Изв. Акад. Наук СССР
Сер. Матем. Том 51(1987), № 2

Math. USSR Izvestiya
Vol. 30(1988), No. 2

PROOF OF A CONDITIONAL THEOREM OF LITTLEWOOD ON THE DISTRIBUTION OF VALUES OF ENTIRE FUNCTIONS

UDC 517.53

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ABSTRACT. It is proved that for any entire function f of finite nonzero order there is a set S *in the plane with density zero and such that for any* $a \in \mathbf{C}$ *almost all the roots* of the equation $f(z) = a$ belong to S. This assertion was deduced by Littlewood from an unproved conjecture about an estimate of the spherical derivative of a polynomial. This conjecture is proved here in a weakened form.

Bibliography: 11 titles.

§1. Introduction

For a meromorphic function f, denote by ρ_f its spherical derivative,

$$\rho_f = |f'|/(1+|f|^2).$$

Let $D(z,R) = \{w \in \mathbf{C}: |w-z| < R\}$ and $D(R) = D(0,R)$, and let m_2 be Lebesgue measure on the plane $\mathbf{C}$. In [1] Littlewood considered the quantities

$$\varphi(n) = \sup_P \iint\limits_{D(1)} \rho_P \, dm_2 = \sup_P \sup_{r>0} \frac{1}{r} \iint\limits_{D(r)} \rho_P \, dm_2.$$

Here the supremum is over all polynomials P of degree n, $n = 1, 2, \cdots$. The analogous quantities for rational functions will be denoted by $\psi(n)$. It follows from the Schwartz-Bunyakovskiĭ inequality that

$$\psi(n) \le \left\{ \iint\limits_{D(1)} dm_2 \cdot \sup_f \iint\limits_{D(1)} \rho_f^2 \iint\limits_{D(1)} \rho_f^2 dm_2 \right\}^{1/2}.$$

The second integral is none other than the spherical area (with multiplicity counted of the image of the disk $D(1)$ under the action of a rational function of degree n. Thus, this integral does not exceed πn, and we get that

$$\varphi(n) \le \psi(n) \le \pi\sqrt{n}. \tag{1.1}$$

The best-known lower estimates were obtained by Hayman [2]:$\psi(n) \ge A_1\sqrt{n}$ and $\varphi(n) \ge A_2 \log n$. Here A_1 and A_2 are absolute constants. Thus, inequality (1.1) gives the correct order of $\psi(n)$. Up to the present time it was not known whether estimate (1.1) can be

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30D35; Secondary, 30C10.

strengthened for $\varphi(n)$. It was conjectured in [1] that there are absolute constants A and α such that

$$\varphi(n) \leq An^{1/2-\alpha}. \tag{1.2}$$

Littlewood deduced [1] the following remarkable consequence of this conjecture: *For any entire function f of finite nonzero order there exists a small portion S of the plane such that for any $a \in \mathbf{C}$ the roots of the equation $f(z) = a$ belong to S with a negligible exception.* For example, for $f(z) = e^z$ we can take $S = \{x + iy: |y| > x^2\}$. Then the set S has zero density, i.e.,

$$m_2(S \cap D(r)) = o(r^2), \qquad r \to \infty,$$

and for any a all the roots of the equation $e^z = a$ fall in S with finitely many exceptions.

In this article we prove this corollary (Theorem 2) and get the estimate $\varphi(n) = o(\sqrt{n})$, $n \to \infty$ (Theorem 1). Theorems 1 and 2 are contained in §§3 and 4, respectively. The proofs of both theorems use two lemmas from potential theory which are contained in §2. These lemmas may also be of independent interest.

§2

LEMMA 1. *Suppose that $u \geq 0$ is a subharmonic function on a domain $G \subset \mathbf{C}$, $\mu = \mu_u$ is the associated measure in the Riesz sense, and $N = \{z \in G: u(z) = 0\}$. Then there exist Borel sets E and L such that $N = E \cup L$, $m_2L = 0$, and $\mu E = 0$.*

PROOF Assume that G and u are bounded. The transition to the general case does not involve difficulties. We take E to be the set of points of density of N which belong to N. Let $L = N \backslash E$. By Lebesgue's theorem (see, for example, [3]), $m_2L = 0$.

We show that $\mu E = 0$. Fix a point $z_0 \in E$ and an arbitrarily small number $\varepsilon > 0$. Let r_0, $0 < r_0 < \varepsilon$, be such that $D(z_0, r_0) \subset G$ and

$$m_2((G\backslash N) \cap D(z_0, r)) < \delta r^2, \qquad r < r_0,$$

where $\delta = (4^4 \cdot 3 \cdot 5 \log 2)^{-1}$. Denote by $\theta(r)$ the angular measure of the set $(G\backslash N) \cap \partial D(z_0, r_0)$. We show that on each interval $[r, 2r] \subset (0, r_0]$ there is a point r^* such that $\theta(r^*) \leq \eta = 4^{-4} \cdot 3^{-1}$. Indeed, if $\theta(r^*) \geq \eta$, $r \leq r^* \leq 2r$, then, by the Schwarz-Bunyakovskiĭ inequality,

$$\begin{aligned} r^2 = \left(\int_r^{2r} dt\right)^2 &\leq \int_r^{2r} t\theta(t)\, dt \cdot \int_r^{2r} \frac{dt}{t\theta(t)} \\ &\leq m_2((G\backslash N) \cap D(z_0, 2r))\eta^{-1} \log 2 \leq 4\delta r^2 \eta^{-1} \log 2 = \tfrac{4}{5} r^2, \end{aligned}$$

a contradiction.

Thus, on each interval $[r, 2r] \subset (0, r_0]$ there is a point r^* such that $\theta(r^*) < \eta$. Therefore, there exists a sequence (r_k) with the following properties:

$$1/4 \leq r_k/r_{k-1} \leq 1/2, \qquad k = 1, 2, \ldots, \tag{2.1}$$

$$\theta(r_k) < \eta, \qquad k = 1, 2, \ldots. \tag{2.2}$$

Let $M = \sup\{u(z): z \in G\}$ and $M_k = \sup\{u(z): |z - z_0| = r_k\}$, $k = 0, 1, 2, \cdots$. Using the Poisson formula, we get by (2.2) that

$$\begin{aligned} M_{k+1} &\leq \frac{r_k + r_{k+1}}{r_k - r_{k+1}} \int_0^{2\pi} u(z_0 + r_k e^{i\theta})\, d\theta \\ &\leq 3M_k\theta(r_k) \leq 3\eta M_k \leq (3\eta)^{k+1} M_0 \leq (3\eta)^{k+1} M. \end{aligned}$$

It follows from (2.1) that $4^{-k-1} \le r_{k+1}/r_0$; therefore,

$$M_{k+1} \le (3 \cdot 4^4 \eta)^{k+1} (r_{k+1}/r_0)^4 M = (r_{k+1}/r_0)^4 M \le r_{k+1}^3 \tag{2.3}$$

for sufficiently large k.

Let $n(t) = \mu D(z_0, t)$. It follows from the Poisson-Jensen formula, the fact that $u(z_0) = 0$, and (2.3) that for sufficiently large k

$$n(r_k/e) \le \int_{r_k/e}^{r_k} \frac{n(t)}{t}\, dt \le \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + r_k e^{i\theta})\, d\theta \le M_k \le r_k^3.$$

Thus, each point $z_0 \in E$ can be included in a disk $D(z_0, R(z_0))$ such that $\mu D(z_0, R(z_0)) \le e^3 (R(z_0))^3$, and $R(z_0) \le \varepsilon e^{-3}$ for all $z_0 \in E$. According to a theorem on coverings (see, for example, [3], p. 5), there exists a countable covering of E by these disks with multiplicity at most six: $E \subset \bigcup_k D(z_k, R(z_k))$. We have that

$$\mu E \le \sum_k \mu D(z_k, R(z_k)) \le e^3 \sum_k (R(z_k))^3 \le \varepsilon \sum_k (R(z_k))^2 \le \frac{6\varepsilon}{\pi} m_2 E.$$

Since $\varepsilon > 0$ is arbitrary, this leads us to conclude that $\mu E = 0$, which is what was required to prove.

We show the connection between Lemma 1 and [4]–[6]. Let D be a domain having a Green's function, and assume that $\infty \in D$. Extending to $\mathbf{C} \backslash D$ the Green's function with a pole at ∞ by assigninig the value zero, we get a function $u \ge 0$ which is subharmonic in $\mathbf{C}$. The Riesz measure μ of this function coincides with the harmonic measure at ∞ with respect to D. In this case Lemma 1 becomes a result of Øksendal (see [4] and [5]) asserting that the harmonic measure μ and Lebesgue measure m_2 are mutually singular. If it is assumed in addition that D is simply connected, then a stronger result is known [6]: μ is singular with respect to the Hausdorff measure $m_{1+\varepsilon}$ for any $\varepsilon > 0$.

The following "stable" variant of Lemma 1 appears to be plausible.

CONJECTURE. *There exist absolute constants $B > 0$ and $\beta > 0$ such that for any function u subharmonic in $D(1)$ with $0 < u < 1$ and any $\varepsilon > 0$*

$$\{z \in D(1) \colon u(z) \le \varepsilon\} \subset L_\varepsilon \cup E_\varepsilon,$$

where $m_2 L_\varepsilon < B\varepsilon^\beta$ and $\mu E_\varepsilon \le B\varepsilon^\beta$.

Simple examples show that this conjecture can be true only for $\beta \le 1/2$. It is possible to deduce (1.2) with any $\alpha < 1/2 - \beta/2$ from our conjecture by the method used below in proving Theorem 1.

To formulate the next lemma we require some definitions.

Fix a number $C > 0$ and denote by U the set of functions u subharmonic in $D(2)$ with the properties $u(z) \le C$, $z < 2$, and $u(0) \ge -C$.

The following facts from potential theory can be found, for example, in [7] and [8]. The set U is a compact subset of the space $L^1 = L^1(D(2), dm_2)$. All the topological terms below relate to the topology of L^1 unless otherwise stated. The convergence $u_n \to u$ implies weak convergence of the associated measures in the Riesz sense: $\mu_n \to \mu$. This means that

$$\iint_{D(2)} g\, d\mu_n \to \iint_{D(2)} g\, d\mu$$

for each continuous function g with compact support in $D(2)$. If $\mu_n \to \mu$, then $\overline{\lim}_{n\to\infty} \mu_n K \le \mu K$ for each compact set $K \subset D(2)$.

The subset $U^+ \subset U$ consisting of nonnegative functions is closed in U; hence it is compact.

LEMMA 2. *Suppose that* $u \in U^+$ *and* $\delta > 0$. *Then there exist a set* $E \subset \overline{D(1)}$ *and a number* $\varepsilon > 0$ *with the following properties. For any function* $v \in U$ *in the ball* $||u - v|| < \varepsilon$

$$\mu_v E \le \delta, \tag{2.4}$$

$$\{z \in \overline{D(1)}: v(z) < \varepsilon\} \subset E \cup L_v, \tag{2.5}$$

where

$$m_2 L_v \le \delta. \tag{2.6}$$

PROOF. Applying Lemma 1 to the function u, we find a set E (change its notation to E^*) and a set L with the properties in Lemma 1. Let

$$M_1 = L \cap \overline{D(1)}, \quad M_2 = E^* \cap \overline{D(1)}, \quad M_3 = \{z \in \overline{D(1)}: u(z) > 0\}.$$

Choose a number $\varepsilon > 0$ such that the closed set $K = \{z \in \overline{D(1)}: u(z) \ge 2\varepsilon_1\} \subset M_3$ satisfies the condition

$$m_2(M_3 \backslash K) < \delta/4. \tag{2.7}$$

Choose E to be a compact subset of M_2 such that

$$m_2(M_2 \backslash E) < \delta/4. \tag{2.8}$$

Since $\mu_u M_2 = 0$ and $E \subset M_2$,

$$\mu_u E = 0. \tag{2.9}$$

In view of (2.9) there exists an $\varepsilon_2 > 0$ such that if $v \in U$ and $||u - v|| \le \varepsilon_2$, then

$$\mu_v E < \delta. \tag{2.10}$$

Further, if $||u - v|| < \varepsilon_1 \delta/4$ and $X_v = \{z \in \overline{D(1)}: |u(z) - v(z)| > \varepsilon_1\}$, then

$$m_2 X_v \le \frac{1}{\varepsilon_1} \iint_{X_v} |u - v| \, dm_2 \le \frac{1}{\varepsilon_1} ||u - v|| < \delta/4. \tag{2.11}$$

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2, \varepsilon_1 \delta/4)$. If $v \in U$, $||u - v|| \le \varepsilon$, then (2.4) holds because of (2.10). Moreover,

$$\begin{aligned} \{z \in \overline{D(1)}: v(z) < \varepsilon\} &\subset (\overline{D(1)} \backslash K) \cup X_v \\ &= X_v \cup (M_3 \backslash K) \cup (M_2 \backslash E) \cup M_1 \cup E =: L_v \cup E, \end{aligned}$$

and, by (2.7), (2.8), and (2.11),

$$m_2 L_v \le m_2 X_v + m_2(M_3 \backslash K) + m_2(M_2 \backslash E) + m_2 M_1 < \delta.$$

Lemma 2 is proved.

§3.

THEOREM 1. $\varphi(n) = o(\sqrt{n})$ *as* $n \to \infty$.

PROOF. By the Schwarz-Bunyakovskiĭ inequality, for any measurable set $K \subset \overline{D(1)}$

$$\iint_K \rho_j \, dm_2 \le \left\{ m_2 K \cdot \iint_K \rho_j^2 \, dm_2 \right\}^{1/2} \tag{3.1}$$

(cf. §1). If the theorem is false, then there exist arbitrarily large indices n, nth-degree polynomials P_n, and a number $x > 0$ such that

$$\iint\limits_{D(1)} \rho_{P_n}\, dm_2 \geq x\sqrt{n}. \tag{3.2}$$

We consider the sequence of nonnegative subharmonic functions

$$v_n(z) = \frac{1}{n}\log\sqrt{1+|P_n(z)|^2} = \iint\limits_{D(1)} \log|z-\xi|\, d\mu_n(\xi)$$
$$+ \iint\limits_{|\xi|\geq 1} \log\left|1-\frac{z}{\xi}\right| d\mu_n(\xi) + C_n.$$

A direct computation shows that

$$\Delta v_n(z) = 2\rho^2_{P_n}(z)/n \tag{3.3}$$

(see, for example, [9], p. 19). In particular, $\mu_n(\mathbf{C}) = 1$. Passing to a subsequence if necessary, we assume that $\mu_n \to \mu$ weakly in each disk $D(R)$, $R > 0$.

We now consider two cases.

1°. $\underline{\lim}\, C_n < +\infty$. Again choosing a subsequence, we assume that $v_n \to u \in U^+$. Let us apply Lemma 2 with $\delta = x^2/(16\pi^2)$ to the function u. We get a partition of $\overline{D(1)}$ into three sets E, $L_n = L_{v_n}$, and $M_n = \{z \in \overline{D(1)}\colon v_n(z) \geq \varepsilon\}$, and for sufficiently large n in the chosen sequence

$$\mu_n E \leq \delta, \qquad m_2 L_2 \leq \delta. \tag{3.4}$$

By (3.1), (3.3), and (3.4),

$$\begin{aligned}\iint\limits_{E\cup L_n} \rho_{P_n}\, dm_2 &\leq \left(\pi \iint\limits_{E} \rho^2_{P_n}\, dm_2\right)^{1/2} + (\pi n m_2 L_n)^{1/2} \\ &= (\pi^2 n\mu_n E)^{1/2} + (\pi n m_2 L_n)^{1/2} \leq 2\pi\sqrt{n\delta} = x\sqrt{n}/2.\end{aligned} \tag{3.5}$$

Further, the image of M_n under the mapping P_n is contained in the exterior of the disk of radius $\sqrt{e^{2n\varepsilon}-1}$ about zero. Therefore, the spherical area of this image (with multiplicity taken into account) tends to 0 as $n \to \infty$ along the chosen sequence. Hence, by (3.1),

$$\iint\limits_{M_n} \rho_{P_n}\, dm_2 = o(1), \qquad n \to \infty. \tag{3.6}$$

But (3.5) and (3.6) contradict (3.2).

2°. $C_n \to +\infty$. Then it follows from the Cartan-Ahlfors lower estimate for the potentials (see, for example, [8] or [7]) that for sufficiently large n in the chosen sequence

$$v_n(z) \geq 1, \qquad z \in \overline{D(1)}\backslash L_n,$$

when $m_2 L_n < \delta$. Of course, as in the first case, we again contradict (3.2), and Theorem 1 is proved.

§4.

Let f be an entire function of order $\lambda < \infty$. We consider a *comparison function* $V(r) = r^\lambda l(r)$ such that

$$\overline{\lim_{r\to\infty}} \frac{\log M(r,f)}{V(r)} = 1, \qquad M(r,f) = \max_{|z|\leq r}|f(z)|,$$

and $l(r) \sim l(2r)$, $r \to \infty$. Such a function $V(r)$ always exists [9].

Denote by $n(r,a)$ the number of roots of the equation $f(z)=a$ (counting multiplicity) in the disk $D(r)$, and by $n(r,a;S)$ the number of roots of this equation on the set $S\cap D(r)$.

THEOREM 2. *Suppose that f is an entire function of order $\lambda<\infty$, and let V be a comparison function. Then there exists a set S of density zero such that for all $a\in\mathbf{C}$*

$$n(r,a)=n(r,a;S)+o(V(r)), \qquad r\to\infty.$$

REMARK 1. A consideration of elliptic functions shows that Theorem 2 ceases to be true for meromorphic functions.

REMARK 2. Theorem 2 loses its content if $\lambda=0$. In this case, as shown by Gol′dberg and Zabolotskiĭ [10], we have that $n(r,a)=o(V(r))$, $r\to\infty$, for all $a\in\mathbf{C}$. But if $\lambda>0$, then it is well known [9] that $\overline{\lim}_{r\to\infty} n(r,a)/V(r)>0$ for all but possibly one exceptional value of $a\in\mathbf{C}$.

We remark that a stronger assertion than Theorem 2 both about the characteristic of S and about an estimate for the remainder is deduced in [1] from the conjecture (1.2).

PROOF OF THEOREM 2. We use the notation $x^+=\max(x,0)$ and $x^-=(-x)^+$. Let $\mathbf{E}_L(f)$ be the set of $a\in\mathbf{C}$ such that

$$\int_0^{2\pi}\log^+\frac{1}{|f(re^{i\theta})-a|}\,d\theta\geq\lambda\log r \tag{4.1}$$

for an unbounded set of values r. It is known [11] that

$$m_2\mathbf{E}_L(f)=0. \tag{4.2}$$

Assume without loss of generality that $f(0)=1$ and that $0\notin\mathbf{E}_L(f)$. We choose a large number $M>0$ and consider on $D(2)$ the family of subharmonic functions

$$\begin{gathered}v_{n,a}(z)=\frac{\log|f(2^nz)-a|}{V(2^n)}, \qquad n=1,2,\dots,\\ a\in Q=\{a\in\mathbf{C}\colon |a|\leq M, |a-1|\geq\frac{1}{M}\}.\end{gathered} \tag{4.3}$$

This family is contained in a certain set U (defined before Lemma 2).

It follows from (4.1) and (4.3) that for $a\notin\mathbf{E}_L(f)$

$$\begin{aligned}&\operatorname{dist}(v_{n,a},U^+)\leq\|v_{n,a}-v_{n,a}^+\|=\|v_{n,a}^-\|\\ &=\frac{1}{V(2^n)}\int_0^2\int_0^{2\pi}\log^+\frac{1}{|f(2^nre^{i\theta})-a|}r\,dr\,d\theta\leq C_1n(V(2^n))^{-1}, \quad n\geq n_0(a),\end{aligned} \tag{4.4}$$

where C_1 is a constant depending only on V.

For each $\delta>0$ and each function $u\in U^+$ we choose a number $\varepsilon=\varepsilon(\delta,u)<\delta$ according to Lemma 2 and consider the covering of the set $U^+\subset U$ by balls of radii $\varepsilon(\delta,u)/3$ about each point $u\in U^+$. In view of compactness there exists a finite subcovering by balls about some points $u_{i,\delta}$, $1\leq i\leq N_\delta$. Let

$$\gamma(\delta)=\min\{\varepsilon(\delta,u_{i,\delta})/3\colon 1\leq i\leq N_\delta\}\to 0, \qquad \delta\to 0.$$

Choose a sequence $\delta_n\to 0$ decreasing so slowly that

$$C_1n/V(2^n)<\gamma(\delta_n)=o(1), \qquad n\to\infty. \tag{4.5}$$

Let $v_n=v_{n,0}$. The nonnegative function v_n^+ is contained in one of the N_{δ_n} balls in the finite covering constructed above. Denote the center of this ball by u_n and its radius by ε_n. By construction, $3\varepsilon_n=\varepsilon(\delta_n,u_n)$ and $\gamma(\delta_n)<\varepsilon_n$

$$\|u_n-v_n^+\|<\varepsilon_n. \tag{4.6}$$

Let us apply Lemma 2 to $u = u_n$, $\delta = \delta_n$. By the choice of ε_n, the assertion of the lemma holds with $\varepsilon = 3\varepsilon_n$. Lemma 2 gives us sets E_n and $L_n = L_{v_n^+}$ such that for any function $v \in U$ with $\|v - u_n\| < 3\varepsilon_n$

$$\mu_v E_n < \delta_n \tag{4.7}$$

and, moreover, by (4.6),

$$\{z \in \overline{D(1)}: v_n^+(z) < 3\varepsilon_n\} \subset E_n \cup L_n, \tag{4.8}$$

$$m_2 L_n < \delta_n. \tag{4.9}$$

Using the inequality $\log^+ |a+b| \le \log^+ |a| + \log^+ |b| + \log 2$ and (4.5), we get that

$$|v_{n,a}^+(z) - v_n^+(z)| \le C_2(V(2^n))^{-1} < \gamma(\delta_n) < \varepsilon_n \tag{4.10}$$

for $z \in D(2)$ and $n > n_0(a)$. By (4.10) and (4.8), for any $a \in Q$ we have that

$$\{z \in \overline{D(1)}: v_{n,a}(z) < \varepsilon_n\} \subset \{z \in \overline{D(1)}: v_n(z) < 2\varepsilon_n\} \subset E_n \cup L_n \tag{4.11}$$

from some index on.

Now let $S_n = \{z: 2^{n-1} < |z| \le 2^n,\ 2^{-n}z \in L_n\}$ and $S^Q = \bigcup_1^\infty S_n$. In view of (4.9)

$$m_2(S^Q \cap D(2^n)) = m_2\left(\bigcup_{k=1}^n S_k\right) \le \sum_{k=1}^n 2^k \delta_k = o(2^n), \qquad n \to \infty.$$

Therefore, the set S^Q has zero density.

Let $a \in Q \backslash \mathbf{E}_L(f)$. We estimate the number of a-points of f in the set $\{z: 2^{n-1} < |z| \le 2^n\} \backslash S_n$. This number is equal to

$$V(2^n)\mu_{v_{n,a}}(\{z: 1/2 < |z| \le 1\} \backslash L_n).$$

Note first that the function $v_{n,a}$ is harmonic on the set $\{z \in D(2): v_{n,a}(z) \ge \varepsilon_n\}$. Therefore, by (4.11),

$$\mu_{v_{n,a}}(\{z: 1/2 < |z| \le 1\} \backslash L_n) \le \mu_{v_{v,a}}(E_n). \tag{4.12}$$

Further, it follows from (4.4), (4.5), (4.10), and (4.6) that for $a \in Q \backslash \mathbf{E}_L(f)$

$$\begin{aligned} \|v_{n,a} - u_n\| &\le \|v_{n,a} - v_{n,a}^+\| + \|v_{n,a}^+ - v_n^+\| + \|v_n^+ - u_n\| \\ &\le \gamma(\delta_n) + \varepsilon_n + \varepsilon_n \le 3\varepsilon_n, \qquad n > n_0(a). \end{aligned}$$

Consequently, (4.7) is applicable to $v_{n,a}$, and we get that

$$\mu_{v_{n,a}} E_n < \delta_n. \tag{4.13}$$

Using (4.12), (4.13), and the properties of the comparison function V, we get that the number of a-points of f in $\overline{D(2)} \backslash \bigcup_1^n S_k$ is

$$\sum_{k=1}^n V(2^k)\mu_{v_{k,a}} E_k \le \sum_{k=1}^n \delta_k V(2^k) = o\left(\sum_{k=1}^n V(2^k)\right) = o(V(2^n)), \qquad n \to \infty.$$

Therefore, for $a \in Q \backslash \mathbf{E}_L(f)$

$$n(r,a) = n(r,a;S^Q) + o(V(r)), \qquad r \to \infty.$$

We now consider the countable family of sets

$$Q_k = \{a \in \mathbf{C}: |c| \le k, |a-1| \ge 1/k\},$$

which together cover the whole plane $\mathbf{C}$ except for the point 1. To each Q_k there corresponds a set $S(k) = S^{Q_k}$ of density zero such that for all $a \in Q_k \backslash \mathbf{E}_L(f)$

$$n(r,a) = n(r,a; S(k)) + o(V(r)), \qquad r \to \infty. \tag{4.14}$$

We choose an increasing sequence of positive numbers (r_k), $r_k \to \infty$, such that

$$m_2((S(1) \cup \cdots \cup S(k)) \cap D(r)) \leq 2^{-k} r^2, \qquad r > r_k. \tag{4.15}$$

Let $S_0 = \bigcup_{k=1}^{\infty}(S(k) \backslash \overline{D(r_k)})$. If $r_{k-1} \leq r < r_k$, then $m_2(S_0 \cap D(r)) \leq 2^{-k+1} r^2$ by (4.15). Consequently, the density of S_0 is equal to 0. It follows from (4.14) that for $a \notin \mathbf{E}_L(f)$, $a \neq 1$,

$$n(r,a) = n(r,a,S_0) + o(V(r)), \qquad r \to \infty.$$

Finally, adding to S_0 the set $f^{-1}(\mathbf{E}_L(f) \cup \{1\})$ of measure zero (by (4.2)), we get the desired set S. The theorem is proved.

The authors thank V. S. Azarin, A. L. Vol′berg, A. A. Gol′dberg, and S. Yu. Favorov for a discussion of this work and for valuable comments.

Received 30/JAN/85

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Translated by H. H. McFADEN