On the number of solutions of some transcendental equations

Walter Bergweiler and Alexandre Eremenko*

Dedicated to Dima Khavinson on the occasion of his 60th birthday

Abstract

We give upper and lower bounds for the number of solutions of the equation $p(z)\log|z| + q(z) = 0$ with polynomials p and q.

1 Introduction and main result

Holomorphic functions are sense-preserving. This allows, for a holomorphic function f and $c \in \mathbb{C}$, to estimate the number of solutions of the equation $f(z) = c$ from above by the topological degree. This method does not work when f is just smooth, or real analytic, unless f is sense-preserving. For the equation

$$\overline{z} = f(z), \tag{1}$$

where f is holomorphic, a remarkable argument combining topological degree considerations with Fatou's theorem from holomorphic dynamics was invented by Khavinson and Świątek [8]. In this paper f was a polynomial; later the argument was extended to rational f by Khavinson and Neumann [6]. The latter result found an important and unexpected application in astronomy. For transcendental meromorphic f the equation (1) was considered in [2, 3, 5], motivated by certain applications. For a description of the method initiated in [8] and its applications to astronomy we also refer to the survey [7].

This paper is a part of our efforts to understand the scope of applicability of the method. The following question was asked on Math Overflow [12]. Let

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p and q be coprime polynomials of degrees m and n, respectively, with at least one of the polynomials non-constant. How many solutions can the equation

$$p(z)\log|z| + q(z) = 0 \tag{2}$$

have?

Theorem. *The number N of solutions of equation* (2) *satisfies*

$$\max\{m,n\} \leq N \leq 3\max\{m,n\} + 2m. \tag{3}$$

The proof of the upper bound, given in section 2, combines the computation of a topological degree with Fatou's theorem as in the paper [8] mentioned above. The difference of our argument in comparison with previous applications of the method is that we transform (2) to an equation with infinitely many solutions, but it is still possible to obtain the desired estimate.

The computation of the topological degree also yields the lower bound, but only if solutions are counted with multiplicities. In order to obtain a lower bound for the number of *distinct* solutions we study the curves where the rational function q/p is real.

In section 3 we give examples to show that the estimate is sharp, at least for many values of m and n.

2 Proof of the theorem

We put

$$f(z) = \frac{2q(z)}{p(z)}$$

and rewrite our equation (2) as

$$g(z) := \log|z|^2 + f(z) = \log|z|^2 + \frac{2q(z)}{p(z)} = 0.$$

The function g is a continuous map of the Riemann sphere $\overline{\mathbb{C}}$ into itself, satisfying $g(0) = g(\infty) = \infty$.

We recall the definition of the topological (or Brouwer) degree; see [13, Chapter II, §2] or [10, §5]. A value w is *regular* for g if for all solutions z of the equation $g(z) = w$ the map g is continuously differentiable near z and the Jacobian determinant $J_g(z)$ does not vanish. Then

$$\deg g = \sum_{z\in f^{-1}(w)} \operatorname{sgn} J_g(z)$$

is the *topological degree* of g. This definition does not depend on w. (We note that e.g. in [4, §§1–2] the topological degree is introduced for functions on bounded domains, but this could be achieved by considering g as map from $\{z \in \mathbb{C}\colon |f(z)| < R\}$ onto $\{z \in \mathbb{C}\colon |z| < R\}$ for some large R.)

Taking $w = ir$ with large real r we find $\max\{n-m, 0\}$ preimages near ∞ and m preimages near the poles. Evidently $J_g(z) > 0$ at all these preimages. So with $d := \max\{m, n\} = m + \max\{n - m, 0\}$ we have $\deg g = d$.

For $w \in \mathbb{C}$ we denote by N_w the number of solutions of

$$g(z) = w \tag{4}$$

so that $N = N_0$. Suppose first that w is a regular value of g. We denote by N^+ and N^- the numbers of solutions of (4) where $J_g(z)$ is positive and negative, respectively. Then

$$N^+ - N^- = d \tag{5}$$

by the definition of the topological degree.

Computing the Jacobian we obtain

$$J_g(z) = |g_z(z)|^2 - |g_{\overline{z}}(z)|^2 = \frac{1}{|z|^2}\left(|1 + zf'(z)|^2 - 1\right). \tag{6}$$

We put

$$h(z) = \frac{e^{-f(z)+w}}{z}$$

and note that if z satisfies (4), then z also satisfies

$$k(z) := h(z) - \overline{z} = 0. \tag{7}$$

Note that the set of solutions of (4) is, in general, not equal to but only *contained* in the set of solutions of (7). The equation (7) can have infinitely many solutions; for example this is the case for the equation $\overline{z} = e^z/z$.

Since

$$h'(z) = -\frac{e^{-f(z)+w}}{z^2}(1 + zf'(z)) \tag{8}$$

the Jacobian of k is given by

$$J_k(z) = |h'(z)|^2 - 1 = \frac{|e^{-2f(z)+2w}|}{|z|^4}|1 + zf'(z)|^2 - 1. \tag{9}$$

If z is a solution of (4), then $|z|^2 = \exp(-f(z) + w)$. We deduce from (6) and (9) that for such z we have $J_g(z) = J_k(z)$. In particular, the Jacobians $J_g(z)$ and $J_k(z)$ have the same sign for z satisfying (4).

Thus

$$N^- \leq n^-,$$

where n^- is the number of solutions of $k(z) = h(z) - \overline{z} = 0$ with negative Jacobian. For these solutions we have $|h'(z)| < 1$, so they are exactly the *attracting fixed points* of the antiholomorphic function $\overline{h(z)}$.

Now we use the generalized Fatou theorem which says that the number of attracting fixed points of a holomorphic or antiholomorphic function does not exceed the number of singular values; see [11, Lemma 8.5] for rational functions, [1, Lemma 10 (i)] for functions which are meromorphic in $\overline{\mathbb{C}}$ except for a compact, totally disconnected set (and thus in particular our function h), and [3, p. 2914] for a version for self-maps of a Riemann surface (which also applies to our function h).

The number of singular values of h is easy to estimate. By (8), the critical points of h are the zeros of $1 + zf'(z)$ in $\mathbb{C}$, so there are at most $\max\{m + n, 2m\} = d + m$ of them. The asymptotic values of h can be only 0 and ∞, so they do not contribute. Thus

$$N^- \leq n^- \leq d + m.$$

Combining this with (5) we find that the number N_w of solutions of (4) satisfies

$$N_w = N^+ + N^- = 2N^- + d \leq 2(d + m) + d = 3d + 2m. \tag{10}$$

This proves the upper estimate in (3) if 0 is a regular value of g.

To deal with the case that 0 is not regular we use the following lemma proved in [2, Proposition 3].

Lemma. *Let D be a region in $\mathbb{C}$ and let $g\colon D \to \mathbb{C}$ be harmonic. Suppose that there exists $M \in \mathbb{N}$ such that every $w \in \mathbb{C}$ has at most M preimages under g. Then the set of points which have M preimages is open.*

We show that our function g satisfies the hypothesis of this lemma for a suitable domain D. In order to do this we note that if z satisfies (4), then z is a fixed point of the function

$$\zeta \mapsto \overline{h\left(\overline{h(\zeta)}\right)}.$$

This function is holomorphic in $\mathbb{C}$ except for singularities at 0 and the poles of f. So the solutions of (4) form a discrete set. Since the solutions of (4) do not accumulate at 0, ∞ or a pole of f, we conclude that (4) has only finitely many solutions, for each $w \in \mathbb{C}$. We thus have $N_w < \infty$ also if w is not regular; that is, for each $w \in \mathbb{C}$ the function g has only finitely many w-points. In order to apply the lemma we still have to show that the number of w-points is uniformly bounded by some $M \in \mathbb{N}$, at least after restricting to a suitable domain D.

We denote by $N_w(D)$ the number of w-points of g in a domain D. We choose a bounded domain D containing all solutions of the equation $g(z) = 0$ such that the closure of D does not contain 0 or a pole of f. By the choice of D we then have $N = N_0(D)$. If $\zeta \in D$ is such that $J_g(\zeta) \neq 0$, then ζ clearly has a neighborhood U_ζ such that $N_w(U_\zeta) \leq 1$ for all $w \in \mathbb{C}$. Moreover, it follows from results of Lyzzaik [9, Theorems 5.1 and ??6.1] that if $\zeta \in D$ with $J_g(\zeta) = 0$, then there exist a neighborhood U_ζ of ζ and $M_\zeta \in \mathbb{N}$ such that $N_w(U_\zeta) \leq M_\zeta$ for all $w \in \mathbb{C}$. (The results of Lyzzaik give precise information about the value M_ζ, but this is irrelevant for our purposes.) Since D can be covered by finitely many neighborhoods U_ζ, we deduce that there exists $M \in \mathbb{N}$ such that $N_w(D) \leq M$ for all $w \in \mathbb{C}$. We may assume that M has been chosen minimal. Then the set of all $w \in \mathbb{C}$ with $N_w(D) = M$ is a non-empty open subset of $\mathbb{C}$ by the lemma. This implies that there exists a regular value w with $N_w(D) = M$. Combining this with (10) we thus have

$$N = N_0(D) \leq M = N_w(D) \leq N_w \leq 3d + 2m.$$

This shows that the upper estimate in (3) also holds if 0 is not a regular value.

To prove the lower estimate in (3) we put

$$F(z) = \frac{q(z)}{p(z)} = \frac{1}{2} f(z).$$

So F is a rational function of degree d. If F has no real critical values, the preimage of $\mathbb{R}$ under F is a union of d disjoint curves in $\overline{\mathbb{C}}$. The start and end point of such a curve are (not necessarily distinct) poles. If $F(\infty)$ is finite and real, then at least one and possibly several of these curves pass through ∞.

If F has real critical values, we consider these curves for the function $F - i\varepsilon$ instead of F, for some small positive ε. Taking the limit as $\varepsilon \to 0$ we find that $F^{-1}(\mathbb{R})$ is still the union of d curves $\gamma_1, \dots, \gamma_d$, with each γ_j starting and ending at a pole of F, but now these curves are not disjoint anymore.

Indeed, let $w_0 \in \mathbb{R}$ be a critical value of F, say $w_0 = F(z_0)$ where $z_0 \in \mathbb{C}$ with $F'(z_0) = 0$. Let L be the multiplicity of the w_0-point z_0; that is, $L = \min\{k \in \mathbb{N} \colon F^{(k)}(z_0) \neq 0\}$. Then there exist L curves passing through z_0, and we may assume that the curves are numbered so that this is the case for the curves $\gamma_1, \dots, \gamma_L$. Choosing parametrizations $\gamma_j \colon I_j \to \overline{\mathbb{C}}$ with intervals I_j we thus have $\gamma_j(t_j) = z_0$ for some $t_j \in I_j$. We may assume that the parametrizations γ_j are chosen such that $F(\gamma_j(t))$ increases with t. The directions of the curve γ_j at the point z_0 are given by the one-sided derivatives $\gamma_j'(t_j^{\pm})$ of γ_j at t_j. The left and right derivative are related by

$$\arg \gamma_j'(t_j^+) = \arg \gamma_j'(t_j^-) + \frac{\pi}{L} - \pi. \tag{11}$$

Moreover, for a suitable permutation $\sigma \in S_L$ we have

$$\arg \gamma_j'(t_j^+) = \frac{2\pi\sigma(j)}{L} - \alpha \tag{12}$$

where $\alpha = \arg F^{(L)}(z_0)$.

We now consider the function

$$G_j \colon I_j \to \mathbb{R}, \quad G_j(t) = F(\gamma_j(t)) + \log|\gamma_j(t)|. \tag{13}$$

Noting that there are poles p_j^+ and p_j^- such that $\gamma_j(t) \to p_j^+$ as $t \to \sup I_j$ while $\gamma_j(t) \to p_j^-$ as $t \to \inf I_j$ we can deduce that $G_j(t) \to \pm\infty$ as $t \to \sup I_j$ or $t \to \inf I_j$, respectively. This is clear if $p_j^- \neq 0$ and $p_j^+ \neq \infty$, but it also follows if $p_j^- = 0$ or $p_j^+ = \infty$, since then $F(\gamma_j(t))$ tends to $\pm\infty$ faster than $\log|\gamma_j(t)|$.

Thus there exists $s_j \in I_j$ such that G_j changes its sign from $-$ to $+$ at s_j; that is, there exists $\delta > 0$ such that $G_j(s) < 0$ for $s_j - \delta < s < s_j$ while $G_j(s) > 0$ for $s_j < s < s_j + \delta$. It follows that

$$G_j'(s_j^-) \geq 0 \quad \text{and} \quad G_j'(s_j^+) \geq 0. \tag{14}$$

If γ_j passes through ∞, which can happen only if $F(\infty)$ is finite and real, then $F(z) - \log|z|$ is negative for all z on this curve of sufficiently large modulus. This implies that $\gamma_j(s_j) \in \mathbb{C}$ and hence $z = \gamma_j(s_j)$ is a solution of our equation (2). If all the points $\gamma_j(s_j)$ are distinct we thus have d solutions. This is clearly the case if none of the points $\gamma_j(s_j)$ is a critical point.

Suppose now that $z_0 = \gamma_j(s_j)$ is a critical point for some j. Using the notation above we thus have $j \in \{1, \dots, L\}$ and $s_j = t_j$.

Noting that

$$\frac{d}{dt} \log|\gamma_j(t)| = \operatorname{Re} \frac{\gamma_j'(t)}{\gamma_j(t)}$$

and $F'(z_0) = 0$ we then have

$$\begin{aligned} G_j'(t_j^\pm) &= F'(\gamma_j(t_j))\gamma_j'(t_j^\pm) + \operatorname{Re} \frac{\gamma_j'(t_j^\pm)}{\gamma_j(t_j)} \\ &= F'(z_0)\gamma_j'(t_j^\pm) + \operatorname{Re} \frac{\gamma_j'(t_j^\pm)}{z_0} \\ &= \operatorname{Re} \frac{\gamma_j'(t_j^\pm)}{z_0}. \end{aligned}$$

Put $\beta = \arg z_0$. In view of (14) the last equation yields that

$$\cos\bigl(\arg \gamma_j'(t_j^+) - \beta\bigr) \geq 0 \quad \text{and} \quad \cos\bigl(\arg \gamma_j'(t_j^-) - \beta\bigr) \geq 0. \tag{15}$$

Since

$$\begin{aligned} \cos\bigl(\arg \gamma_j'(t_j^-) - \beta\bigr) &= \cos\Bigl(\arg \gamma_j'(t_j^+) - \frac{\pi}{L} + \pi - \beta\Bigr) \\ ?? &= -\cos\Bigl(\arg \gamma_j'(t_j^+) - \frac{\pi}{L} - \beta\Bigr) \end{aligned}$$

by (11) we deduce from (12) and (15) with $\theta = \alpha + \beta$ that

$$\cos\left(\frac{2\pi\sigma(j)}{L} - \theta\right) \geq 0 \quad \text{and} \quad \cos\left(\frac{2\pi\sigma(j)}{L} - \frac{\pi}{L} - \theta\right) \leq 0. \tag{16}$$

Since in an interval of length 2π there is only point where the cosine changes its sign from $-$ to $+$ there exists at most one value $\sigma(j) \in \{1, \dots, L\}$ that satisfies (16). We conclude that if z_0 is a critical point of F, then $z_0 = \gamma_j(s_j)$ for at most one value of j. Altogether we see that the points $\gamma_j(s_j)$ are all distinct so that our equation has at least $d = \max\{m, n\}$ solutions. This completes the proof of the theorem.

3 Examples

We give several examples to show that the estimates in our theorem are best possible. More specifically, Examples 1 and 2 show that the upper bound is sharp if $m = 0$ or $n = 0$. Example 3 deals with the case $n \leq m$, thus generalizing Example 2. Examples 4 and 5 show that the upper bound is sharp if $n = 2m$ or $n = 3m$. Finally, Example 6 shows that the lower bound is sharp for all m and n.

Example 1. For $p(z) = 1$ and $q(z) = 2\log 2 \cdot (1 - z)$ the equation (2) has the positive solutions $1/2$ and 1, and there is one negative solution ξ by the intermediate value theorem. Computation shows that $\xi \approx -0.191666$. This shows that the upper bound in the theorem is sharp for $m = 0$ and $n = 1$. Considering

$$q_n(z) := \frac{1}{n} q(z^n) = \frac{2\log 2}{n}(1 - z^n)$$

with $n \geq 2$ instead of q we see that the upper bound is sharp for $m = 0$ and arbitrary $n \in \mathbb{N}$.

Indeed, for any n-th root of unity ω the equation $\log|z| + q_n(z) = 0$ has the solutions ω, $\omega/\sqrt[n]{2}$ and $\omega\sqrt[n]{|\xi|}e^{i\pi/n}$ so that there are $3n$ solutions altogether; that is, the equation

$$\log|z| + \frac{2\log 2}{n}(1 - z^n) = 0$$

has $3n$ solutions.

Example 2. For $p(z) = 8z+1$ and $q(z) = 6\log 2$ the equation (2) has the three positive solutions $1/16$, $1/8$ and $1/4$, and two negative solutions $\xi_{1,2}$ by the intermediate value theorem. The numerical values are $\xi_1 \approx -1.471293$ and $\xi_2 \approx -0.0106199$. Similarly as in the previous example we see by considering $p_m(z) = m\, p(z^m)$ with $m \geq 2$ instead of p that the upper bound in our result is sharp for $n = 0$ and arbitrary $m \in \mathbb{N}$; that is, the equation

$$\log|z| + \frac{6\log 2}{m(8z^m + 1)} = 0 \tag{17}$$

has $5m$ solutions.

Example 3. The previous example can be perturbed as follows. Choose a polynomial q of degree $n \leq m$ which is close to 1 on a compact set containing all $5m$ solutions of (17). As all solutions of (17) are non-degenerate, the inverse function theorem will guarantee that the number of solutions of

$$\log|z| + \frac{6\log 2\; q(z)}{m(8z^m + 1)} = 0$$

is at least $5m$ when q is sufficiently close to 1. This shows that the upper estimate in the theorem is best possible for all $n \leq m$.

An explicit example with $m = n$ is

$$\log|z| = 3\log 2 \frac{z^n - 1}{n(z^n + 1)}. \tag{18}$$

When $n = 1$ this equation has 5 real solutions: the positive solutions 1, 2 and 1/2, as well as two negative solutions by the intermediate value theorem, which can be computed to be $\xi_1 \approx -11.770347$ and $\xi_2 \approx -0.0849592$. Making the change of the variable $z \mapsto z^n$ we see that (6) has $5n$ solutions.

Example 4. Take $a = 0.015$, and consider the equation

$$\log|z| = 3\log 2 \cdot (1 - a(z-1))\frac{z-1}{z+1}. \tag{19}$$

This is a small perturbation of (18) with $n = 1$. Again $z = 1$ is clearly a solution and one can check that it has 4 further real solutions near the solutions of (18). Moreover, the intermediate value theorem yields that it has one more negative solution. The numerical values of these 6 real zeros $\xi_1, \ldots, \xi_6$ are at $\xi_1 \approx -58.249375$, $\xi_2 \approx -20.915701$, $\xi_3 \approx -0.0826000$, $\xi_4 \approx 0.466285$, $\xi_5 = 1$ and $\xi_6 \approx 1.780021$.

Let f be the right hand side of (19). Then f has two real critical points $x_1 \approx -12.718930$ and $x_2 \approx 10.718930$ with critical values $y_1 \approx 2.935272$ and $y_2 \approx 1.473143$.

This shows that there is a curve γ in the upper half-plane with endpoints x_1 and x_2 on which f is real. As $\log|x_1| \approx 2.543091478 < y_1$ and $\log|x_2| \approx$

$2.372011 > y_2$ we conclude that the equation (19) must have a solution in the upper half-plane and, by symmetry, another one in the lower half-plane. Numerically these two solutions are $\xi_{7,8} \approx -5.705306 \pm 10.732819i$. Altogether the total number of solutions of (19) is thus 8.

Making the change of variable $z \mapsto z^m$, we obtain an equation with $n = 2m$ having $8m = 3 \cdot 2m + 2m$ solutions. This shows that the upper estimate in the theorem is exact when $n = 2m$.

Example 5. This example is again a small perturbation of the previous example. As there we take $a = 0.015$, put $b = 0.00185$ and and consider the equation

$$\log|z| = 3\log 2 \cdot (1 - a(z-1)) \cdot (1 - b(z-1))\frac{z-1}{z+1}. \tag{20}$$

The equation has 7 real solutions, 6 of which correspond to the solutions of (19). The numerical values are $\xi_1 \approx -198.8150$, $\xi_2 \approx -176.4617$, $\xi_3 \approx -17.8054$, $\xi_4 \approx 0.08289$, $\xi_5 \approx 0.4704$, $\xi_6 = 1$ and $\xi_7 \approx 1.8020$. Denoting by f the right hand side of (20) we see that f has two critical points near those found in the previous example, and there is a curve connecting these points in the upper half-plane on which f is real. On this curve we then have a solution of (20). Together with its complex conjugate this yields the two solutions $\xi_{8,9} \approx 8.6167 \pm 10.2654i$.

Moreover, f has one critical point at $x_0 \approx -234.2572$, and we have $f(x_0) < \log|x_0|$. This yields that there exists a curve in the upper half-plane connecting x_0 with ∞ on which f is real. This curve then contains a solution of (20). Together with its complex conjugate we obtain the solutions $\xi_{10,11} \approx -234.2803 \pm 43.6244i$.

Altogether we thus have 11 solutions. The change of variable $z \mapsto z^m$ then yields an equation with $n = 3m$ having $11m = 3 \cdot 3m + 2m$ solutions. Thus the upper estimate in the theorem is exact when $n = 3m$.

Example 6. Let p and q be polynomials of degrees m and n, respectively. Suppose that $F(z) := q(z)/p(z) \in \mathbb{C}\backslash\mathbb{R}$ for $|z| \leq 1$. If $F(\infty) \in \mathbb{C}\backslash\{0\}$, assume in addition that $F(\infty) \notin \mathbb{R}$. It is clear that polynomials p and q with these properties exist. In fact, if p_0 and q_0 are polynomials satisfying $p_0(0) \neq 0$ and $q_0(0) \neq 0$, then there exists $\varphi \in \mathbb{R}$ such that $p(z) = e^{i\varphi}p_0(\delta z)$ and $q(z) = e^{i\varphi}q_0(\delta z)$ have the above properties for all small positive δ.

We show that if c is a large positive number, then the equation

$$cp(z)\log|z| + cq(z) = 0$$

has $\max\{m,n\}$ solutions. This shows that the lower bound in our theorem is best possible.

As before we put $d = \max\{m,n\}$. For $1 \le j \le d$ we choose the curves $\gamma_j\colon I_j \to \overline{\mathbb{C}}$ as in the proof of the theorem. Since $F(z) \in \mathbb{C}\backslash\mathbb{R}$ for $|z| \le 1$ we find that the curves γ_j are contained in $\{z\colon |z| > 1\} \cup \{\infty\}$. Since F has no real critical values, we have

$$|F'(z)| > 0 \quad \text{if } z \in \mathbb{C} \text{ and } F(z) \in \mathbb{R}.$$

Suppose first that j is such that the curve γ_j does not pass through ∞. For a sufficiently large positive constant c we then can have

$$c|F'(\gamma_j(t))| > \frac{1}{|\gamma_j(t)|} \quad \text{for } t \in I_j. \tag{21}$$

Note that this also works if ∞ is an endpoint of γ_j since then $F(\infty) = \infty$. Let $G_{c,j}$ be defined as in (13), with F replaced by cF; that is,

$$G_j\colon I_j \to \mathbb{R}, \quad G_j(t) = cF(\gamma_j(t)) + \log|\gamma_j(t)|.$$

We deduce from (21) that

$$G'_{c,j}(t) = cF'(\gamma_j(t))\gamma_j'(t) + \operatorname{Re}\frac{\gamma_j'(t)}{\gamma_j(t)} \ge |\gamma_j'(t)|\left(c|F'(\gamma_j(t))| - \frac{1}{|\gamma_j(t)|}\right) > 0.$$

Hence $G_{c,j}$ is increasing and thus $G_{c,j}$ has exactly one zero.

Suppose now that γ_j passes through ∞, say $\gamma_j(t_j) = \infty$. Noting that $F(\infty) = F(\gamma_j(t_j)) \notin \mathbb{R}$ if $F(\infty) \in \mathbb{C}\backslash\{0\}$, by our choice of p and q, we see that $F(\infty) = 0$. Next we note that as it approaches ∞, the curve γ_j is asymptotic to a ray from the origin. In fact, it is not difficult to show that

$$\operatorname{Re}\frac{\gamma_j'(t)}{\gamma_j(t)} \sim \left|\frac{\gamma_j'(t)}{\gamma_j(t)}\right| \quad \text{as } t \to t_j,\ t < t_j.$$

In particular, there exists $s_j \in I_j$ with $s_j < t_j$ such that

$$\operatorname{Re}\frac{\gamma_j'(t)}{\gamma_j(t)} > 0 \quad \text{for } s_j \le t < t_j.$$

Since $F(\gamma_j(t))$ increases with t we have

$$\frac{d}{dt}F(\gamma_j(t)) = F'(\gamma_j(t))\gamma_j'(t) \ge 0.$$

The last two inequalities imply that

$$G'_{c,j}(t) > 0 \quad \text{for } s_j \le t < t_j. \tag{22}$$

On the other hand, for $t \in I_j$ with $t \le s_j$ we have

$$F(\gamma_j(t)) \le F(\gamma_j(t_j)) < 0.$$

This implies that if c is sufficiently large, then

$$cF(\gamma_j(t)) < -\log|\gamma_j(t)| \quad \text{for } t \le s_j. \tag{23}$$

It follows from (23) and (22) that $G_{c,j}(t) < 0$ for $t \le s_j$ and that $G_{c,j}$ is strictly increasing in $[s_j, t_j]$. Moreover, $G_{c,j}(t) > 0$ for $t > t_j$ since $F(\gamma_j(t))$ increases with t and $F(\gamma_j(t_j)) = 0$ and since γ_j does not intersect $\{z\colon |z| \le 1\}$. Thus $G_{c,j}$ has exactly one zero also in this case.

Altogether we see that $G_{c,j}$ has exactly one zero for each $j \in \{1, \ldots, d\}$. Thus the equation $cF(z) + \log|z| = 0$ has exactly d zeros, from which the conclusion follows.

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Mathematisches Seminar
Christian-Albrechts-Universität zu Kiel
Ludewig-Meyn-Str. 4
24098 Kiel
Germany

Department of Mathematics
Purdue University
West Lafayette, IN 47907
USA