq \int_{\Gamma_r} \frac{|dz|}{|z|} = \int_{\Gamma_r} \sqrt{\lambda} \frac{1}{\sqrt{\lambda}|z|} |dz|,$$

By Cauchy-Schwarz:

$$4\pi^2 \leq \int_{\Gamma_r} \lambda|dz| \int_{\Gamma_r} \frac{\lambda|dz|}{|z|^2\lambda^2} = L(r) \int_{\Gamma_r} \frac{ds}{|z|^2\lambda^2}.$$

Dividing on $L(r)$ and integrating from some $R_0 > 0$ to R we obtain

$$2\pi^2 \int_{R_0}^{R} \frac{dr}{L(r)} \leq \int_{R_0}^{R} dr \int_{\Gamma_r} \frac{ds}{|z|^2\lambda^2} = \int\int_{W_R \setminus W_{R_0}} \frac{dr ds}{|z|^2\lambda^2} = \int_{W_R \setminus W_{R_0}} \frac{dxdy}{|z|^2}.$$

the last integral equals to the "logarithmic area". Since this area is finite if $R < \infty$, divergence of the integral in the left hand side gives a sufficient condition of parabolic types.

On the other hand if $R = \infty$, then there exists a conformal metric, namely $ds = |dz|/|z|$ which is complete, and for which the integral in the LHS of (4) is divergent. So we obtain:

Theorem 4. *A simply connected open Riemann surface is of parabolic type if and only if it supports a complete conformal metric such that*

$$\int^{\infty} \frac{dr}{L(r)} = \infty, \tag{4}$$

where $L(r)$ is the length of the part of the circle of radius r centered at a fixed point which separates this point from infinity.

As an example, let us consider the radial metric

$$ds = \sqrt{dr^2 + L(r)d\theta^2}$$

in the plane, where r and θ are the polar coordinates. (It is not conformally equivalent to the standard metric!) Milnor [28] proved independently of Ahlfors that this surface is parabolic if and only if the integral (4) diverges. He also obtained the following criterion in terms of the curvature

$$K = -\frac{d^2L}{Ldr^2} :$$

If $K \geq -1/(r^2 \log r)$ for large r, then the surface is parabolic. If $K \leq -(1+\epsilon)/(r^2 \log r)$ for large r and if g is unbounded then the surface is hyperbolic. (If g is bounded, it is parabolic, by the first part.)

The result is based on the observation that for rotationally symmetric metrics, (4) is actually necessary and sufficient for parabolicity. So parabolicity follows from the Ahlfors criterion by a simple argument relating K and L. The hyperbolicity criterion was generalized by Peter Doyle [14] as follows. Let r, θ be polar coordinates on a simply connected Riemannian surface, that is we fix a point, and issue a geodesic from this point with argument θ. Then r is a distance along this geodesic, and we assume that such a global coordinate system exists, that is geodesics do not intersect except at the origin.

This happens for example when the curvature is non-positive. Then the expression of the metric in geodesic coordinates has the form

$$\sqrt{dr^2 + g^2(r,\theta)d\theta^2}$$

with some non-negative function g. Doyle's criterion of hyperbolic type is

$$\int_0^{2\pi} \frac{d\theta}{\int_0^\infty \frac{dr}{g(r,\theta)}} > 0, \tag{5}$$

which means that $\int_0^\infty dr/g(r,\theta) < \infty$ on a set of positive measure. When g is independent of θ one recovers the Milnor criterion.

We also mention the result of Ch. Blanc and F. Fiala [9], that when a complete Riemannian surface S with curvature ω satisfies

$$\int_S \omega^- d\sigma < \infty,$$

where $\omega^- = -\max\{-\omega, 0\}$ and $d\sigma$ is the area element, then S is parabolic. This can be obtained from Ahlfors' parabolicity criterion.

Theorem 4 can be variously modified, to make it more flexible in applications. First of all, one may allow λ to have singularities at infinite (metric) distance from the origin. Second, one may start instead of the metric with a function U, which is the case of the metric is the distance from the origin. Here is a version of Ahlfors' criterion:

Theorem $4'$**.** *Let U be a real-valued function in the disk $\{z : |z| < R\}$, continuous except isolated points, and such that $U(z) \to +\infty$ as z tends to the points of discontinuity or $|z| \to R$.*

Suppose in addition that U has continuous partial derivatives, except on some smooth arcs, such that every point of the disk has a neighborhood intersecting finitely many of these arcs.

Let

$$L(r) = \int_{U=r} |\nabla U||dz| = \int_{U=r} \left|\frac{\partial U}{\partial n}\right| |dz|.$$

If (4) holds then $R = \infty$.

The proof is essentially the same: we use $\{z : U(z) = r\}$ instead of Γ_r, and $|\partial U/\partial n|$ instead of λ.

As Ahlfors says, this result does not solve the problem but "indicates the direction of research". Concrete criteria of parabolic type can be stated by choosing the metric.

In his first paper of 1931 he used the pull back of the Euclidean metric and obtained a rather poor criterion. In the commentaries to his collected work he writes: "it took me five years to realize that I had chosen the wrong metric. In the next paper I did exactly the same thing, but this time for the covering of the Riemann sphere with distances counted in the spherical metric".

Let $\lambda|dz|$ be the spherical metric on a simply connected surface S spread over the Riemann sphere. Fix a point $p \in S$ and denote by ν_k the number of critical points (counting multiplicities) whose spherical distance from p_0 is in the interval $[\pi k, \pi(k+1)]$. Also, let $n(t)$ be the number of critical points at the distance at most πt from p.

Theorem 5. (Ahlfors) *Suppose that S has no transcendental singularities. Then each of the following conditions is sufficient for parabolic type:*

$$\sum_{k=1}^{\infty} \frac{1}{\nu_k} = \infty, \quad \int^{\infty} \frac{t dt}{n(t)} = \infty. \tag{6}$$

The assumption that S has no transcendental singularities is needed to ensure that the pull back of the spherical metric is complete. Theorem 2 easily follows from Theorem 1 if one notices that a circle of radius r in S with respect to the spherical metric consists of arcs of circles centered at the critical points whose distance to p is between r and $r - \pi$. The second condition in (6) implies the first one by Calculus.

For example, the surface S corresponding to an elliptic function, we have $n(t) \sim ct^2$, so conditions (6) hold.

The condition that S has no transcendental singularities was relaxed by Z. Kobayashi. He assumes only that singularities of S are isolated (thus admitting logarithmic branch points). With this assumption he defines the "Kobayashi net" (which is also known as Voronoi diagram) as the subset of S consisting of the points whose spherical distance from the set of singularities is attained on at least two singularities. In other words, if a is a singularity, one defines $W(a)$ as the set of points in S which are closer to a than to any other singularity, and the Kobayashi net is the union of the boundaries of those polygons $W(a)$.

Then he defines a Kobayashi metric in the following way:

a) on the net, it coincides with the spherical metric.

b) to each cell $W(a)$ it is extended as a singular metric with the singularity at a which makes $W(a)$ isometric to a half of a cylinder. In the local coordinate at a the length element has the form

$$ds = \left|\log \frac{w-a}{1+\overline{a}w}\right|.$$

By applying Ahlfors criterion to this metric, he obtains the following

Theorem 6. (Kobayashi) *Let S be an open Riemann surface spread over the sphere whose singularities are isolated. Let $\nu(t)$ be the number of points of the net at the distance t from a fixed point p. If*

$$\int^{\infty} \frac{dt}{\displaystyle\int_0^t \nu(\tau)d\tau} = \infty,$$

then S is of parabolic type.

For example, every surface with finitely many singularities is of parabolic type, the result previously proved by Nevanlinna and Elfving with a more complicated argument.

The following convenient sufficient condition of parabolic type is due to Nevanlinna and Wittich. It applies to Riemann surfaces of class **S** (with singularities lying over a finite set of points in $\overline{\mathbf{C}}$.

Let us fix a vertex v_0 of the line complex and consider the subset W_n of the line complex at combinatorial distance n. This is a finite connected graph, and let σ_n be the number of vertices of W_n which can be connected to infinity by curves not intersecting W_n. Then the sufficient condition of parabolic type (due to Nevanlinna and Wittich) is

$$\sum_{n=1}^{\infty} \frac{1}{\sigma_n} = \infty.$$

The proof is uses a version of Kobayashi argument, simplified by applying some auxiliary quasiconformal mappings.

One historically important class consists of surfaces satisfying the following conditions:

a) all singularities are logarithmic and they lie over a finite set,

b) there are no logarithmic ends, and

c) every vertex of the reduced line complex has degree 2 or 3.

The reduced line complex is a tree, whose branches have integer length. Fixing as ramified vertex we order the branches (sequences of edges and vertices of degree 2) according to the distance from this fixed vertex. Suppose that all these branches of a given generation k have the same length ℓ_k, and the sequence ℓ_k is non-decreasing, we obtain a *necessary and sufficient* criterion of parabolic type:

$$\sum_{n=1}^{\infty} \frac{\log \ell_n}{2^n} = \infty. \tag{7}$$

The sufficient part can be obtained from Kobayashi's criterion, while the necessary part, which is due to Le Van Thiem (after previous weaker criteria of Teichmüller and Kobayashi). This is obtained by a quasiconformal surgery mapping pieces of the Riemann surface onto the pieces of Riemann surface of the modular function.

Milnor's theorem mentioned above suggests type criteria in terms of curvature. There is no conformal metric in the plane whose curvature is bounded from above by a negative constant. Let S be a surface spread over the plane equipped with the pull back ρ of the spherical metric. This metric has curvature 1 at all points except the critical points, while at a point where the local degree is n, its integral curvature has an atom of mass $2\pi(1-n)$. Now one can exhaust our surface S by metric disks V_t of radii t centered at a fixed point t. Let $A(t)$ be the area of the disk, and $K(t)$ its integral curvature. Consider the quantity

$$V = \limsup \frac{A(t)}{V(t)} \tag{8}$$

one can conjecture that the surface is parabolic if $V = 0$ and hyperbolic if $V < 0$. When S is in Speiser class we may assume that the base curve is a great circle so to each vertex of line complex corresponds integral curvature 2π, the area of a hemisphere. Now we spread the negative curvature coming from a face of the line complex equally between its $2n$ vertices, and this gives the amount

$$K(v) = \pi \left(2 - \sum_{\{F : v \in \overline{F}\}} \left(1 - \frac{1}{k(F)} \right) \right)$$

attached to each vertex; here k is the number of edges on the boundary of a face, and summation is over all faces F whose closures contain v. Nevanlinna proves that if $K(v) = K$ is the same for all vertices, then the surface is of hyperbolic or parabolic type depending on whether $K < 0$ or $K = 0$.

Then he considered the number of vertices $A(n)$ at the distance at most n from a fixed vertex, and $V(n) = \sum_{v \in A(n)} K(v)$, and

$$V = \limsup_{n \to \infty} \frac{V(n)}{A(n)},$$

and conjectured that the type is hyperbolic or parabolic if $V < 0$ or $V = 0$, respectively.

Exercise. Prove that $V \leq 0$ for every infinite graph embedded in the sphere.

It was recently proved that the largest number of vertices in a planar graph with all vertices and faces of degree at least 3 and $K(v) > 0$ at each vertex has 208 vertices, except two explicitly described families of graphs, "prisms" and "antiprisms" [21].

Nevanlinna's conjecture turned out to be wrong. Teichmüller [36] constructed a line complex corresponding to a surface of hyperbolic type, with $V = 0$. Actually this is a line complex of the type discussed in section discussed in section 4 above. More recently, Benjamini, Merenkov and Schramm constructed an example of line complex with $V < 0$ corresponding to a parabolic surface. They also constructed an Aleksandrov smooth metric on a parabolic surface for which the quantity V defined in (8) is negative. This example can be described as follows: the conformal metric is defined in the plane by the formula $\lambda(z)|dz|$, where

$$\lambda(x + iy) = \begin{cases} y^{-1}, & y > 1, \\ e^{1-y}, & y < 1. \end{cases}$$

Hyperbolicity criterion based on curvature becomes true if one replaces the condition $V < 0$ by certain uniform condition of negativity of curvature.

In this way one obtains an exact correspondence. We state it for Aleksandrov surfaces.

Theorem 7. *Let S be an open simply connected Aleksandrov surface of curvature bounded from above. The following conditions are equivalent:*
a) There are constants B and ϵ such that the integral curvature of every disk of radius B is at most $-\epsilon$.
b) A linear isoperimetric inequality holds on S.
c) S is hyperbolic and conformal maps from the unit disk to S are satisfy the Lipschitz condition with respect to the hyperbolic metric in the unit disk and the metric in S.

All three constants involved in a), b), c) are bounded in terms of each other.

In the case of non-positively curved surfaces this result was proved in [12], and the general case in [17]. Duval's proof was non-constructive and did not give any explicit bounds for the constants. A constructive proof was proposed by Bruce Kleiner (unpublished).

This result applies to the Riemann surfaces spread over the plane with the pullback of the Euclidean metric, and to the surfaces spread over the sphere with the pullback of the spherical metric. Functions satisfying c) are called Bloch functions in the first case and normal functions in the second case. So this theorem gives an exact geometric description of Bloch and normal functions.

A necessary and sufficient conditions of the conformal type of simply connected Riemann surface can be stated in terms of *extremal length.*

Let D be a closed disk in the surface, then the type is parabolic or hyperbolic depending on whether the extremal distance from D to ∞ is finite or infinite.

Extremal distance has a nice physical interpretation: it is nothing but the electrical resistance of a thin homogeneous plate modeled by our surface [16]. Doyle [14] interpreted his criterion (5) in terms of the familiar laws of resistance of a system of conductors connected in parallel circuit, while Ahlfors' criterion (4) corresponds to a connection in a sequence.

Another necessary and sufficient condition for a simply connected surface with Riemannian metric can be given in probabilistic terms:

A simply connected Riemann surface is parabolic iff the Brownian motion on *it is recurrent.*

The question may be asked whether this criterion can be restated in terms of a random walk on a graph associated with a surface. One candidate is the Speiser graph, but recurrence or transience of the random walk on the Speiser graph is not the same as recurrence or transience of the Brownian motion (an example is given in Merenkov's thesis, Appendix A. It is essentially the example described above in connection with (7)).

Doyle [15] found an appropriate extension of the Speiser graph for the type criteria. His extension is similar to the Kobayashi "cylindrical surface".

The *half-plane lattice* Λ is the graph whose vertices are $\mathbf{Z} \times Z_{\geq 0}$ and a vertex (x, y) is connected to (x', y') if $(x - x', y - y') \in \{(\pm 1, 0), (0, \pm 1)\}$. The boundary of this graph is the infinite chain $\mathbf{Z} \times \{0\}$. The group $\mathbf{Z}$ acts on the half-plane lattice by shifts, and the *half-cylinder* lattice Λ_n is $\Lambda / n\mathbf{Z}$.

Let $n \geq 1$ be given. For $k \geq n$, we replace each face of the Speiser graph with $2k$ edges by the half-cylinder lattice Λ_{2k}, and each face with infinitely many edges by a half-plane lattice Λ, identifying the boundaries of the faces with the boundaries of the lattices. The result of this construction is called the *extended Speiser graph* Γ_n. It has a bounded degree and all faces have at most $\max\{2(n-1), 4\}$ edges.

Theorem 8. *For every $n \geq 1$, a surface (X, p) of Speiser class has parabolic type iff the simple random walk on Γ_n is recurrent.*

This theorem is essentially given by Doyle [15], but we state it in a modified form proposed by Merenkov in his thesis [27]. The idea of the proof is construction of conformal metric on X which is makes X roughly isometric to the extended Speiser graph. This conformal metric is essentially the Kobayashi cylindrical metric.

7. Nevanlinna theory

Conformal type criteria constitute only a part of geometric theory of meromorphic functions. Suppose that the type is known to be parabolic. How geometric properties influence the the asymptotic behavior of the uniformizing function?

In this section we introduce some notions which characterize this asymptotic behavior. Consider a rational function first. The main characteristic is the degree, which can be defined in two very different ways.

a) As the topological degree of the map $f : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$, or put it simply the number of preimages of a point, counted with multiplicity.

b) As $\max\{\deg p, \deg q\}$ where $f = p/q$ is an irreducible representation of two polynomials.

For a polynomial, the degree can be defined as a rate of convergence to ∞ as $z \to \infty$.

The Nevanlinna characteristic is a measure of complexity of a meromorphic function which is similar to the degree of a rational function.

It can be also defined in two ways, and equivalence of these definitions is an important fact.

For the first definition, we consider the average covering number of the sphere by disks $|z| \leq r$:

$$A(r,f) = \frac{1}{\pi}\int_0^r \int_{-\pi}^{\pi} \frac{|f'(te^{i\theta})|^2}{\left(1+|f(te^{i\theta})|\right)^2} tdtd\theta. \tag{9}$$

Here $2f'/(1+|f|^2)$ is the spherical derivative, and the total spherical area of the image of the disk $|z| \leq r$ is divided by the area of the sphere 4π. So for a rational function we obtain $A(\infty, f) = \deg f$.

Unfortunately, this characteristic $A(r,f)$ does not have reasonable behavior with respect to addition and multiplication of functions[3]

It turns out that a remedy of this defect is an averaging with respect to r:

$$T_0(r,f) = \int_0^t \frac{A(t,f)}{t} dt. \tag{10}$$

This is called the Nevanlinna characteristic (in the form of Ahlfors–Shimizu). It has nice algebraic properties which are similar to the properties of the degree:

$$\begin{aligned} T(r, f+g) &\leq T(r,f) + T(r,g) + O(1), & (11)\\ T(r, fg) &\leq T(r,f) + T(r,g) + O(1), & (12)\\ T(r, f^n) &= nT(r,f) + O(1), & (13)\\ T(r, 1/f) &= T(r,f) + O(1). & (14) \end{aligned}$$

[3]The following anecdote is told about V. Arnold. In a lecture he asked the audience: what is a rational function? Answer: a ratio of two polynomials. Arnold: This is wrong! You were taught incorrectly! A rational function is a holomorphic map of the sphere to itself! Question: Professor, could you please explain what is the sum of two rational functions?

The error term is considered irrelevant since $T(r, f) \to +\infty$ for all non-constant functions. More precisely, for a rational function of degree d we have $T(r, f) = d \log r + O(1)$ while for every transcendental f, $T(r, f)/\log r \to \infty$.

Exercise. Let $f \mapsto T(f)$ be a map from the field of rational functions to $\mathbf{R}_{\geq 0}$ satisfying (11)–(14), and $T(c) = 0$ for every constant. Prove that $T(f) = c \deg f$ for some constant $c > 0$.

The proof of those algebraic properties directly from the definition (9), (10) seems is far from trivial. This is does with the help of a different, equivalent definition, which was the original definition of Nevanlinna.

Let $n(r, f)$ be the counting function of poles of f, counting multiplicity in the disks $|z| \leq r$, then

$$N(r, f) = \int_0^\infty (n(t, f) - n(0, f)) \frac{dt}{t} + n(0, f) \log r,$$

and

$$m(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta.$$

Then the Nevanlinna characteristic is defined as

$$T(r, f) = m(r, f) + N(r, f). \tag{15}$$

For this characteristic, properties (11)-(13) are trivial, while (14) is Jensen's formula in disguise.

Now a theorem of Ahlfors and Shimizu says that

$$T(r, f) = T_0(r, f) + O(1),$$

so for most questions it does not matter which form of the characteristic is used. Now $T_0(r, f)$ is invariant with respect to rotations of the sphere, and almost invariants (up to an $O(1)$ summand) with respect to arbitrary linear-fractional transformations, so from (15) we obtain

$$T(r, f) = m(r, (f - a)^{-1}) + N(r, f - a)^{-1}) + O(1),$$

which is called the First Fundamental Theorem of Nevanlinna. We introduce the convenient notation

$$N(r, a, f) = N(r, (f - a)^{-1}), \quad m(r, a, f) = m(r, (f - a)^{-1}),$$

and write the First Main Theorem in the form

$$T(r,f) = m(r,a,f) + N(r,a,f) + O(1).$$

It is instructive to see what this says about rational functions: the degree is equal to the number of poles in $\mathbf{C}$ plus the multiplicity of a pole at ∞.

So for rational function, $m(r,a,f) = O(1)$ for all a except one. It turns out that for arbitrary meromorphic function $T(r,f) \sim N(r,a,f)$ for most $a \in \overline{\mathbf{C}}$, namely for all a except a set of zero logarithmic capacity. A more subtle result in the Second Main Theorem of Nevanlinna:

Theorem 8. *For every meromorphic function f in the plane, and any finite set $a_1, a_2, \ldots, a_q$ in $\overline{\mathbf{C}}$, we have*

$$\sum_{j=1}^{q} m(r,a_j,f) + N_1(r,f) \leq 2T(r,f) + S(r,f). \tag{16}$$

Here $N_1(r,f)$ is a function defined similarly to $N(r,f)$ but counting the critical points of f (including multiplicity), and $S(r,f)$ is a small error term,

$$S(r,f) = O(\log(rT(r,f))), \quad r \to \infty, \quad r \notin E,$$

where E is a set of finite length.

Exercise. Show that $N_1(r,f) = N(r, \frac{1}{f'}) + 2N(r,f) - N(r,f')$.

One consequence of (16) is the *defect relation*. The defect is defined by

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r,f)}{T(r,f)}, \quad \text{and} \quad \theta(f) = \liminf_{r\to\infty} \frac{N_1(r,f)}{T(r,f)}.$$

and we have

$$\sum_{a \in \overline{\mathbf{C}}} \delta(a,f) + \theta(a,f) \leq 2.$$

Second Main Theorem also implies

$$T(r,f') \leq 2T(r,f) + S(r,f),$$

which is analogous to $\deg f' \leq 2 \deg f$ for rational functions.

Prior to Nevanlinna theory, the characteristic $\log M(r,f)$, where $M(r,f) = \max\{|f(z)| : |z| leqr\}$ was used to measure the growth of an entire function. In the case of entire functions, both characteristics are comparable:

$$\log M(r,f) \geq T(r,f) = m(r,f) \geq \frac{r-t}{r+t} \log M(t,f), \quad t < r.$$

The *order* and of a meromorphic function f is defined as

$$\rho(f) = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r}.$$

If in this definition lim sup is replaced by lim inf we obtain *lower order*. Using $A(r,f)$ or, in the case of entire functions $\log M(r,f)$ does not affect the order and lower order.

8. Direct singularities and the Denjoy–Carleman–Ahlfors Theorem

We begin with a general result on direct singularities of a parabolic surface:

Theorem 9. (M. Heins) *On a surface of parabolic type, the set of asymptotic values corresponding to direct singularities is at most countable.*

It is not true that the set of direct singularities itself is countable on a surface of parabolic type: one can have an uncountable set of direct singularities with the same asymptotic values. An example of such a surface was constructed by M. Heins in [23]. Here is an explicit example of an entire function with uncountable set of logarithmic singularities with asymptotic value 0 is

$$f(z) = \sum_{k=1}^{\infty} \left(\frac{z}{2^k} \right)^{2^k},$$

see [8] for the proof.

Heins's theorem is derived from the following "Local Picard theorem":

Theorem 10. (Heins) *Let f be a non-constant meromorphic function in a region $D \subset \mathbf{C}$ mapping it into a region $\Omega \subset \overline{\mathbf{C}}$, such that f is continuous on $D \cup \partial_{\mathbf{C}} D$, and maps $\partial_{\mathbf{C}} D into \partial\Omega$. Then f cannot omit three values in Ω. If Ω and D are simply connected then f cannot omit two values.*

Here $\partial_{\mathbf{C}} D$ means the boundary with respect to $\mathbf{C}$, while $\partial\Omega$ is the boundary in $\overline{\mathbf{C}}$, as usual.

Proof. The first step reduces the theorem to the case that D and Ω are simply connected. Let Ω_1 be a Jordan region, whose closure is contained in Ω, and such that $\{a, b\} \in \Omega_1$ but $c \in \Omega\backslash\overline{\Omega_1}$. Suppose also that $\partial\Omega_1$ contains no critical values of f. Let $D_1 \subset D$ be a component of $f^{-1}(\Omega_1)$. Then D_1 is bounded by disjoint simple curves, and is Ω_1 unbounded. (Otherwise $f : D_1 \to \Omega_1$ would be a covering which contradicts the assumption that it omits $\{a, b\} \subset \Omega_1$).

We claim that D_1 is simply connected. To prove the claim, assume the contrary. Then ∂D_1 contains a Jordan curve γ on which f is continuous and maps it into $\partial\Omega_1$. If any part of ∂D lies inside γ, then f is continuous on this part and maps it into $\partial\Omega$. So if we denote the interior component of Γ by G, then f maps $G \cap D$ into $\Omega\backslash\overline{\Omega_1}$ and $\partial(G \cap D)$ into $\partial(\Omega\backslash\overline{\Omega_1}$, so the restriction of f onto $G\cap D$ is a covering, and this contradicts the assumption that f omits c.

So it remains to prove the last statement of the theorems (the case when Ω and D are simply connected and $f : D \to \Omega$ omits two values. We may assume without loss of generality that $\partial_{\mathbf{C}} D$ and $\partial\Omega$ are smooth curves (shrink Ω if necessary). Let $\phi : \mathbf{U} \to D$ be a uniformizing map, and $\psi : \Omega \to U$ a Riemann map. Then $g = \psi \circ f \circ \phi$ maps the unit disk into itself, and according to a theorem of Beurling, there is a closed set $E \subset \partial U$ of zero capacity, such that g is continuous on every arc $\partial U\backslash E$ and maps this arc into ∂U. Let $\delta : \mathbf{U} \to U\backslash\{a, b\}$ be the universal cover. This universal cover can be explicitly described, and the limit sets F of the corresponding Fuchsian group has positive capacity (even positive Hausdorff dimension), see, for example [19]. By the universality of δ, there exists a function $h : \mathbf{U} \to \mathbf{U}$ such that $g = \delta \circ h$. Using reflection in the arcs $\partial\mathbf{U}\backslash E$ of the unit circle where $|g| = 1$ and thus $|h| = 1$ we extend g and $\delta \circ h$ to a map $\overline{\mathbf{C}}\backslash E \to \overline{\mathbf{C}}\backslash F$. Since E has zero capacity and F has positive capacity, this is a contradiction. (One can also use Hausdorff dimension to obtain this contradiction, see, for example [37]).

The next two results concern meromorphic functions of finite order.

Theorem 11. *For a meromorphic function in the plane of finite lower order λ, the number of direct singularities is at most*

$$\min\{2\lambda, 1\}.$$

This is usually called the Denjoy–Carleman–Ahlfors theorem, though Beurling also had an independent claim, so the proper name would be ABCD-Theorem. There are two different proofs: one is based on potential theory (Carleman's inequality), another on the Ahlfors's distortion theorem. Both proofs have their advantages. Potential theoretic proof allows multi-dimensional generalizations, while the conformal mapping proof permits to improve the result in the plane by taking into account the spiraling behavior of the tracts. Because of the great importance of both methods, we will discuss both proofs.

A simple example of indirect singularities is the two singularities that $(\mathbf{C}, (\sin)/z)$ has over 0. In this example, the indirect singularities are accumulation points of critical point. This is not always so:

Example. (Volkovyskii) Let E be a Cantor set on the unit circle. We start with the unit disk, and to each complementary interval (a, b) we attach a logarithmic end with asymptotic values a, b. The resulting surface has no critical points at all, and the set of singularities has the power of continuum: they lie over every point of E. By choosing very small E, one can achieve parabolic type. One can construct a similar example spread over the plane by starting with a star consisting of all rays from zero through the complementary points of the Cantor set, and attaching logarithmic ends to the part of every ray from an endpoint of the Cantor set to infinity.

However it turns out that for functions of finite lower order, all indirect singularities are indeed accumulation points of critical points:

Theorem 12. (Bergweiler–Eremenko-Hinchlif) *For a meromorphic function in the plane of finite lower order, every indirect singularity over a point a is a limit of critical points whose critical values are distinct from a.*

This theorem is useful since it implies existence of infinitely many critical points in certain situations. See, for example, section 9, where all functions of finite lower order with only direct singularities are completely described.

9. Speiser class

9.1. Let S be the Speiser class of meromorphic functions $\mathbf{C} \to \overline{\mathbf{C}}$. This means that there exists a finite set A such that

$$f : \mathbf{C}\backslash f^{-1}(A) \to \overline{\mathbf{C}}\backslash A \tag{17}$$

is a covering. If $|A| = q$ we will say that $f \in S_q$. It is intuitively clear that the elements of A can serve as local parameters for f. To make this precise, we fix a function $g \in S$ and define M_g as the set of all functions $f : \mathbf{C} \to \overline{\mathbf{C}}$ which are topologically equivalent to g in the sense that homeomorphisms $\phi : \mathbf{C} \to \mathbf{C}$ and $\psi : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ such that

$$\psi \circ g = f \circ \phi. \tag{18}$$

We assume that the set A is minimal for which (17) is a covering. We will show that M_g is a complex analytic manifold of dimension $q + 2$.

Choose β_1 and β_2 such that $g(\beta_j)$ is not a singular value of g. Let $M_g(\beta_1, \beta_2)$ be the set of functions f for which the homeomorphisms in (18) can be chosen so that $\phi(\beta_j) = \beta_j$. One can easily check that $m_g = \cup_{\beta_1,\beta_2} M_g(\beta_1, \beta_2)$. Let $A = (a_1, \ldots, a_q)$ and $g(\beta_1) = a_{q+1},\ g(\beta_2) = a_{q+2}$. We will show that $(a_1, \ldots, a_{q+2})$ are local coordinates in the chart $M_g(\beta_1, \beta_2)$ of M_g.

Lemma. *For $j \in \{0, 1\}$ let*

$$\psi_j \circ g = f_j \circ \phi_j, \quad \phi_j(b_i) = \beta_i, \quad i \in \{1, 2\}.$$

Assume that there exists an isotopy ϕ_t connecting ψ_0 and ψ_1, and such that $\psi_t(a_k) = a_k$ for $0 \le t \le 1$ and $1 \le k \le q + 2$. Then $f_0 = f_1$.

Proof. By the Covering Homotopy Theorem, there exists a continuous family of homeomorphisms h_t such that $h_1 = \phi_1$ and $\psi_t \circ g = f_1 \circ h_t$, $0 \le t \le 1$. The functions $t \mapsto h_t(\beta_i)$ are continuous and their values are in a discrete set, therefore $h_t(\beta_i) = \beta_i$. Putting $t = 0$, we obtain $f_0 \circ \phi_0 = \psi_0 \circ g = f_1 \circ h_0$, thus $f_0 = f_1 \circ h_0 \circ \phi_0^{-1}$. The homeomorphism $h_0 \circ \phi_0^{-1}$ of the plane has two fixed points and is conformal outside a discrete set. Therefore it is identity and $f_0 = f_1$.

Let us now define an analytic structure on $M_f(\beta_1, \beta_2)$. Consider the space Y of homeomorphisms $\mathbf{C} \to \mathbf{C}$ modulo the following equivalence relation: $\psi_0 \sim \psi_1$ if there exists an isotopy $\psi_t : \mathbf{C} \to \mathbf{C}$ such that $\psi_t(a_k) = a_k,\ 0 \le t \le 1,\ 1 \le k \le k+2$. The map $Y \to \mathbf{C}^{q+2}$ $\psi \mapsto (\psi(a_1), \ldots, \psi(a_{q+2})$ being a local homeomorphism defines on Y the structure of a complex analytic manifold of dimension $q + 2$. Let us construct a map $\pi : Y \to M_g(\beta_1, \beta_2)$. Observe that every element of Y can be represented by a quasiconformal homeomorphism. Consider the map $\phi \circ g$, where ψ is a quasiconformal representative. By the Measurable Riemann theorem, there exists a homeomorphism $\phi : \mathbf{C} \to \mathbf{C}$ such that $\phi(\beta_i) = \beta_i$ and $\psi \circ g \circ \phi = f$ is analytic. Let $\pi(\psi) = f$. Then π is correctly defined (by the Lemma), and singular values of f are $\psi(a_1), \ldots, \psi(a_q)$.

Notice that π is surjective and locally injective. So π indices a complex analytic structure on $M_g(\beta_1, \beta_2)$.

Let us show that the map

$$M_g \times \mathbf{C} \to \overline{\mathbf{C}}, \quad (f, z) \mapsto f(z)$$

is complex analytic. To prove this we apply the operator $\partial/\partial\overline{\lambda}$ to the equation

$$\psi_\lambda \circ g = f_\lambda \circ \phi\lambda.$$

Since ψ_λ is holomorphic, we obtain

$$\frac{\partial f_\lambda}{\partial \overline{\lambda}} = -\frac{\partial f_\lambda}{\partial z} \frac{\partial \phi_\lambda}{\partial \overline{\lambda}} = 0,$$

the last inequality holds because ψ_λ is holomorphic in λ.

References

[1] L. Ahlfors, Sur le type de surface de Riemann, C. R., 201 (1935) 30–32.

[2] L. Ahlfors, Zur Theorie der Überlagerungsflächen, Acta math., 65 (1935) N1 157–194.

[3] L. Ahlfors, An extension of Schwarz's lemma, Trans. Amer. Math. Soc. 43 (1938), no. 3, 359–364.

[4] L. Ahlfors, Untersuchungen zur Theorie der konformen Abbildung und der ganzen Funktionen, Acta Soc. Sci. Fenn., A1 (1930) 1-40.

[5] I. Benjamini, S. Merenkov and O. Schramm, A negative answer to Nevanlinna's type question and a parabolic surface with a lot of negative curvature, Proc. AMS 132 (2003) 641–647.

[6] W. Bergweiler, A new proof of the Ahlfors five islands theorem. J. Anal. Math. 76 (1998) 337–347.

[7] W. Bergweiler and A. Eremenko, On the singularities of the inverse to a meromorphic function of finite order, Rev. Mat. Iberoamericana 11 (1995) no. 2 355–373.

[8] W. Bergweiler, Eremenko, Direct singularities and completely invariant domains of entire functions, Illinois J. Math. 52 (2008), no. 1, 243–259.

[9] Ch. Blank and F. Fiala, Le type d'une surface et sa curbure totale, Comm. math. helv., 14 (1941) 230–233.

[10] A. Bloch, La conception actuelle de la théorie des fonctions entières et méromorphes, Enseignement math., 25 (1926) 83–103.

[11] M. Bonk and A. Eremenko, Covering properties of meromorphic functions, negative curvature and spherical geometry. Ann. of Math. (2) 152 (2000) no. 2, 551–592.

[12] M. Bonk and A. Eremenko, Univormly hyperbolic surfaces, Indiana Univ. Math. J. 49 (2000), no. 1, 61–80.

[13] M. Bonk, Singular surfaces and meromorphic functions, Notices Amer. Math. Soc. 49 (2002) no. 6, 647–657.

[14] P. Doyle, On deciding whether a surface is hyperbolic or parabolic, Geometry of random motion (Ithaca, N.Y., 1987), 41–48, Contemp. Math., 73, Amer. Math. Soc., Providence, RI, 1988.

[15] P. Doyle, Random walk on the Speiser graph of a Riemann surface, Bull. Amer. Math. Soc., 11, 2 (1984) 371–377.

[16] R. Duffin, The extremal length of a network, J. Math. Anal. Appl., 5 (1962) 200–215.

[17] J. Duval, Sur le lemme de Brody, Invent. Math. 173 (2008), no. 2, 305–314.

[18] J. Duval and B. da Costa, Sur les courbes de Brody dans $P^n(\mathbf{C})$, Math. Ann. 355 (2013), no. 4, 1593–1600.

[19] J. Fernandez, Singularities of inner functions, Math. Z. 193 (1986) no. 3, 393–396.

[20] L. Geyer and S. Merenkov, Hyperbolic surface with a square net, A hyperbolic surface with a square grid net, J. Anal. Math. 96 (2005), 357–367.

[21] L. Ghidelli, On the largest planar graphs with everywhere positive combinatorial curvature, J. Combin. Theory Ser. B 158 (2023), part 2, 226–263.

[22] H. Habsch, Die Theorie der Grundkurven und das Äquivalenzproblem bei der Darstellung Riemannscher Flächen, Mitt. Math. Sem. Giessen No. 42 1952 (1952), i+51 pp.

[23] M. Heins, Asymptotic spots of entire and meromorphic functions, Ann. Math., 66 (1957) 430–439.

[24] Zen-ichi Kobayashi, Theorems on the conformal representation of Riemann surfaces, Sc. Rep. Tokyo Bunrika Daigaku A 2, (1935) 125–166.

[25] S. Mazurkiewicz, Sur les points singuliers s'une fonction analytique, Fundamenta math., 17 (1931) 26–29.

[26] R. Nevanlinna, Eindeutige analytische Funktionen, Springer, 1953.

[27] S. Merenkov, Determining biholomorphic type of a manifold using combinatorial and algebraic structures. Thesis (Ph.D.)–Purdue University. 2003. 65 pp.

[28] J. Milnor, On deciding whether a surface is parabolic or hyperblic, Amer. Math. Monthly, 84 (1977) 43–46.

[29] K. Oikawa, Welding of polygons and the type of Riemann surfaces, Kodai Math. Sem. Rep. 13 (1961), 37–52.

[30] Ch. Pommerenke, Estimates for normal meromorphic functions, Ann. Acad. Sci. Fenn. Ser. AI 476 (1970).

[31] Yu. Reshetnyak, Two-dimensional manifolds of bounded curvature, in the book: Geometry IV. Non-regular Riemannian geometry, Encycl. Math. Sci. 70, 3–163 (1993).

[32] P. Seibert, Sur les chemins asymptotiques de fonctions méromorphes, C. R., 244 (1957) 1443–1445.

[33] P. Seibert, Über die Randstrukturen von überlagerungsflächen, Math. Nachr., 19 (1958) 339–352.

[34] P. Seibert, Typus und topologische Randstruktur einfach-zusammenhängender riemannscher Flächen, Ann. Acad. Sci. Fenn., Ser. A I 250/34 (1958) 10 S.

[35] P. Seibert, On a problem of Mazurkiewicz concerning the boundary of a covering surface, Proc. Natl. Acad. Sci., 45 (1959) 50–54.

[36] O. Teichmüller, Untersuchungen über konforme und quasikonforme Abbildung, Deutsche Math., 3 (1938) 621–678.

[37] S. Toppila, On exceptional values of functions meromorphic outside a linear set, Ann. Acad. Sci. Fenn., Ser. A I. Math., 5 (1980) 115–119.

[38] L. I. Volkovyskii, Convergent sequences of Riemann surfaces (in Russian), Mat. Sbornik, 23, 3 (1943) 361–382.

[39] L. I. Volkovyskii, Investigations on the type problem of a simply connected Riemann surface (Russian), Trudy Mat. Inst. Steklova, 34 (1950).

[40] H. Wittich, Neuere Untersuchingen über eindeutige analytische Funktionen, Springer, 1955.

Purdue University, West Lafayette, IN 47907 USA
eremenko@math.purdue.edu