Complex Variables, 1985, Vol. 4, pp. 305–309
0278-1077/85/0404-0305 $10.00

On the Natural Asymptotic Curves of Meromorphic Functions

A. E. EREMENKO
Institute for Low Temperature Physics and Engineering, Ukrainian SSR Academy of Science, 47 Lenin Avenue, Kharkov 310164, U.S.S.R.

Communicated by S. Hellerstein

(*Received January 19, 1984*)

Let f be a meromorphic function of finite lower order. Suppose that the inverse function f^{-1} has a direct transcendental singularity at $w = \infty$. Then there exists an asymptotic curve Γ with the property: $\operatorname{Re} f(z) \to +\infty$, $\operatorname{Im} f(z) = 0$ if $z \in \Gamma$. This is not true for the functions having only indirect singularities at $w = \infty$.

There exists an example of an entire function (of infinite lower order) such that $\operatorname{Re} f$ is bounded on every connected component of the set $\{z\colon \operatorname{Im} f(z) = 0\}$. This example answers a question of S. Hellerstein.

AMS (MOS): 30D30, 30F99

Let f be a meromorphic function in C and $a \in R$ be its asymptotic value. We shall investigate the following problem: Does there exist an asymptotic curve $\Gamma \subset C$ such that $f(z) \to a$, $z \in \Gamma$, $z \to \infty$ and $f(\Gamma) \subset R$? This problem (in a slightly different form) was posed for entire functions by S. Hellerstein [1], who proposed to call such curves the natural ones.

In the sequel we assume without loss of generality that $a = \infty$. It was pointed out in [1] that an entire function of finite order always has a natural asymptotic curve. The following theorem generalizes the latter fact. We use the terms and notions from the theory of simply connected Riemann surfaces from [2], Ch. XI. Denote by π the projection of the Riemann surface to the plane.

THEOREM *Let f be a meromorphic function of finite lower order. Suppose that the inverse function f^{-1} has a direct singularity over ∞. Then there exists a curve $\Gamma \subset C$ such that $f(z) \to \infty$, $z \in \Gamma$, $z \to \infty$ and $f(\Gamma)$ is a ray $[w_0, \infty) \subset R$.*

Proof It follows from the Denjoy–Carleman–Ahlfors theorem that the function f^{-1} has finitely many direct singularities. Thus there exists a neighborhood V of a direct singularity over ∞ on the Riemann surface of f^{-1} which is the neighborhood of exactly one direct singularity, $\infty \in \overline{\pi V}$.

Let $w_1 \in V$ be a point, $\pi w_1 > 0$. Consider the curve $\gamma_1 \subset V$ beginning at w_1 and having the following properties: π is one-to-one on γ_1, $\pi\gamma_1 = [\pi w_1, a_1)$, $\pi w_1 < a_1 \leqslant \infty$; γ_1 has the maximal property, i.e. if $\gamma_1 \subset \gamma' \subset V, \pi\gamma' \subset [\pi w_1, \infty)$ then $\gamma' = \gamma_1$. It is clear that such a γ_1 actually exists. If $a_1 = \infty$ then the curve $f^{-1}(\gamma_1) \subset C$ is the desired asymptotic curve. If $a_1 < \infty$ we choose a point $w_2 \in V, \pi w_2 > a_1$. Consider the curve γ_2 beginning at w_2 such that π is one-to-one on γ_2, $\pi\gamma_2 = [\pi w_2, a_2)$, $\pi w_2 < a_2 \leqslant \infty$, and γ_2 has the maximal property. If $a_2 = \infty$ then $f^{-1}(\gamma_2)$ is the desired asymptotic curve. We shall show that the assumption that $a_2 < \infty$ leads to a contradiction. Indeed if $a_2 < \infty$ we connect the points w_1, w_2 by a curve $\gamma_3 \subset V$. It is easy to see that the curve $\gamma_1 \cup \gamma_2 \cup \gamma_3$ divides V in two parts. Each of them contains its own neighborhood of some direct singularity over ∞. This contradicts the assumption that V is a neighborhood of exactly one direct singularity. Q.E.D.

We shall give two examples showing that neither of the two assumptions of the theorem may be removed.

Example 1 $f(z) = iz/\sin z$. This function has ∞ as an asymptotic value. We may take $\Gamma = [0, +\infty)$ for an asymptotic curve. There is no natural asymptotic curve. The inverse function has an indirect singularity over ∞.

Example 2 We shall construct an entire function f (of infinite lower order) having no natural asymptotic curves. First of all we construct a simply connected Riemann surface F, $\infty \in \overline{\pi F}$ containing no rays with real projection. Further we shall show that F is of the parabolic type.

1. Consider the set of all finite sequences $K = (0, k_1, \ldots, k_n)$ where $k_j \in Z\backslash\{0\}, j = 1, \ldots, n$. Denote $|K| = n$, $n \in Z_+$. With each sequence K

we associate the open half-plane

$$\Delta(K) = \{w\colon \operatorname{Im} w < 0\} \qquad \text{if} \quad |K| \text{ is even}$$

and

$$\Delta(K) = \{w\colon \operatorname{Im} w > 0\} \qquad \text{if} \quad |K| \text{ is odd.}$$

If $n = |K| > 0$, consider the intervals $I(K, s) \subset \partial\Delta(K)$, $s \in Z$. The left endpoint of $I(K, s+1)$ coincides with the right endpoint of $I(K, s)$, $s \in Z$. To determine uniquely this system of intervals it is sufficient to determine $I(K, 0)$ and the length $|I(K, s)|$ of all $I(K, s)$, $s \neq 0$.

Consider the intervals $I((0), s) \subset \partial\Delta(0)$, $s \in Z\backslash\{0\}$:

$$\pi I((0), s) = (s-1, s) \qquad \text{if} \quad s > 0$$

and

$$\pi I((0), s) = (s, s+1) \qquad \text{if} \quad s < 0.$$

Let $I(K, s) \subset \partial\Delta(K)$ be already determined for $|K| = n \geqslant 0$. Put

$$\pi I((0, k_1, \ldots, k_{n+1}), 0) = \pi I((0, k_1, \ldots, k_n), k_{n+1}) \tag{1}$$

and

$$|I((0, k_1, \ldots, k_{n+1}), s)| = 2 \prod_{j=1}^{n+1} (1 + 2|k_j|), \qquad s \in Z\backslash\{0\}. \tag{2}$$

The desired Riemann surface F is the result of the sewing of $\Delta(K)$'s. We sew $\Delta(0, k_1, \ldots, k_{n+1})$ to $\Delta(0, k_1, \ldots, k_n)$ along the interval $I((0, k_1, \ldots, k_{n+1}), 0) \subset \partial\Delta(0, k_1, \ldots, k_{n+1})$ which is identified with $I((0, k_1, \ldots, k_n), k_{n+1}) \subset \partial\Delta(0, k_1, \ldots, k_n)$. This is possible in view of (1). The endpoints of the intervals $I(K, s)$ do not belong to F.

It is easily seen that defined in such a way the Riemann surface is simply connected and does not cover ∞. The surface F has no algebraic branch points and no critical points over the finite plane except for the logarithmic singularities. The latter lie over the real axis. The surface F does not contain any ray over the real axis.

Every logarithmic singularity is the endpoint of a countable set of intervals. There are exactly two of these intervals which have non-zero numbers s. These two intervals are:

$$I(K, s),\ I(K, s+1),\ s(s+1) \neq 0 \qquad \text{if} \quad K \neq (0)$$

or

$$\pi^{-1}(\nu - 1, \nu),\ \pi^{-1}(\nu, \nu + 1) \subset \partial\Delta(0), \qquad \nu \in Z.$$

With each logarithmic singularity we associate the common endpoint of the before mentioned pair of intervals in $\partial\Delta(K)$. Thus we obtain the one-to-one correspondence between the set of the logarithmic singularities of F and the set of the endpoints of the intervals $I(K, s)$, $s \neq 0$ which are not endpoints of the intervals $I(K, 0)$.

Let us show that there is a finite set of logarithmic singularities over each disk of the form $\{w: |w| < r\}$. For any interval $I \subset F$ set $\|I\| = \sup\{|w|: w \in \pi I\}$. We claim that

$$\|I((0, k_1, \ldots, k_{n-1}), s)\| \leqslant \prod_{j=1}^{n-1} (1 + 2|k_j|)(1 + 2|s|). \tag{3}$$

This is obvious for $n = 1$. Suppose that (3) is true for $n = m$. In view of (1)–(3) we have

$$\begin{aligned}\|I((0, k_1, \ldots, k_m), s)\| &\leqslant \|I((0, k_1, \ldots, k_m), 0)\| + |s||I((0, k_1, \ldots, k_m), 1)| \\ &\leqslant \prod_{j=1}^{m} (1 + 2|k_j|) + 2|s| \prod_{j=1}^{m} (1 + 2|k_j|) \\ &= \prod_{j=1}^{m} (1 + 2|k_j|)(1 + 2|s|)\end{aligned}$$

which proves (3) for $n = m + 1$ and consequently for all $n \in N$.

Consider now a point $a \in \partial\Delta(0, k_1, \ldots, k_n)$ which is the endpoint of $I((0, k_1, \ldots, k_n), s)$ but is not the endpoint of $I((0, k_1, \ldots, k_n), 0)$. Denote by x the projection of the endpoint of $I((0, \ldots, k_n), 0)$ with the maximal modulus. We have $|\pi a| \geqslant |\pi a - x| - |x|$. Applying (1)–(3) we conclude

$$|\pi a| \geqslant 2 \prod_{j=1}^{n} (1 + 2|k_j|) - \prod_{j=1}^{n} (1 + 2|k_j|) = \prod_{j=1}^{n} (1 + 2|k_j|).$$

This implies that the set of such a's over any disk $\{w: |w| < r\}$ is finite and consequently the set of the logarithmic singularities over this disk is finite.

2. Let (d_j) be the projections of all logarithmic singularities of F, $|d_1| \leqslant |d_2| \leqslant \cdots \to \infty$. There is the following sufficient condition for parabolicity of F ([3], §10):

$$n^{-1} \ln \ln d_n \to \infty. \tag{4}$$

Consider the homeomorphism of the w-plane

$$w = r\, e^{i\theta} \mapsto g(r)\, e^{i\theta},$$

g is a continuous increasing function. We choose g in such a way that the sequence $|d_j'| = g(|d_j|)$ satisfies (4). The image F' of F under this homeomorphism is the parabolic Riemann surface having all the desired properties.

Remark Using the method of constructing Example 2 it is possible to construct an entire function with finite Picard exceptional value having no natural asymptotic curves. This example will answer the question due to S. Hellerstein in the original form. To find such a function we construct first of all a simply connected Riemann surface F covering $\{w: |w| < \infty\}$ with the following properties: There are no algebraic branch points and no other singularities except for the logarithmic branch points. The set of the logarithmic branch points over each strip of the form $\{w: -A \leqslant \operatorname{Re} w \leqslant A\}$ is finite. There are no rays on F projecting to the set $\{w = u + iv: -\infty < u < +\infty; v = 2k\pi, k \in Z\}$. The construction of such a surface F is essentially the same as in Example 2. Let F' be the image of F under the homeomorphism $u + iv \mapsto g(u) + iv$. We choose g in such a way that F' satisfies (4). If f_1 maps conformally C to F' then $f = \exp f_1$ is the desired function.

The author is deeply grateful to Professor A. A. Gol'dberg for his advice.

References

[1] D. M. Campbell, J. C. Clunie, and W. K. Hayman, Research problems in complex analysis, *Aspects of Contemporary Complex Analysis*, Academic Press, London, 1980, 527–572.

[2] R. Nevanlinna, *Eindeutige Analytische Funktionen*, 2. Aufl., Springer, Berlin, 1953.

[3] L. I. Volkovyskii, *Researches on the Type Problem of Simply Connected Riemann Surface*, Trudy MIAN im. V. A. Steklov, Moscow–Leningrad, 1950.