# Uniform approximation of $\operatorname{sgn}(x)$ on two intervals by polynomials 

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$$
\begin{aligned}
& I=[-A,-1] \cup[1, B], \operatorname{sgn}(x)=x /|x|, x \neq 0 . \\
& L_{n}=\inf _{\operatorname{deg} P=n} \sup _{x \in I}|P(x)-\operatorname{sgn}(x)| . \\
& \text { If } A=B \text { (Eremenko-Yuditskii, 2001), then } \\
& L_{2 m+1}= \\
& L_{2 m+1} \sim \frac{\sqrt{2}(A-1)}{\sqrt{\pi A}}(2 m+1)^{-1 / 2}\left(\frac{A-1}{A+1}\right)^{m} .
\end{aligned}
$$

W. H. J. Fuchs (1978):

$$
C^{\prime} n^{-1 / 2} e^{-\eta n} \leq L_{n} \leq C^{\prime \prime} n^{-1 / 2} e^{-\eta n} .
$$

N. I. Akhiezer (1928-1922):
a) Polynomials of least deviation from 0 ,
b) Uniform approximation of $1 /(x-c)$.

In both cases he obtained:

- Extremal polynomials are expressed in terms of Abelian integrals of genus 0,1 or 2 .
- Asymptotic of the error in terms of elliptic theta-functions. It oscillates.


## Notation

$$
G(x)=G(x, \infty)=\Re \int_{-1}^{x} \frac{C-t}{\sqrt{R(t)}} d t
$$

where

$$
\begin{gathered}
R(t)=\left(1-t^{2}\right)(t+A)(B-t) . \\
C=\frac{\int_{-1}^{1} \frac{t d t}{\sqrt{R(t)}}}{\int_{-1}^{1} \frac{d t}{\sqrt{R(t)}}}, \\
\eta=G(C), \eta_{1}=-\frac{1}{2} G^{\prime \prime}(C)=\frac{1}{2 \sqrt{R(C)}},
\end{gathered}
$$

$G(x, C)$ is the Green function of $\bar{C} \backslash I$ with the pole at $C$,

$$
G(z, C)=-\ln |x-C|+\eta_{2}+O(x-C), x \rightarrow C .
$$

$\omega(x)=\omega(x,[-A,-1], \bar{C} \backslash I)$ is the harmonic measure.

All these quantities can be expressed in terms of elliptic integrals or theta-functions. They depend on $A$ and $B$.

Theorem.
$L_{n} \sim \frac{c}{\sqrt{n} e^{\eta n}}\left|\frac{\theta_{0}\left(\left.\frac{1}{2}(\{n \omega(\infty)+\omega(C)\}-\omega(C)) \right\rvert\, \tau\right)}{\theta_{0}\left(\left.\frac{1}{2}(\{n \omega(\infty)+\omega(C)\}+\omega(C)) \right\rvert\, \tau\right)}\right|$,
where

$$
\begin{gathered}
c=2\left(\pi \eta_{1}\right)^{-1 / 2} e^{-\eta_{2}}, \\
\tau=i \frac{\int_{1}^{B} \frac{d t}{\sqrt{-R(t)}}}{\int_{-1}^{1} \frac{d t}{\sqrt{-R(t)}}},
\end{gathered}
$$

\{.\} is the fractional part and
$\theta_{0}(t \mid \tau)=1-2 h \cos 2 \pi t+2 h^{4} \cos 4 \pi t-2 h^{9} \cos 6 \pi t+\ldots$
is the theta-function, and

$$
h=e^{\pi i \tau}
$$

Chebyshev theorems. For every continuous function $f$ on a compact set $E \subset \mathbf{R}$ and for every $n$, the polynomial $P_{n}$ of the best uniform approximation exists and is unique.

It is characterized by the following property: there exist at least $n+2$ points

$$
x_{1}<x_{2}<\ldots<x_{m}
$$

in $E$, such that

$$
P_{n}\left(x_{j}\right)-f\left(x_{j}\right)= \pm(-1)^{j} L_{n}, \quad 1 \leq j \leq m .
$$

where $L_{n}$ is the approximation error.

Such set $\left\{x_{1}, \ldots, x_{m}\right\}$ is called an alternance set. It is not necessarily unique.

Example. Chebyshev polynomial:
G. MacLane's theorem. For every finite updown sequence of real numbers,

$$
y_{1} \leq y_{2} \geq y_{3} \leq y_{4} \geq \ldots,
$$

or

$$
y_{1} \geq y_{2} \leq y_{3} \geq \ldots,
$$

there exists a real polynomial $f$ with all critical points

$$
x_{1} \leq x_{2} \leq \ldots
$$

real such that

$$
f\left(x_{j}\right)=y_{j}, \quad j=1,2,3, \ldots
$$

(Multiple critical points and their critical values are repeated according to their multiplicity).

Such polynomial is unique up to an increasing real affine change of the independent variable.

MacLane also proved it for multiple critical points and for a class of entire functions.

## Extremal polynomials.

Properties of extremal polynomials:
a) all critical points real and simple
b) $\pm 1$ are alternance points
c) at most one critical point is not an alternance point; this exceptional critical point is either the leftmost one or the rightmost one.

An entire function $S(z, a)$ of exponential type 1 of the best uniform approximation to $\operatorname{sgn}(x)$ on $(-\infty,-a] \cup[a, \infty)$
$L$ is uniquely defined by $a$,

$$
L(a) \sim \sqrt{\frac{2}{\pi a}} e^{-a}
$$

Construction:

$$
\gamma(t)=\arccos \frac{\operatorname{ch} b}{\operatorname{ch} t}+i t, t \geq b
$$

$$
\psi(0)=i b, \quad \psi(z) \sim z, \quad z \rightarrow \infty
$$

$$
a:=\psi^{-1}(0), \quad L(a)=\frac{1}{\operatorname{ch} a} .
$$

$S(z, a)=1-L(a) \cos \psi(z) \quad$ in the 1-st quadrant.

Verify the properties:
$S$ is an odd real entire function of exponential type one,
critical points are real and simple,
critical values $1 \pm L$ on the positive ray and $-1 \pm L$ on the negative ray,
$S(a, a)=1$.
$S^{-1}(\mathbf{R})$ consists of the real line, some curves $\delta_{k}$ in the upper half-plane and their reflection, crossing the real line at the critical points $c_{k}$, $k= \pm 1, \pm 2 a \ldots$

Make cuts along $\delta_{-m}, \delta_{k}$ and the interval $\left[-c_{m}, c_{k}\right]$, and denote the resulting region $\Omega_{m, k}$.

Let $\Theta_{m, k}$ be the conformal map of the upper half-plane onto $\Omega_{m, k}$ such that

$$
\Theta(\infty)=\infty \quad \text { and } \quad \Theta( \pm 1)= \pm a
$$

Then $S(\Theta(z), a)$ is the extremal polynomial of the first kind with $-A=\Theta^{-1}\left(-c_{m}\right)$ and $B=$ $\Theta^{-1}\left(c_{k}\right)$.

To construct a polynomial of the second kind, make an additional slit of finite height from $-c_{m-1}$ along $\delta_{m-1}$ or from $c_{k-1}$ along $\delta_{k}$.

