POLE PLACEMENT BY STATIC OUTPUT FEEDBACK FOR GENERIC LINEAR SYSTEMS

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Abstract. We consider linear systems with m inputs, p outputs and McMillan degree n, such that $n = mp$. If both m and p are even, we show that there is a non-empty open (in the usual topology) subset U of such systems, where the real pole placement map is not surjective. It follows that for each system in U, there exists an open set of pole configurations, symmetric with respect to the real line, which cannot be assigned by any real static output feedback.

Key words. linear systems, static output control feedback, pole placement

AMS subject classifications. 14N10, 14P99, 14M15, 30C99, 26C15

1. Introduction. We consider *linear systems* $S = (A, B, C)$ described by the equations

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx \end{array} . \tag{1.1}$$

Here the state x, the input u and the output y are functions of a real variable t (time), with values in $\mathbf{R}^n$, $\mathbf{R}^m$ and $\mathbf{R}^p$, respectively, the dot denotes the derivative with respect to t, and A, B, C are real matrices of sizes $n \times n$, $n \times m$ and $p \times n$, respectively.

Assuming zero initial conditions and applying the Laplace transform, we obtain

$$Y(s) = C(sI - A)^{-1} B\, U(s),$$

so the behavior of our linear system is described by the rational matrix-function $G(s) = C(sI - A)^{-1}B$ of size $p \times m$ of a complex variable s, which is called the (open loop) *transfer function* of S. It is clear that $G(\infty) = 0$. The poles of the transfer function are the eigenvalues of the matrix A.

For a given $p \times m$ matrix function G with the property $G(\infty) = 0$ there exist infinitely many representations of G in the form $G(s) = C(sI - A)^{-1}B$. The smallest integer n over all such representations is called the *McMillan degree* of G.

We consider the possibility to control a given system S by attaching a feedback. This means that the output is sent to the input after a preliminary linear transformation, called a *compensator*. The compensator may be another system of the form (1.1) (dynamic output feedback) or just a constant matrix (static output feedback). In this paper we consider only static output feedback, referring for the recent results on dynamic output feedback to [14, 11].

A static output feedback is described by the equation

$$u = Ky, \tag{1.2}$$

where K is an $m \times p$ matrix which is usually called a *gain* matrix. Eliminating u and y gives

$$\dot{x} = (A + BKC)x,$$

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whose characteristic polynomial is

$$\varphi_K(s) = \det(sI - A - BKC). \tag{1.3}$$

It is called the *closed loop characteristic polynomial.*

The pole placement problem is formulated as follows:

Given a system $S = (A, B, C)$, and a set of points $\{s_1, \ldots, s_n\}$ in $\mathbf{C}$ (listed with multiplicities) symmetric with respect to the real axis, find a real matrix K such that the zeros of φ_K are exactly $s_1, \ldots, s_n$.

For a fixed system S, we define the (real) *pole placement map*

$$\chi_S : \mathrm{Mat}_{\mathbf{R}}(m \times p) \to \mathrm{Poly}_{\mathbf{R}}(n), \quad \chi_S(K) = \varphi_K, \tag{1.4}$$

where $\mathrm{Mat}_{\mathbf{R}}(m \times p)$ is the set of all real matrices of size $m \times p$, $\mathrm{Poly}_{\mathbf{R}}(n)$ the set of all real monic polynomials of degree n, and the polynomial φ_K is defined in (1.3). Thus to say that for a system S, an arbitrary symmetric set of poles can be assigned by a real gain matrix, is the same as to say that the real pole placement map χ_S is surjective. Extending the domain to complex matrices K and the range to complex monic polynomials gives the *complex pole placement map*

$$\mathrm{Mat}_{\mathbf{C}}(m \times p) \to \mathrm{Poly}_{\mathbf{C}}(n),$$

defined by the same formula as the real one.

It is easy to see that for every m, n, p there are systems for which the pole placement map is not surjective. For example, one can take $B = 0$ or $C = 0$. A necessary condition of surjectivity proved in [13] is that S is observable and controllable. This is equivalent to saying that the McMillan degree of the transfer function is equal to n, the dimension of the state space. Notice that this property is *generic*: it holds for an open dense subset of the set

$$\mathfrak{A} = \mathrm{Mat}_{\mathbf{R}}(n \times n) \times \mathrm{Mat}_{\mathbf{R}}(n \times m) \times \mathrm{Mat}_{\mathbf{R}}(p \times n)$$

of all triples (A, B, C). All topological terms in this paper refer to the usual topology.

In this paper we consider the following problem: *for a given triple of integers (m, n, p), does there exist an open dense subset $V \subset \mathfrak{A}$, such that the real pole placement map χ_S is surjective for $S \in V$?* If this is the case, we say that the real pole placement map is *generically surjective* for these m, n and p.

We briefly recall the history of the problem, referring to a comprehensive survey [2]. The pole placement map defined by (1.3) and (1.4) is a regular map of affine algebraic varieties. Comparing the dimensions of its domain and range, we conclude that $n \leq mp$ is a necessary condition for generic surjectivity of the pole placement map, real or complex. In the complex case, this condition turns is to be also sufficient [7]. To show this, one extends the pole placement map to a regular map between compact algebraic manifolds and verifies that its Jacobi matrix is of full rank. In the case when $n = mp$ we have the following precise result:

Theorem A [1] *For $n = mp$, the complex pole placement map is generically surjective. Moreover, it extends to a finite regular map between projective varieties and has degree*

$$d(m, p) = \frac{1!2! \ldots (p-1)! \, (mp)!}{m!(m+1)! \ldots (m+p-1)!}.$$

It follows that for a generic system (A, B, C) with $n = mp$ and a generic monic complex polynomial φ of degree mp, there are $d(m, p)$ complex matrices K such that $\varphi_K = \varphi$.

The numbers $d(m, p)$ occur as the solution of the following problem of enumerative geometry: how many m-subspaces intersect mp given p-subspaces in $\mathbf{C}^{m+p}$ in general position? The answer $d(m, p)$ was obtained by Schubert in 1886 (see, for example, [9]).

The real pole placement map is harder to study. For a survey of early results we refer to [2, 12]. X. Wang [16] proved that $n < mp$ is sufficient for generic surjectivity of real (or complex) pole placement map. A simplified proof of this result can be found in [17, 12].

From now on we only discuss the so-called critical case, that is we assume

$$n = mp$$

in the rest of the paper. In addition, we may assume without loss of generality that $p \leq m$, in view of the symmetry of our problem with respect to the interchange of m and p (see, for example, [15, Theorem 3.3]).

One corollary from Theorem A is that the real pole placement map is generically surjective if $d(m, p)$ is odd. This number is odd if and only if one of the following conditions is satisfied [2]: a) $\min\{m, p\} = 1$, or b) $\min\{m, p\} = 2$, and $\max\{m, p\} + 1$ is an integral power of 2.

In the opposite direction, Willems and Hesselink [18] found by explicit computation that the real pole placement map is not generically surjective for $(m, p) = (2, 2)$. A closely related fact, that the problem of enumerative geometry mentioned above, may have no real solutions for the case $(m, p) = (2, 2)$ even when the given 2-subspaces are real, is mentioned in [8].

In [13] Rosenthal and Sottile found with a rigorous computer-assisted proof that the real pole placement map is not generically surjective in the case $(m, p) = (4, 2)$, thus disproving a conjecture of Kim, that $(2, 2)$ is the only exceptional case.

In [6] we showed that the real pole placement map is not generically surjective when $p = 2$ and m is even, thus extending the negative results for the cases $(2, 2)$ and $(4, 2)$ stated above.

In the present paper we extend this result to all cases when both m and p are even.

Theorem 1 *If $n = mp$, and m and p are both even, then the real pole placement map is not generically surjective.*

Our proof of Theorem 1 gives explicitly a system $S_0 \in \mathfrak{A}$, and a polynomial $u(s) = s(s^2 + 1)^{mp/2-1}$, such that for any S' in a neighborhood of S_0, the real pole placement map $\chi_{S'}$ omits a neighborhood of u.

Our proofs in [6] depend on a hard analytic result from [5], related to the so-called B. and M. Shapiro conjecture which is stated below in Section 2. The proofs in the present paper are new, even in the case $\min\{m, p\} = 2$, and they are elementary.

We conclude the Introduction with an unsolved problem.

A system S is called stabilizable (by real static output feedback), if there exists a gain matrix $K \in \mathrm{Mat}_{\mathbf{R}}(m \times p)$ such that all zeros of the closed loop characteristic polynomial φ_K belong to the left half-plane. From the positive results on pole placement stated above, it follows that generic systems with m inputs, p outputs and state of dimension n are stabilizable if $n < mp$, or if $n = mp$ and $d(m, p)$ is odd. We ask whether generic systems with $n = mp$ and even m and p are stabilizable. The answer is known to be negative in the case $(m, p) = (2, 2)$ [3]. For complex output feedback, with static or dynamic compensators, the problem of generic stabilizability was solved in [10].

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2. A class of linear systems. We begin with a well-known transformation of the closed loop characteristic polynomial (1.3). The open loop transfer function of a system of McMillan degree n, equal to the dimension of the state space, can be factorized as

$$C(sI - A)^{-1}B = D(s)^{-1}N(s), \quad \det D(s) = \det(sI - A), \tag{2.1}$$

where D and N are polynomial matrix-functions of sizes $p \times p$ and $p \times m$, respectively. For the possibility of such factorization for systems (1.1) of McMillan degree n we refer to [4, Assertion 22.6]. Using (2.1), and the identity $\det(I - PQ) = \det(I - QP)$, which is true for all rectangular matrices of appropriate dimensions, we write

$$\begin{aligned}\varphi_K(s) &= \det(sI - A - BKC) = \det(sI - A)\det(I - (sI - A)^{-1}BKC)\\ &= \det(sI - A)\det(I - C(sI - A)^{-1}BK)\\ &= \det D(s)\det(I - D(s)^{-1}N(s)K) = \det(D(s) - N(s)K).\end{aligned}$$

This can be rewritten as

$$\varphi_K(s) = \det\left([D(s), N(s)]\begin{bmatrix} I \\ -K \end{bmatrix}\right). \tag{2.2}$$

Now we extend $\chi_S : K \mapsto \varphi_K$ to a map between compact manifolds. For this purpose, we allow an arbitrary $(m + p) \times p$ complex matrix L of rank p in (2.2) instead of

$$\begin{bmatrix} I \\ -K \end{bmatrix}, \tag{2.3}$$

and define

$$\varphi_L(s) = \det\left([D(s), N(s)]\ L\right), \tag{2.4}$$

A system S represented by $[D(s), N(s)]$ is called *non-degenerate* if $\varphi_L \neq 0$ for every $(m + p) \times p$ matrix L of rank p. Such matrices are called equivalent, $L_1 \sim L_2$ if $L_1 = L_2U$ where $U \in GL_p(\mathbf{C})$. The set of equivalence classes is the Grassmannian $G_{\mathbf{C}}(p, m+p)$ which is a compact algebraic manifold of dimension mp. If $L_1 \sim L_2$, we have $\varphi_{L_1} = c\varphi_{L_2}$, where $c \neq 0$ is a constant. The space of all non-zero polynomials of degree at most mp, modulo proportionality, is identified with the projective space $\mathbf{CP}^{mp}$, coefficients of the polynomials serving as homogeneous coordinates. Monic polynomials represent the points of an open dense subset of $\mathbf{CP}^{mp}$, a so-called "big cell", which consists of polynomials of degree mp. This construction extends the complex pole placement map of a non-degenerate system to a regular map of compact algebraic manifolds

$$\chi_S : G_{\mathbf{C}}(p, m + p) \to \mathbf{CP}^{mp}, \tag{2.5}$$

where $\chi_S(L)$ is the proportionality class of the polynomial φ_L in (2.4), and L is a matrix of rank p representing a point in $G_{\mathbf{C}}(p, m + p)$. The set $\mathfrak{B}$ all non-degenerate systems is open and dense in the set $\mathfrak{A}$ of all systems, and the map

$$X \times G_{\mathbf{C}}(p, m + p) \to \mathbf{CP}^{mp}, \quad (S, L) \mapsto \chi_S(L) \tag{2.6}$$

is continuous. Notice that the subset of $G_{\mathbf{R}}(p, m+p)$ consisting of points which can be represented by matrices L of the form (2.3) is open and dense. It corresponds via χ_S to the big cell in $\mathbf{CP}^{mp}$ consisting of polynomials of degree mp.

We consider a system $S_0 = (A_0, B_0, C_0)$ represented by the following polynomial matrix $[D(s), N(s)]$

$$(2.7) \quad = \begin{bmatrix} 1 & s & \dots & s^{m+p-2} & s^{m+p-1} \\ 0 & 1 & \dots & (m+p-2)s^{m+p-3} & (m+p-1)s^{m+p-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & (m+1)\dots(m+p-1)s^m \end{bmatrix}.$$

The first row of $[D(s), N(s)]$ consists of monic monomials, and the k-th row is the $(k-1)$-st derivative of the first, for $2 \le k \le p$. This system S_0 has McMillan degree mp, and the matrices A_0, B_0, C_0 can be recovered from $[D, N]$, by [4, Theorem 22.18]. Let $L = (a_{i,j})$. Introducing polynomials

$$(2.8) \qquad f_j(s) = a_{1,j} + a_{2,j}s + \dots + a_{m+p-1,j}s^{m+p-2} + a_{m+p,j}s^{m+p-1},$$

for $1 \le j \le p$, we can write (2.4) as

$$\varphi_L = W(f_1, \dots, f_p) = \begin{vmatrix} f_1 & \dots & f_p \\ f_1' & \dots & f_p' \\ \dots & \dots & \dots \\ f_1^{(p-1)} & \dots & f_p^{(p-1)} \end{vmatrix}.$$

Thus for our system (A_0, B_0, C_0), the pole placement map becomes the *Wronski map*, which sends a p-vector of polynomials into their Wronski determinant. We say that two p-vectors of polynomials are equivalent, $(f_1, \dots, f_p) \sim (g_1, \dots, g_p)$, if $(g_1, \dots, g_p) = (f_1, \dots, f_p)U$, where $U \in GL_p(\mathbf{C})$. Equivalent p-vectors have proportional Wronski determinants. Equivalence classes of p-vectors of linearly independent polynomials of degree at most $m+p-1$ parametrize the Grassmannian $G_{\mathbf{C}}(p, m+p)$. A p-vector of complex polynomials will be called *real* if it is equivalent to a p-vector of real polynomials. The system represented by (2.7) is non-degenerate. This is a consequence of the well-known fact that the Wronski determinant of p polynomials is zero if and only if the polynomials are linearly dependent.

To prove Theorem 1, we use the following general result (compare [13, Thm. 3.1]):

Proposition 1. *If, for some (m, n, p), there exists a real non-degenerate system $S_0 = (A_0, B_0, C_0)$ such that the real pole placement map χ_{S_0} in* (2.5) *is not surjective, then for these (m, n, p) the real pole placement map is not generically surjective.*

Indeed, if χ_{S_0} omits one point u, it omits a neighborhood of u, because the image of a compact space under a continuous map is compact. Using continuity of the map (2.6) we conclude that for all S in a neighborhood of S_0 the maps χ_S omit a neighborhood of u. □

In view of Proposition 1, to prove Theorem 1, it is enough to find a non-zero real polynomial of degree at most mp which cannot be represented as the Wronski determinant of p real polynomials of degree at most $m+p-1$. Thus Theorem 1 follows from Proposition 1 and

Proposition 2. *If $m \geq p \geq 2$ are even integers, then the polynomial $u(s) = s(s^2 + 1)^{mp/2-1}$ is not proportional to the Wronski determinant of any p real polynomials of degree at most $m + p - 1$.*

Proposition 2 is motivated by a conjecture of B. and M. Shapiro (see, for example [15]), which says: *If the Wronskian determinant of a polynomial p-vector has only real roots, then this p-vector is real.* In [5] we proved this conjecture for $p = 2$, and used this result in [6] to derive the case $p = 2$ of Theorem 1. In the present paper, we prove a result, Proposition 3 in Section 3, which is a very special case of the B. and M. Shapiro conjecture, but still it permits to derive Proposition 2.

3. The Wronski map. A p-vector of linearly independent polynomials of degree at most $m + p - 1$ can be represented by a $(m + p) \times p$ matrix L of rank p, whose columns are composed of the coefficients of the polynomials as in (2.8).

The group $GL_p(\mathbf{C})$ acts on such matrices by multiplication from the right. This action is equivalent to the usual column operations on matrices: interchange of two columns, multiplication of a column by a non-zero constant, and adding to a column a multiple of another column. For each column j of L, we introduce two integers $1 \leq e_j \leq d_j \leq m + p$, which are the positions of the first and last non-zero elements of this column, counted from above. Thus $\deg f_j = d_j - 1$, and the order of a root of f_j at zero is $e_j - 1$. It is easy to see that by column operations, every $(m + p) \times p$ matrix $L = (a_{i,j})$ of rank p can be reduced to the following

canonical form:

(i) $d_1 > d_2 > \ldots > d_p$,
(ii) $a_{e_j,j} = 1$, for every $j \in [1, p]$,
(iii) $a_{e_k,j} = 0$ for $1 \leq j < k \leq p$.

The elements $a_{e_j,j} = 1,\ 1 \leq j \leq p$ of the canonical form will be called the *pivot* elements. It follows from (iii) that all numbers e_j are distinct.

Proposition 3. *Suppose that mp is even. Then every polynomial p-vector $(f_1, \ldots, f_p)$ of degree at most $m + p - 1$ in canonical form, which satisfies*

$$(3.1) \qquad W(f_1 \ldots, f_p) = \lambda w, \quad \textit{where} \quad w(s) = s^{mp/2+1} - s^{mp/2-1}, \quad \lambda \in \mathbf{C}^*$$

has only real entries.

Corollary. *All polynomial p-vectors of degree at most $m + p - 1$ satisfying (3.1) are real.*

This Corollary confirms a special case of the B. and M. Shapiro Conjecture, when the Wronskian determinant of a polynomial p-vector is $w(s) = s^{mp/2+1} - s^{mp/2-1}$, which is a polynomial with real roots $0, \pm 1$.

The properties of the Wronskian determinants used here are well-known and easy to prove:

Lemma. *The Wronski map $(f_1, \ldots, f_p) \mapsto W(f_1, \ldots, f_p)$ is linear with respect to each f_j, and*

$$W(s^{n_1}, \ldots, s^{n_p}) = V(n_1, \ldots, n_p) s^{n_1 + \ldots + n_p - p(p-1)/2},$$

where

$$V(n_1,\ldots,n_p)=\prod_{k<j}(n_j-n_k)$$

is the Vandermonde determinant. □

Using this Lemma, we compute the Wronskian determinant of a polynomial p-vector in canonical form, and conclude that

$$\deg W(f_1,\ldots,f_p)=d_1+\ldots+d_p-p(p+1)/2, \tag{3.2}$$

and

$$\operatorname{ord} W(f_1,\ldots,f_p)=e_1+\ldots+e_p-p(p+1)/2, \tag{3.3}$$

where ord denotes the multiplicity of a root at zero.

Proof of Proposition 3. According to (3.1), $\deg w=mp/2+1$, and $\operatorname{ord} w= mp/2-1$. So (3.2) and (3.3) imply

$$d_1+\ldots+d_p=p(p+1)/2+mp/2+1,$$

$$e_1+\ldots+e_p=p(p+1)/2+mp/2-1.$$

Subtracting the second equation from the first, we get

$$\sum_{j=1}^{p}(d_j-e_j)=2.$$

As all the summands are non-negative, there are two possibilities:

Case 1. In all columns but one, all elements, except the pivot elements, are equal to zero, and for the exceptional column j, $d_j-e_j=2$. Computing the Wronskian and comparing with (3.1), we obtain

$$\begin{aligned} & V(\ldots,e_j-1,\ldots)s^{mp/2-1}\\ + \;& V(\ldots,e_j,\ldots)a_{e_j+1,j}s^{mp/2}\\ + \;& V(\ldots,e_j+1,\ldots)a_{e_j+2,j}s^{mp/2+1}\\ = \;& -\lambda s^{mp/2-1}+\lambda s^{mp/2+1}. \end{aligned}$$

Here and in what follows, the notation $V(\ldots,e_j+m,\ldots)$ means the Vandermonde determinant of p arguments, whose k-th argument is e_k-1 for $k\neq j$, and the j-th argument is e_j+m.

Comparing the terms with $s^{mp/2-1}$ we conclude that λ is real. Comparing the terms with $s^{mp/2+1}$ we conclude that $V(\ldots,e_j+1,\ldots)\neq 0$, and thus $a_{e_j+2,j}$ is real. Now we consider the middle term in the expansion of the Wronskian determinant. If $V(\ldots,e_j,\ldots)=0$ then $e_k=e_j+1$ for some k. As $d_k=e_k$, and $d_j=e_j+2$, we conclude that $d_k=d_j-1$, so $k>j$ by (i) in the definition of the canonical form. Now (iii) from the definition of the canonical form implies that $a_{e_j+1,j}=0$. If $V(\ldots,e_j,\ldots)\neq 0$, we also conclude that $a_{e_j+1,j}=0$. Thus all entries of L are real.

Case 2. In all columns but two, all non-pivot elements are equal to zero, and the two exceptional columns contain one extra non-zero element each. Let $j < k$ be the positions of the exceptional columns, and $a = a_{e_j+1,j}$ and $b = a_{e_k+1,k}$ the non-zero, non-pivot elements of these columns. Computing the Wronskian and comparing with (3.1), we obtain

$$\begin{aligned} & V(\ldots, e_j - 1, \ldots) s^{mp/2-1} \\ + \quad & (aV(\ldots, e_j, \ldots) + bV(\ldots, e_k, \ldots))\, s^{mp/2} \\ + \quad & abV(\ldots, e_j, \ldots, e_k, \ldots) s^{mp/2+1} \\ = \quad & -\lambda s^{mp/2-1} + \lambda s^{mp/2+1}, \end{aligned} \tag{3.4}$$

where $V(\ldots, e_j, \ldots, e_k \ldots)$ denotes the Vandermonde determinant of p arguments, whose j-th argument is e_j, k-th argument is e_k and for all other indices $l \notin \{j, k\}$, the l-th argument is $e_l - 1$.

Our first conclusions are

$$V(\ldots, e_j - 1, \ldots) = -\lambda. \tag{3.5}$$

and

$$V(\ldots, e_j, \ldots, e_k, \ldots) \neq 0. \tag{3.6}$$

It follows from (3.5) that λ is real. If exactly one of the numbers $V(\ldots, e_j, \ldots)$ and $V(\ldots, e_k, \ldots)$ is zero, then (3.4) implies that at least one of the numbers a or b is zero. Then the third term in the expansion of the Wronskian is zero, which contradicts (3.4). If both $V(\ldots, e_j, \ldots)$ and $V(\ldots, e_k, \ldots)$ are zero, then $V(\ldots, e_j, \ldots, e_k, \ldots) = 0$, and this contradicts (3.6). So both $V(\ldots, e_j, \ldots)$ and $V(\ldots, e_k, \ldots)$ are non-zero. This means that there are no pivot elements in the rows $e_j + 1$ and $e_k + 1$. Using (3.6) we conclude that $V(\ldots, e_j - 1, \ldots)$, $V(\ldots, e_j, \ldots)$, $V(\ldots, e_k, \ldots)$ and $V(\ldots, e_j, \ldots, e_k, \ldots)$ have the same sign, and by (3.5), all these numbers have the sign of $-\lambda$. As $V(\ldots, e_j, \ldots)$ and $V(\ldots, e_k, \ldots)$ are of the same sign, (3.4) implies that $a = -cb$, where $c > 0$, and from the equations

$$V(\ldots, e_j, \ldots, e_k, \ldots)ab = \lambda$$

and (3.5) we conclude that a and b are real. □

The group $\mathrm{Aut}(\mathbf{CP}^1)$ of fractional-linear transformations acts on the space $\mathbf{CP}^k$ of proportionality classes of non-zero polynomials of degree at most k by the following rule. Let

$$\ell(s) = \frac{as + b}{cs + d}, \quad ad - bc \neq 0$$

represent a fractional-linear transformation. For a polynomial $r(s)$, we put

$$\ell r(s) = (-cs + a)^k r \circ \ell^{-1}(s).$$

That this is indeed a group action, can be verified as follows. The space of proportionality classes of non-zero polynomials of degree at most k can be canonically identified with the symmetric power $\mathrm{Sym}^k(\mathbf{CP}^1)$, which is the set of unordered k-tuples of points in $\mathbf{CP}^1$. To each polynomial r one puts into correspondence its roots, counted with multiplicity, and the point ∞ with multiplicity $k - \deg r$. Then the action of $\ell \in \mathrm{Aut}(\mathbf{CP}^1)$ on such k-tuple is simply

$$(s_1, \ldots, s_k) \mapsto (\ell(s_1), \ldots, \ell(s_k)).$$

It is easy to verify that this action of $\mathrm{Aut}(\mathbf{CP}^1)$ extends to the space $G_{\mathbf{C}}(p, m+p)$ of equivalence classes of polynomial p-vectors of degree at most $m+p-1$. Furthermore, this extended action is respected by the Wronski map:

$$W(\ell g_1, \ldots, \ell g_p) = \ell W(g_1, \ldots, g_p). \tag{3.7}$$

Of course, in the left hand side of this equality, the group $\mathrm{Aut}(\mathbf{CP}^1)$ acts on $\mathrm{Sym}^{m+p-1}(\mathbf{CP}^1)$, while in the right hand side it acts on $\mathrm{Sym}^{mp}(\mathbf{CP}^1)$. Equation (3.7) permits to simplify the polynomial equation

$$W(g_1, \ldots, g_p) = v, \quad v(s) \sim s(s^2 - 1)^{mp/2-1} \tag{3.8}$$

which will be used to prove Proposition 2.

Consider the fractional-linear transformation

$$\ell(s) = \ell^{-1}(s) = \frac{1-s}{1+s}. \tag{3.9}$$

We have $\ell : (0, 1, \infty, -1) \mapsto (1, 0, -1, \infty)$, and $\ell(\overline{\mathbf{R}}) = \overline{\mathbf{R}}$.

Using (3.8) and (3.9) we obtain

$$\ell v(s) = (s+1)^{mp} v \circ \ell^{-1}(s) \sim s^{mp/2+1} - s^{mp/2-1} = w(s),$$

where "$\sim$" means "proportional". Thus, with $f_j = \ell g_j$, the equation (3.8) is equivalent to the equation

$$W(f_1, \ldots, f_p) = w, \quad w(s) \sim s^{mp/2+1} - s^{mp/2-1}, \tag{3.10}$$

which we solved in Proposition 3. The conclusion is that

(3.11) *all solutions of (3.8) in canonical form have real coefficients.*

Proof of Proposition 2. Suppose that $(f_1, \ldots, f_p)$ is a real polynomial p-vector in canonical form satisfying

$$W(f_1, \ldots, f_p) = u, \quad u(s) = \lambda s(s^2 + 1)^{mp/2-1}, \quad \lambda \neq 0. \tag{3.12}$$

Then (3.3) implies

$$e_1 + \ldots + e_p = 1 + p(p+1)/2,$$

As $(e_j)_{j=1}^p$ are distinct positive integers, the only possibility is

$$\{e_1, \ldots, e_p\} = \{1, 2 \ldots, p-1, p+1\}. \tag{3.13}$$

Similarly, (3.2) implies

$$d_1 + \ldots + d_p = mp + p(p+1)/2 - 1.$$

As $(d_j)_{j=1}^p$ are distinct integers in the interval $[1, m+p]$, the only possibility is that

$$\{d_1, \ldots, d_p\} = \{m, m+2, m+3, \ldots, m+p\}. \tag{3.14}$$

Notice that the sequence (3.13) contains $p/2+1$ odd numbers and $p/2-1$ even numbers. On the other hand, the sequence (3.14) contains $p/2-1$ odd numbers and $p/2+1$ even numbers. This implies that at least for one j

$$d_j - e_j \quad \text{is odd.} \tag{3.15}$$

This means that the polynomial f_j contains both even and odd powers of s with non-zero coefficients. So the polynomial $g_j(s) = f_j(is)$, $i = \sqrt{-1}$, is not proportional to any polynomial with real coefficients. On the other hand, the polynomial p-tuple $(g_1, \ldots, g_p)$, where $g_j(s) = \epsilon_j f_j(is)$ with appropriate $\epsilon_j \in \{\pm 1, \pm i\}$ is a solution of (3.8) in canonical form, and we know from (3.11) that all such solutions have real coefficients. This contradiction completes the proof of Proposition 2. □

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