

A generalization of Malmquist's Theorem of 1941

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1. Fields of algebroid functions.

All fields in this text are of characteristic 0.

Let k be the field of all meromorphic functions on the complex line $\mathbf{C}$. We equip it with the standard differentiation d/dz , and consider as a differential field.

There is a convenient way to define differential subfields of k .

We fix a positive, increasing continuous function ϕ on $[0, +\infty)$ with the property

$$\phi\left(r + \frac{1}{\phi(r)}\right) = O(\phi(r)), \quad (1)$$

The symbols O and o in this paper always refer to $r \rightarrow +\infty$. Then the field k_ϕ is defined as

$$k_\phi = \{f \in k : T(r, f) = O(\phi(r))\}, \quad (2)$$

where T is the Nevanlinna characteristic, whose definition we recall. If $f = g_1/g_0$, where $g = (g_0, g_1)$ is a pair of entire functions without common zeros, then

$$T(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \|g(re^{it})\| dt - \log \|g(0)\|,$$

where $\|\cdot\|$ is the ℓ^2 norm. This characteristic function has the following properties:

$$T(r, f + g) \leq T(r, f) + T(r, g) + O(1), \quad (3)$$

$$T(r, fg) \leq T(r, f) + T(r, g) + O(1), \quad (4)$$

$$T(r, 1/f) = T(r, f) + O(1), \quad (5)$$

$$T(r, f^n) = nT(r, f) + O(1), \quad (6)$$

which imply that k_ϕ is a field. These algebraic properties can be derived from the alternative definition of the characteristic as

$$T_N(r, f) = m(r, f) + N(r, f),$$

where

$$m(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{it})| dt,$$

and

$$N(r, f) = \int_0^r (n(t, f) - n(0, f)) \frac{dt}{t} + n(0, f) \log r,$$

where $n(t, f)$ is the number of poles of f in $|z| \leq t$ counted with multiplicity. Then we have $T(r, f) = T_N(r, f) + O(1)$. Since we are only interested in the rates of growth of $T(r, f)$, the $O(1)$ term plays no role, and these two definitions of the characteristic can be considered as equivalent.

All k_ϕ are differential fields. This is proved using the lemma on the logarithmic derivative; property (1) is used to make sure that the exceptional set in this lemma does not affect the conclusion.

Examples. If $\phi \equiv 1$ then $k_\phi = \mathbf{C}$. If $\phi(r) = \log^+ r$, then $k_\phi = \mathbf{C}(z)$.

Now we fix an algebraic closure K of k which is called the field of all algebroid functions. An algebroid function $w \in K$ is defined by an algebraic equation of the form

$$a_m w^m + a_{m-1} w^{m-1} + \dots + a_0 = 0, \quad a_j \in k, \quad (7)$$

so it is a multi-valued function on $\mathbf{C}$. We can assume that this equation is irreducible over k . By multiplying equation (7) on an element of k we can ensure that $a_0, \dots, a_m$ are entire functions with no common zeros of all of them. Then the Nevanlinna characteristic can be defined as

$$T(r, w) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \|a(re^{it})\| - \log \|a(0)\|,$$

where $a = (a_0, \dots, a_m)$ and $\|a\|$ is the ℓ^2 norm. There is an alternative definition as above,

$$T_N(r, w) = m(r, w) + N(r, w),$$

where

$$m(r, w) = \frac{1}{2\pi m} \int \log^+ |w(re^{it})| dt,$$

where the integration is over the curve on the Riemann surface of w which projects to the circle $|z| = r$. And $N(r, w)$ is defined by

$$N(r, w) = \frac{1}{m} \int_0^r (n(t, w) - n(0, w)) \frac{dt}{t} + n(0, w) \log r,$$

where $n(t)$ counts the poles of w on the part of the Riemann surface of w which lies over $|z| \leq t$, with multiplicities counted with respect to a local parameter on this Riemann surface.

Nevanlinna characteristics are similar to the heights in Diophantine Geometry, and their extension to algebraic extensions of fields is also similar.

Nevanlinna characteristic on K enjoys the same algebraic properties (3)-(6); this was proved by Mokhonko in [4].

We define subfields $K_\phi \subset K$ in the same way as k_ϕ in (2). These are differential fields. This follows from the generalization of the lemma on the logarithmic derivative to algebroid functions due to Valiron [6], see also [5] and [2] for other proofs.

Proposition 1. *The field K_ϕ is the algebraic closure of k_ϕ in K .*

Example. If $\phi = \log^+ r$ then $K_\phi = \overline{\mathbf{C}(z)}$, the field of all algebraic functions on $\mathbf{C}$.

2. First order differential equations in the fields of algebroid functions.

We fix the function ϕ as in the previous section.

Let A be (an abstract) extension of K_ϕ of transcendence degree 1 over K_ϕ . It can be constructed as follows. Start with an irreducible polynomial $F \in K_\phi[t_1, t_2]$, and consider the factor ring $K_\phi[t_1, t_2]/(F)$, then A is the quotient field of this ring.

To make a differential field out of A , we choose two elements $x, y \in A$, where x is transcendental over K_ϕ , and set $Dx = y$, and $D = d/dz$ on K_ϕ . We denote the resulting differential field by (A, D) . The field K_ϕ is called the coefficient field of (A, D) .

Definition. *A differential field (A, D) is called Fuchsian, if the differentiation D maps all local rings of A into themselves.*

Fuchsian fields have been classified by Fuchs, Schwarz and Poincaré, see, for example [3]. They correspond to differential equations which can be either reduced to linear ones (Riccati fields), or solved in elliptic functions (Poincaré fields), or solved algebraically (Clairaut fields).

Conjecture. *Let $E : (A, D) \rightarrow K$ be an embedding of differential fields which sends every element of K_ϕ to itself. Then $E(A)$ contains a Fuchsian subfield.*

This generalizes Malmquist's 1941 theorem, which corresponds to the case $\phi(r) = \log^+ r$, and contains Malmquist's theorem of 1913 (case when $\phi(r) = \log^+ r$, and the genus of A is zero). The case that $E(A) \subset k \cdot K_\phi$, composite of k and K_ϕ , is the theorem of Eremenko [1]. The case $\phi = \log^+ r$ of Eremenko's theorem is due to Malmquist (1920).

References

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