A Picard type theorem for holomorphic curves*

A. Eremenko†

Let $\mathbf{P}^m$ be complex projective space of dimension m, $\pi : \mathbf{C}^{m+1}\backslash\{0\} \to \mathbf{P}^m$ the standard projection and $M \subset \mathbf{P}^m$ a closed subset (with respect to the usual topology of a real manifold of dimension $2m$). A hypersurface in $\mathbf{P}^m$ is the projection of the set of zeros of a non-constant homogeneous form in $m+1$ variables. Let n be a positive integer. Consider a set of hypersurfaces $\{H_j\}_{j=1}^{2n+1}$ with the property

$$M \cap \left(\bigcap_{j \in I} H_j \right) = \emptyset \quad \text{for every} \quad I \subset \{1, \ldots, 2n+1\}, \quad |I| = n+1. \qquad (1)$$

This means that no more than n of the restrictions of our hypersurfaces to M may have non-empty intersection.

Theorem 1. *Every holomorphic map* $f : \mathbf{C} \to M\backslash \left(\bigcup_{j=1}^{2n+1} H_j \right)$ *is constant.*

Using an argument based on Brody's reparametrization lemma [7, Theorem 3.6] we obtain from Theorem 1

Corollary 1. *Let* $M \subset \mathbf{P}^m$ *be a projective variety, and suppose a collection of hypersurfaces* $\{H_j\}_{j=1}^{2n+1}$ *satisfies* (1). *Then* $M\backslash \left(\bigcup_{j=1}^{2n+1} H_j \right)$ *is complete hyperbolic and hyperbolically imbedded to* M.

Remark. Neither the dimension of M nor the dimension of the ambient projective space are important in this formulation. Only the intersection pattern (1) is relevant.

---

*AMS Classification 32C10, 32H15

†Supported by NSF grant DMS-9800084

Corollary 2. *The complement of* $2n+1$ *hypersurfaces in projective space, such that any* $n+1$ *of them have empty intersection, is complete hyperbolic and hyperbolically imbedded.*

The assumptions of Corollary 1 can be satisfied only if $n \geq m$. For $m = n$ and assuming in addition normal intersections of hypersurfaces Corollary 2 was proved in [1]. In the special case when the hypersurfaces are hyperplanes, Corollary 2 can be deduced from a result of Zaidenberg [11] (see also [6, (3.10.16)]).

The method of the proof used here first appeared in [3]. It also provides a new proof of the classical Picard theorem [8, 9] as well as its generalizations to quasiregular maps in $\mathbf{R}^n$ [2, 8, 4]. One of the purposes of this paper is to explain the idea in its simplest form, not obscured by technical details as in [3].

Proof of Theorem 1. Let $P_1, \ldots, P_{2n+1}$ be the forms in $m+1$ variables defining the hyperplanes, $d_j = \deg P_j$. We consider a homogeneous representation $F = (f_0 : \ldots : f_m)$ of the curve f, where f_j are entire functions without common zeros. The function

$$u = \log \|F\| = \frac{1}{2} \log(|f_0|^2 + \ldots + |f_m|^2)$$

is subharmonic, and the functions

$$u_j = d_j^{-1} \log |P_j \circ F| = d_j^{-1} \log |P_j(f_0, \ldots, f_m)|, \quad j = 1, \ldots 2n+1,$$

are harmonic in $\mathbf{C}$.

Let $I \subset \{1, \ldots, 2n+1\}$, $|I| = n+1$. The set $K = \{z \in \pi^{-1}M : \|z\| = 1\}$ is compact, so for some positive constants C_1 and C_2 we have

$$C_1 \leq \max_{j \in I} |P_j(z)|^{1/d_j} \leq C_2 \quad \text{for} \quad z \in K.$$

This follows from the assumption (1). Using homogeneity we conclude that

$$C_2 \|F(z)\| \leq \max_{j \in I} |P_j \circ F(z)|^{1/d_j} \leq C_2 \|F(z)\|, \quad z \in \mathbf{C},$$

so

$$\max_{j \in I} u_j = u + O(1), \quad |I| = n+1. \tag{2}$$

In particular

$$\max_{1\le j\le 2n+1} u_j = u + O(1). \tag{3}$$

We use the notation $D(a,r) = \{z \in \mathbf{C} : |z-a| < r\}$. Let us denote by $\mu = (2\pi)^{-1}\Delta\, u$ the Riesz measure of u.[1]

Suppose that f is not constant. Then at least one of the harmonic functions u_j is not constant, assume without loss of generality that $u_1 \neq$ const. Then

$$B(r,u_1) := \max_{|z|=r} u_1(z) \ge cr \quad \text{for} \quad r > r_0,$$

where $c > 0$. Now (3) implies $B(r,u) \ge c_1 r$ for $r > r_0$ with some $c_1 > 0$ and using Jensen's formula we have

$$c_1 r \le B(r,u) \le \frac{3}{2\pi}\int_{-\pi}^{\pi} u(2re^{i\theta})\,d\theta \le \int_0^{2r} \mu(D(0,t))\frac{dt}{t} + u(0).$$

In particular, $\mu(\mathbf{C}) = \infty$.

Lemma. *Let μ be a Borel measure in $\mathbf{C}$, $\mu(\mathbf{C}) = \infty$. Then there exist sequences $a_k \in \mathbf{C}$, $a_k \to \infty$ and $r_k > 0$ such that*

$$M_k = \mu(D(a_k,r_k)) \to \infty \tag{4}$$

and

$$\mu(D(a_k,2r_k)) \le 200\mu(D(a_k,r_k)). \tag{5}$$

This Lemma is due to S. Rickman [10]. A proof is included in the end of this paper for completeness.

Applying the Lemma to the Riesz measure μ of the function u, we obtain two sequences a_k and r_k, such that (4) and (5) are satisfied. Consider the functions defined in $D(0,2)$:

$$u_k(z) = \frac{1}{M_k}(u(a_k + r_k z) - \tilde{u}(a_k + r_k z))$$

and

$$u_{j,k}(z) = \frac{1}{M_k}(u_j(a_k + r_k z) - \tilde{u}(a_k + r_k z)), \quad 1 \le j \le 2n+1,$$

[1] It coincides with the pull back of the Fubini–Study $(1,1)$ form. Notice the formula $T(r,f) = \int_0^r \mu(D(0,t))dt/t$.

where $\tilde{u}$ is the smallest harmonic majorant of u in the disc $D(a_k, 2r_k)$. The functions u_k are Green's potentials, that is

$$u_k(z) = -\int_{D(0,2)} G(z,.)\, d\mu_k,$$

where $G(z,.)$ is the Green function of $D(0,2)$ with pole at the point z and μ_k is the Riesz measure of u_k.

It follows from (5) that $\mu_k(D(0,2)) \leq 200$ so, after selecting a subsequence, we may assume that $u_k \to v$, where v is a subharmonic function, not identically equal to $-\infty$. Convergence holds in $L^1_{\text{loc}}(D(0,2), dxdy)$, and the Riesz measures converge weakly, see [5, Theorem 4.1.9]. In particular v is *not harmonic* because the Riesz measure of $\overline{D}(0,1)$ is at least 1.

All functions $u_{j,k}$ are harmonic and bounded from above in view of (3) so, after selecting a subsequence, we may assume that $u_{j,k} \to v_j$, each v_j being harmonic or identically equal to $-\infty$ in $D(0,2)$. From (2) and (4) follows

$$\max_{j \in I} v_j = v, \quad |I| = n+1. \tag{6}$$

Thus v is continuous. For every $I \subset \{1, \ldots, 2n+1\}$ of cardinality $n+1$ we consider the set $E_I = \{z \in D(0,2) : v(z) = v_j(z),\ j \in I\}$. From (6) it follows that the union of these sets coincides with $D(0,2)$. Thus at least one set E_{I_0} has positive area. By the uniqueness theorem for harmonic functions all functions v_j for $j \in I_0$ are equal. Applying (6) to I_0 we conclude that v is harmonic. This is a contradiction which proves the theorem.

Proof of the Lemma. We take a number $R > 0$, so that $\mu(D(0, R/4)) > 0$ and denote $\delta(z) = (R - |z|)/4$. Then we find $a \in D(0,R)$ such that

$$\mu(D(a, \delta(a))) > \frac{1}{2} \sup_{z \in D(0,R)} \mu(D(z, \delta(z))). \tag{7}$$

We set $r = \delta(a)$. Then the disc $D(a, 2r)$ can be covered by at most 100 discs of the form $D(z, \delta(z))$, so by (7)

$$\mu(D(a, 2r)) \leq 200 \mu(D(a, r)).$$

Putting $z = 0$ in (7) we get

$$\mu(D(a, r)) \geq \frac{1}{2} \mu(D(0, R/4)).$$

Now we take any sequence $R_k \to \infty$ and construct the discs $D(r_k, a_k)$ as above.

The author thanks László Lempert, Min Ru and Yum-Tong Siu for helpful comments.

References

[1] V. F. Babets, Theorems of Picard type for holomorphic mappings, Siberian Math. J., 25 (1984), 195-200.

[2] A. Eremenko, J. Lewis, Uniform limits of certain A-harmonic functions with applications to quasiregular mappings, Ann. Acad. Sci. Fenn., Ser, A. I. Math., Vol. 16, 1991, 361-375.

[3] A. Eremenko and M. Sodin, The value distribution of meromorphic functions and meromorphic curves from the point of view of potential theory, St. Petersburg Math. J., 3 (1992), 109-136.

[4] I. Holopainen, S. Rickman, A Picard type theorem for quasiregular mappings of $\mathbf{R}^n$ into n-manifolds with many ends, Revista Mat. Iberoamericana, 8 (1992), 131-148.

[5] L. Hörmander, The Analysis of Linear partial Differential Operators I, Springer, NY, 1983.

[6] S. Kobayashi, Hyperbolic Complex Spaces, Springer, 1998.

[7] S. Lang, Introduction to Complex Hyperbolic Spaces, Springer, 1987.

[8] J. Lewis, Picard's theorem and Rickman's theorem by way of Harnack inequality, Proc. Amer. math. Soc., 122 (1994), 199-206.

[9] Th. Ransford, Potential theory in the complex plane, Cambridge Univ. press, Cambridge, 1995.

[10] S. Rickman, On the number of omitted values of entire quasiregular mappings, J. d'Analyse Math., 37, 1980, 100-117.

[11] M. Zaidenberg, Picard's theorem and hyperbolicity, Siberian math. J., 24 (1983), 858-867.

Purdue University, West Lafayette, IN 47907 USA
eremenko@math.purdue.edu