Алгебра и анализ
Том 3 (1991), вып. 1

St. Petersburg Math. J.
Vol. 3 (1992), No. 1

# THE VALUE DISTRIBUTION OF MEROMORPHIC FUNCTIONS AND MEROMORPHIC CURVES FROM THE POINT OF VIEW OF POTENTIAL THEORY

A. E. EREMENKO AND M. L. SODIN

*Dedicated to Anatoliĭ Asirovich Gol'dberg on his sixtieth birthday*

ABSTRACT. This paper presents a new method for proving R. Nevanlinna's second fundamental theorem. This method reduces the problem to a proposition in potential theory. It also lets us establish some generalizations and analogs of the second fundamental theorem. In particular, we prove analogs for mappings $f\colon \mathbb{C} \to \mathbb{P}^n$ and nonlinear divisors without any nondegeneracy conditions (B. Shiffman's conjecture).

## §1. INTRODUCTION

Let $f\colon \mathbb{C} \to \overline{\mathbb{C}} = \mathbb{P}^1$ be a meromorphic function. We represent it in the form $f_0/f_1$, where $f_k$ are entire functions with no zero points in common. The study of the distribution of the $a$-points of $f$ is equivalent to the study of the zeros of linear combinations of the form $f_0 - af_1$. In order to include the case $a = \infty$, we need to consider more general combinations of the form $a^0 f_0 + a^1 f_1$. Let there be given a set of $q \geq 3$ vectors $a_k = (a_k^0, a_k^1)$, and let $a_j$ and $a_k$ be linearly independent when $j \neq k$. We set

$$v = (\log|f_0|) \vee (\log|f_1|)$$

(here and subsequently, the symbols $\vee$ and $\wedge$ denote upper and lower envelopes) and

$$v_k = \log|a_k^0 f_0 + a_k^1 f_1|, \qquad 1 \leq k \leq q.$$

The functions $v$ and $v_k$ are subharmonic in $\mathbb{C}$. It is easily verified that

$$v = v_j \vee v_k + O(1), \qquad z \to \infty, \tag{1.1}$$

for all $j \neq k$. By a theorem of H. Cartan ([1], Chapter I, formula (4.8)), the Nevanlinna characteristic $T(r, f)$ can be represented in the form

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} v(re^{i\theta})\, d\theta. \tag{1.2}$$

If we denote by $N(r, a_k)$ the Nevanlinna function of the number of zeros of the entire function $a_k^0 f_0 + a_k^1 f_1$, we have, by Jensen's formula,

$$N(r, a_k) = \frac{1}{2\pi} \int_0^{2\pi} v_k(re^{i\theta})\, d\theta + O(1), \quad r \to \infty. \tag{1.3}$$

1991 *Mathematics Subject Classification*. Primary 30D35; Secondary 31A05.
*Key words and phrases.* Meromorphic functions, value distribution, R. Nevanlinna's theorem, subharmonic functions, meromorphic curves.

We want to prove Nevanlinna's second fundamental theorem in the form

$$(1.4)\qquad (q-2)T(r,f) \le \sum_{k=1}^{q} N(r,a_k) + o(T(r,f)) \quad \|$$

(the symbol $\|$ at the right of the formula means that it is valid as $r \to \infty$ except for a set $E$ of finite logarithmic measure, i.e., $\int_E d\log t < \infty$). By (1.2) and (1.3), this is the same as

$$\int_0^{2\pi} \left( \sum_{k=1}^{q} v_k(re^{i\theta}) - (q-2)v(re^{i\theta}) \right) d\theta \ge o(1) \int_0^{2\pi} v(re^{i\theta})\, d\theta . \quad \|$$

In fact, we shall establish a stronger proposition, namely that under condition (1.1) the function

$$\sum_{k=1}^{q} v_k - (q-2)v$$

is "almost subharmonic". This result will be obtained from the following "nonasymptotic" proposition: if $u_1, \dots, u_q$, $u$ are subharmonic in a domain $\Omega \subset \mathbb{C}$ and $u = u_j \vee u_k$ for every $j \ne k$, the function

$$w = \sum_{k=1}^{q} u_k - (q-2)u$$

is subharmonic in $\Omega$.

The transition from the asymptotic formulation to a nonasymptotic problem in potential theory is based on the compactness of certain families of subharmonic functions ([2], [3]; also see [4], Volume II, Chapter 16). In the authors' previous papers [5] and [6], this approach was applied to the solution of various problems in value distribution; however, there, as in [2] and [3], we considered only functions of finite lower order. Our method uses the analytic nature of the functions $f_0$ and $f_1$ only to obtain the subharmonicity of $v$ and $v_k$, and is consequently convenient for various generalizations.

We turn to the precise statements of the results.

Let $f: \mathbb{C} \to \mathbb{P}^n$ be a meromorphic curve. We describe it in homogeneous coordinates $f = (f_0, f_1, \dots, f_n)$ so that the entire functions $f_j$ do not vanish simultaneously. We consider the subharmonic function

$$v = \bigvee_{j=0}^{n} \log|f_j| .$$

We call the Riesz measure $\mu$ of $v$ the Cartan measure of $f$. We set $\mathscr{D}(z_0, r) = \{z : |z - z_0| < r\}$, $\mathscr{D}(r) = \mathscr{D}(0, r)$, and $A(r) = A(r, f) = \mu(\overline{\mathscr{D}(r)})$. The Nevanlinna characteristic $T(r) = T(r, f)$ is defined by

$$T(r,f) \overset{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} v(re^{i\theta})\, d\theta = \int_{r_0}^{r} \frac{A(t,f)}{t}\, dt + O(1), \qquad r \to \infty .$$

Let $Q$ be a homogeneous polynomial (form) in $n+1$ variables. We consider the entire function $Q \circ f = Q(f_0, \dots, f_n)$. Let $n(r, Q) = n(r, Q, f)$ be the set of zeros of $Q \circ f$ in the disk $\mathscr{D}(r)$, counting multiplicities, and set

$$N(r,Q) = N(r,Q,f) = \frac{1}{2\pi} \int_0^{2\pi} \log|Q \circ f(re^{i\theta})|\, d\theta = \int_{r_0}^{r} \frac{n(t,Q)}{t}\, dt + O(1), \qquad r \to \infty ,$$

Let $\sigma$ be a function that increases to $+\infty$ on $[0, \infty)$. We consider a field $M_\sigma$ consisting of meromorphic functions $a(z)$,

$$T(r, a) = O(\sigma(r)), \qquad r \to \infty.$$

Let $K_\sigma \subset M_\sigma[x_0, \dots, x_n]$ be the space of homogeneous forms in $n+1$ variables over $M_\sigma$. If $Q \in K_\sigma$, we denote by $Q(z)$ the form over $\mathbb{C}$ obtained by substituting numbers $z \in \mathbb{C}$ for the coefficients of the form $Q$. A finite system $S \subset K_\sigma$ is said to be admissible if, for every $n+1$ forms $Q_1, \dots, Q_{n+1} \in S$, and some $z \in \mathbb{C}$, the system of equations

$$Q_k(z)(w_0, \dots, w_n) = 0, \qquad 1 \le k \le n+1,$$

has only the trivial solution $w_0 = \dots = w_n = 0$ in $\mathbb{C}^{n+1}$. If this condition is satisfied for one $z \in \mathbb{C}$, it is also satisfied for all $z$ except for those belonging to a certain discrete set (see §6).

For a meromorphic curve $f = (f_0, \dots, f_n)$ and a form $Q \in K_\sigma$, we set

$$(Q \circ f)(z) = Q(z)(f_0(z), \dots, f_n(z)).$$

This is a meromorphic function in $\mathbb{C}$. We write

$$N(r, Q) = N(r, Q, f) = \frac{1}{2\pi}\int_0^{2\pi} \log|Q \circ f(re^{i\theta})|\, d\theta.$$

**Theorem 3.** *Let $f: \mathbb{C} \to \mathbb{P}^n$ be a meromorphic curve,*

$$\sigma(r) = \frac{T(r, f)}{\log^\tau T(r, f)}, \qquad \tau > 1, \tag{1.9}$$

*$Q_1, \dots, Q_q \in K_\sigma$ an admissible system of forms, $d_k = \deg Q_k$, and $q > 2n$. If $Q_k \circ f \not\equiv 0$, $1 \le k \le q$, we can carry out the integration of the second fundamental theorem* (1.6).

If $n = 1$ and $d_k = 1$, the conclusion of Theorem 3 becomes a result close to R. Nevanlinna's conjecture, recently proved by Osgood [14] and Steinmetz [15]. In the theorem of Osgood and Steinmetz, instead of (1.9) it is required only that

$$\sigma(r) = o(T(r, f)), \qquad r \to \infty,$$

and that the exceptional set in (1.6) has finite measure.

In the important case when $\sigma(r) = \log r$ (i.e., $K_\sigma$ is the space of homogeneous forms over the field of rational functions), Theorem 3 is not applicable to curves with a slowly increasing characteristic. Nevertheless, we have the following theorem.

**Theorem 4.** *Let $f: \mathbb{C} \to \mathbb{P}^n$ be a "transcendental" meromorphic curve, i.e., $T(r, f) \ne O(\log r)$, $r \to \infty$. If $Q_1 \dots, Q_q$ is an admissible system of forms in $K_{\log}$, $d_k = \deg Q_k$, $q > 2n$, and $Q_k \circ f \not\equiv 0$, $1 \le k \le q$, the integration of the second fundamental theorem* (1.6) *can be carried out for $r \to \infty$, $r \notin E$, where $E$ is a certain set with the property*

$$\varliminf_{r\to\infty} \frac{1}{\psi(\log\log r)} \int_{E\cap[1, r]} d\log t = 0,$$

*for an arbitrary function $\psi$ such that $\psi(x)/x$ tends to $+\infty$ as $x \to +\infty$.*

It is easily proved (see the beginning of §5) that the condition $T(r, f) = O(\log r)$ implies the existence of a representation $f = (f_0, \dots, f_n)$ in which all the $f_j$ are polynomials. Such curves are said to be rational. The following proposition of Picard type follows directly from Theorem 4.

if $Q \circ f \not\equiv 0$. The system $Q_1, \dots, Q_q$ of forms, $q > 2n$, is said to be admissible if no set of $n+1$ forms in this system has common zeros in $\mathbb{C}^{n+1}\backslash\{0\}$.

**Theorem 1.** *Let $f: \mathbb{C} \to \mathbb{P}^n$ be a meromorphic curve, and let $Q_1, \dots, Q_q$ be an admissible system of homogeneous forms of degree $d_k \geq 1$. If the entire functions $Q_k \circ f$ are not identically 0 for $1 \leq k \leq q$, we have*

$$(1.5) \qquad (q-2n)A(r, f) \leq \sum_{k=1}^{q} d_k^{-1} n(r, Q_k) + o(A(r, f)), \quad \|$$

$$(1.6) \qquad (q-2n)T(r, f) \leq \sum_{k=1}^{q} d_k^{-1} N(r, Q_k) + o(T(r, f)). \quad \|$$

We note that $N(r, Q^m) = mN(r, Q)$ and the number $(\deg Q)^{-1}N(r, Q)$ is invariant under the transformation $Q \mapsto Q^m$. Consequently we shall suppose from now on that all the $Q_k$ have the same degree $d$.

The relation (1.6) proves a conjecture made by Shiffman [7]. Special cases of this conjecture, under much stronger restrictions on $f$, were proved in [7]–[9]. If $d = 1$, (1.6) is a weak form of a conjecture of Cartan [10]. This conjecture was recently established in full generality by Nochka [11]. We note that (1.6) follows from (1.5). This elementary fact (Lemma 9) has apparently not been noticed previously.

Following Shiffman, we set

$$\delta(Q, f) = 1 - \varlimsup_{r\to\infty} \frac{N(r, Q)}{dT(r, f)}, \qquad d = \deg Q.$$

The following deficiency relation is a consequence of (1.6): let $S$ be an admissible system of forms, $f: \mathbb{C} \to \mathbb{P}^n$ a meromorphic curve, and $Q \circ f \not\equiv 0$ for all $Q \in S$. Then

$$(1.7) \qquad \sum_{Q \in S} \delta(Q, f) \leq 2n.$$

The deficiency relation (1.7) strengthens a theorem of Picard type that was proved in [12] (also see [13]).

We now present the "nonasymptotic" theorem on subharmonic functions from which Theorem 1 follows.

We denote by $I_k$ the collection of the subsets of $\{1, 2, \dots, q\}$ of cardinality $k$, $0 \leq k \leq q$.

**Theorem 2.** *Let $\Omega$ be a domain in $\mathbb{C}$, and $u, u_1, \dots, u_q$, functions subharmonic in $\Omega$, where $q > 2n$ and $n \in \mathbb{N}$. If, for some $I \in I_{n+1}$, we have*

$$(1.8) \qquad u = \bigvee_{k \in I} u_k,$$

*the function*

$$w \stackrel{\text{def}}{=} \sum_{k=1}^{q} u_k - (q-2n)u = \bigwedge_{I \in I_n} \sum_{k \in I} u_k + nu$$

*is subharmonic in $\Omega$.*

Our method of proof for Theorem 1 lets us also consider forms $Q$ whose coefficients are not constants, but meromorphic functions whose characteristics grow more slowly than $T(r, f)$.

**Corollary.** *Let $Q_1, \dots, Q_{2n+1}$ be an admissible system of forms with polynomial coefficients. If the meromorphic curve $f: \mathbb{C} \to \mathbb{P}^n$ has the property that each entire function $Q_k \circ f$, $1 \le k \le 2n+1$, has finitely many zeros, it follows that $f$ is a rational curve.*

Our outline is as follows. In §2 we prove the nonasymptotic Theorem 2, on which everything else depends. Theorem 2′ in §3 is the "stable" form of Theorem 2. In §4, we have a special partition of unity, of the same type as the continuous partition of Dieudonné and Whitney (see [4], Vol. 1, §1.4), that is needed for the proofs of Theorems 1, 3, and 4. Theorem 1 is proved in §5, and Theorems 3 and 4 in §6.

The authors thank V. S. Azarin, A. A. Gol′dberg, A. F. Grishin, A. U. Rashkovskiĭ, S. Yu. Favorov, A. E. Fryntov, and the participants in B. Ya. Levin's seminar for helpful discussions of this research.

## §2. The nonasymptotic theorem

To begin with, we show that it is enough to prove Theorem 2 for continuous functions $u$ and $u_k$. The authors owe this remark to V. S. Azarin. For a subharmonic function $v$, we set $v^\varepsilon(z) = \max\{v(\xi) : |z-\xi| \le \varepsilon\}$. It is easily seen that $v^\varepsilon(z) \to v(z)$, decreasing monotonically, as $\varepsilon \to 0$. If $v$ is subharmonic in a domain $\Omega$, the function $v^\varepsilon$ is subharmonic in the domain $\Omega^\varepsilon = \{z : \operatorname{dist}(z, \partial\Omega) > \varepsilon\}$. Let us show that $v^\varepsilon$ is continuous. If $|z_1 - z_2| < \delta$, and we set $z' = \frac{1}{2}(z_1 + z_2)$, we obtain

$$\begin{aligned}|v^\varepsilon(z_1) - v^\varepsilon(z_2)| &\le v^{\varepsilon+2\delta}(z') - v^{\varepsilon-2\delta}(z') \\ &= \max_{|\xi|=\varepsilon+2\delta} v(z'+\xi) - \max_{|\xi|=\varepsilon-2\delta} v(z'+\xi),\end{aligned}$$

but $\max_{|\xi|=\rho} v(z'+\xi)$ is a convex function of $\log\rho$ and consequently is continuous. It follows from the hypotheses of Theorem 2 that

$$u^\varepsilon = \Big(\bigvee_{k\in I} u_k\Big)^\varepsilon = \bigvee_{k\in I} u_k^\varepsilon$$

in $\Omega^\varepsilon$, for every $I \in I_{n+1}$. If Theorem 2 holds for continuous functions, we find that the function

$$w_\varepsilon \overset{\text{def}}{=} \bigwedge_{I\in I_n} \sum_{k\in I} u_k^\varepsilon + nu^\varepsilon$$

is subharmonic in $\Omega^\varepsilon$. In addition, $w_\varepsilon \to w$, decreasing monotonically, as $\varepsilon \to 0$. Consequently $w$ is subharmonic.

Therefore we may suppose that $u$ and $u_k$ are continuous. For each set $I \subset \{1, 2, \dots, q\}$ we define the open set

$$\mathscr{D}_I = \operatorname{int}\{z : u_k(z) < u(z),\ k \in I;\ u_k(z) = u(z),\ k \notin I\}.$$

The sets $\mathscr{D}_I$ are pairwise disjoint. If $\mathscr{D}_I \ne \varnothing$, we have $\operatorname{card} I \le n$. This follows from (1.8). To clarify the idea of the proof of Theorem 2, let us suppose that the sets $\mathscr{D}_I$ are bounded by finitely many Jordan curves. For every pair $I, J \in I_n$ with $\operatorname{card}(I \cup J) = m \le 2n$, we set

$$u_{I,J} = \sum_{k\in I\cup J} u_k + (2n-m)u.$$

This is a subharmonic function.

We have $w = u_{I,J}$ in $\mathscr{D}_I \cup \mathscr{D}_J$. Consequently, if a point $z \in \Omega$ has a neighborhood $V$ that intersects at most two sets $\mathscr{D}_I$, it follows that $w$ is subharmonic in $V$.

It is easily seen (see Lemma 3 below) that there are only finitely many exceptional points. Since $w$ is continuous, the removal of a finite set leaves $w$ subharmonic in $\Omega$.

The general case requires a more elaborate analysis. The concepts and theorems that we shall need from potential theory can be found in [16]–[18].

Let $\mathscr{D}$ be a bounded domain in $\mathbb{C}$, and let $G(z_0, z)$ be its Green function. For each given $z_0 \in \mathscr{D}$ the function $z \mapsto G(z_0, z)$ can be continued to a subharmonic function in $\mathbb{C}\backslash\{z_0\}$ with the property that $G(z_0, z) = 0$, $z \notin \overline{\mathscr{D}}$. The continuation is unique. We have the representation

$$G(z_0, z) = -\log|z - z_0| + \int_{\mathbb{C}} \log|z - \xi|\omega(z_0, d\xi), \tag{2.1}$$

where $\omega(z_0, d\xi)$ is harmonic measure on $\partial\mathscr{D}$ with respect to $z_0$ and the domain $\mathscr{D}$.

A point $a \in \partial\mathscr{D}$ is said to be accessible from $\mathscr{D}$ if there is a curve $\gamma\colon [0, 1] \to \mathbb{C}$ such that $\gamma(t) \in \mathscr{D}$ for $0 \le t < 1$ and $\gamma(1) = a$. The set of points that are accessible from $\mathscr{D}$ is a Borel set [19].

**Lemma 1.** *Let $E$ be the set of points of $\mathbb{C}$ that are inaccessible from $\mathscr{D}$. Then $\omega(z_0, E) = 0$ for every $z_0 \in \mathscr{D}$.*

This is a known proposition (see [20], [21]). The most intuitive proof is the one obtained by using Kakutani's theorem on the interpretation of the harmonic measure $\omega(z_0, E)$ as the probability that a Brownian motion, starting from $z_0$, leaves $\mathscr{D}$ for the first time through the set $E$ [22].

The term "quasi-everywhere" means "everywhere except for a set of capacity 0".

A function $v$ defined quasi-everywhere in $\Omega$ is called $\delta$-subharmonic if it can be represented as the difference of two functions that are subharmonic in $\Omega$. The Riesz charge of $v$ is the difference of the Riesz measures. Two $\delta$-subharmonic functions are considered to be equal if they coincide quasi-everywhere. The class of $\delta$-subharmonic functions is closed under taking upper and lower envelopes of finite families. In fact, it is enough to show that the operation $v \mapsto v^+$ preserves $\delta$-subharmonicity. But if $v = v_1 - v_2$, we have $v^+ = (v_1 - v_2)^+ = v_1 \vee v_2 - v_2$.

Let $\Omega_1$ be an open set, $\overline{\Omega}_1 \subset \Omega$. We say that $v^*$ is obtained from $v$ by sweeping out charges from $\Omega_1$ if $v^*(z) = v(z)$ for $z \in \Omega\backslash\Omega_1$, and $v^*|_{\Omega_1}$ is a solution of the (generalized) Dirichlet problem in $\Omega_1$ with the boundary conditions $v(z)$, $z \in \partial\Omega_1$. The sweeping operator preserves subharmonicity, and therefore $\delta$-subharmonicity [18].

**Lemma 2.** *Let $v$ be a continuous $\delta$-subharmonic function in $\Omega = \mathscr{D}(R)$, $E = \{z \in \Omega,\ v(z) = 0\}$, and $E^* \subset E$ a set of points that are inaccessible from $\Omega\backslash E$. Then the restriction of the Riesz charge $\nu$ of the function $v$ to $E^*$ equals 0.*

*Proof.* First we reduce the problem to the case when $v$ is continuous and $\delta$-subharmonic in $\mathscr{D}(R')$ with $R' > R$ and $v(z) = 0$, $|z| = R$. Let $0 < R_1 < R_2 < R_3 < R$, $m = \min\{v(z) : |z| \le R_1\}$, and $M = \max\{v(z) : |z| = R_3\}$. We choose $A > 0$ so large that

$$A\log(R_1/R_2) < m, \quad A\log(R_3/R_2) > M.$$

Now we define a number $B$ by the equality $A\log(R_3/R_2) = B\log(R/R_3)$. We set

$$v_1(z) = \begin{cases} \max\{v(z), A\log|z/R_2|\}, & |z| \le R_3, \\ B\log|R/z|, & R_3 < |z| < \infty. \end{cases}$$

Evidently $v_1$ is continuous and $\delta$-subharmonic in $\mathbb{C}$, $v_1(z) = 0$ for $|z| = R$, and $v_1(z) = v(z)$ in $\mathscr{D}(R_1)$.

We also suppose that $v$ is $\delta$-subharmonic in $\mathscr{D}(R')$, $R' > R$, and $v(z) = 0$ for $|z| = R$.

We now show that the proof can be reduced to the case when $\Omega\backslash E$ is a domain. Let $\{\mathscr{D}_j\}$ denote the collection of connected components of $\Omega\backslash E$. We set

$$v_j(z) = \begin{cases} v(z), & z \in \mathscr{D}_j, \\ 0, & z \notin \mathscr{D}_j. \end{cases}$$

It is evident that the functions $v_j$ are continuous in $\mathscr{D}(R')$. In addition, $v_j$ is a $\delta$-subharmonic function, since it was obtained by applying the sweeping process from $\Omega\backslash\overline{\mathscr{D}}_j$ to $v$. We have $v = \sum v_j$ and $\nu = \sum \nu_j$, where $\nu_j$ is the Riesz charge of $v_j$. It is enough to show that $\nu_j|_{E^*} = 0$.

Consequently, we need to prove the lemma in the case when $\mathscr{D} = \Omega\backslash E$ is a domain, and $v(z) = 0$ on $\partial\mathscr{D}$. Let us show that in this case $v$ is a Green potential,

$$v(z) = -\int_{\mathscr{D}} G(z_0, z)\, d\nu_{z_0}, \tag{2.2}$$

where $G$ is the Green function of $\mathscr{D}$.

In fact, $v = v_1 - v_2$, where the $v_i$ are subharmonic functions, and $v_1 = v_2$ outside $\mathscr{D}$. Let $h_i$ be the best harmonic majorant of $v_i$ in $\mathscr{D}$; then

$$h_i(z) = \overline{\lim_{\xi\to z,\, \xi\in\mathscr{D}}} \inf w(\xi),$$

where the infimum is taken over the class of functions $w(\xi)$ that are superharmonic in $\mathscr{D}$ and have the property that

$$\underline{\lim_{\xi\to z,\, \xi\in\mathscr{D}}}\, w(\xi) \geq v_i(z), \qquad z \in \partial\mathscr{D}, \quad i = 1, 2.$$

These classes are the same for $v_1$ and $v_2$; therefore $h_1 = h_2 = h$, and consequently $v_i = h + \Pi_i$, $i = 1, 2$, in $\mathscr{D}$, where $\Pi_i$ is a Green potential, i.e., $v = v_1 - v_2 = \Pi_1 - \Pi_2$, which is equivalent to (2.2).

We notice that the representation (2.2) is valid throughout $\mathbb{C}$. If we substitute (2.1) into (2.2), we obtain

$$v(z) = \int_{\mathscr{D}} \log|z_0 - z|\, d\nu_{z_0} - \int_{\mathscr{D}} d\nu_{z_0} \int_{\mathbb{C}} \log|z - \xi|\omega(z_0, d\xi).$$

We define the charge $\kappa$ by

$$\kappa(X) = \int_{\mathscr{D}} \omega(z_0, X)\, d\nu_{z_0} \tag{2.3}$$

for an arbitrary Borel set $X \subset \mathbb{C}$. By the generalized Fubini theorem ([23], [16]),

$$v(z) = \int_{\mathscr{D}} \log|z_0 - z|\, d\nu_{z_0} + \int_{\mathbb{C}} \log|z - \xi|\, d\kappa_\xi. \tag{2.4}$$

It follows from (2.4) that the Riesz charge of $v$, restricted to $\Omega\backslash\mathscr{D}$, is $\kappa$. It follows from (2.3) and Lemma 1 that $\kappa(E^*) = 0$, and we obtain the conclusion of the lemma.

**Lemma 3.** *Let $\mathscr{D}_1, \mathscr{D}_2$, and $\mathscr{D}_3$ be pairwise disjoint open subsets of $\mathbb{C}$. Then the set of points that are accessible simultaneously from $\mathscr{D}_1, \mathscr{D}_2$, and $\mathscr{D}_3$ is at most countably infinite.*

*Proof.* First let $\mathscr{D}_1, \mathscr{D}_2$, and $\mathscr{D}_3$ be domains. Let us show that three different points $z_1, z_2, z_3$ cannot simultaneously be accessible from all three domains. In the contrary case we could find points $w_i \in \mathscr{D}_i$, $i = 1, 2, 3$, covered by arcs $\Gamma_{ij} \subset \mathscr{D}_i$ that join $w_i$ to $z_j$, $1 \leq i, j \leq 3$. These arcs can be chosen to be pairwise disjoint. Then the points $w_i$ and $z_j$, together with the arcs $\Gamma_{ij}$, form a graph $K_{3,3}$ which, as is well known, cannot be embedded in the plane [24]. This contradiction establishes our conclusion for the case of domains. In the general case, if $z$ is accessible from $\mathscr{D}_1$, $\mathscr{D}_2$, and $\mathscr{D}_3$, it is accessible from some connected component $\mathscr{D}_i^* \subset \mathscr{D}_i$, $1 \le i \le 3$. For any triple of components, the set of such points $z$ is finite; therefore the set of points accessible from $\mathscr{D}_1$, $\mathscr{D}_2$, and $\mathscr{D}_3$ is at most countable. This completes the proof of Lemma 3.

For a $\delta$-subharmonic function $v$, we denote its Riesz charge by $\mu[v]$. For any charge $\mu$ we have the Jordan decomposition, $\mu = \mu^+ - \mu^-$, where $\mu^+$ and $\mu^-$ are measures. The upper (lower) envelope of two charges $\mu$ and $\nu$ is defined by

$$\mu \vee \nu = (\mu - \nu)^+ + \nu, \qquad \mu \wedge \nu = \mu - (\mu - \nu)^+.$$

This definition can be extended by induction to any finite family of charges. There is a natural order relation on the set of charges: $\mu \ge \nu$ if $\mu(X) \ge \nu(X)$ for every Borel set $X$. Then

$$\bigvee_{j=1}^{m} \mu_j \qquad \left(\bigwedge\nolimits_{j=1}^{m} \mu_j\right)$$

is the supremum (infimum) of the finite family $\{\mu_j\}$ in terms of this order.

**Lemma 4.** *Let $v_1, \dots, v_m$ be continuous $\delta$-subharmonic functions in the domain $\Omega$. Then*

$$\mu\left[\bigwedge\nolimits_{j=1}^{m} v_j\right] \ge \bigwedge\nolimits_{1\le i<j\le m} \mu[v_i \wedge v_j].$$

*Proof.* By the local nature of the lemma, it is enough to consider only the case when $\Omega$ is a disk.

First let $m = 3$. We set

$$w_i = \bigwedge\nolimits_{j\ne i} v_j, \quad \mathscr{D}_i = \{z \in \Omega : v_i(z) < w_i(z)\}, \qquad i = 1, 2, 3.$$

It is evident that the $\mathscr{D}_i$ are pairwise disjoint open sets. If $E_i = \Omega \backslash \mathscr{D}_i$, we have

$$v = \bigwedge\nolimits_{1\le j\le 3} v_j = w_i \quad \text{on } E_i.$$

Let $E_i^* \subset E_i$ be the set of points that are not accessible from $\mathscr{D}_i$. By Lemma 2, applied to $v - w_i$, we have

$$\mu\left[\bigwedge\nolimits_{j=1}^{3} v_j\right] = \mu[w_i] \quad \text{on } E_i^*, \quad 1 \le i \le 3. \tag{2.5}$$

In addition, it follows from Lemma 3 that

$$E_i^* \cup E_2^* \cup E_3^* = \Omega \backslash X,$$

where $X$ is at most countable. For every continuous $\delta$-subharmonic function $v$, we have $\mu[v](X) = 0$. Hence, for $m = 3$, the conclusion of the lemma follows from (2.5).

We now prove the lemma by induction for an arbitrary $m \ge 3$. (The following procedure was suggested by A. Yu. Rashkovskiĭ.) By Lemma 4 with $m = 3$, we have

$$\begin{aligned}
\mu[v_1 \wedge \cdots \wedge v_m] &= \mu[(v_1 \wedge \cdots \wedge v_{m-2}) \wedge v_{m-1} \wedge v_m] \\
&\ge \mu[(v_1 \wedge \cdots \wedge v_{m-2}) \wedge v_{m-1}] \\
&\quad \wedge \mu[(v_1 \wedge \cdots \wedge v_{m-2}) \wedge v_m] \wedge \mu[v_{m-1} \wedge v_m] \\
&= \mu[v_1 \wedge \cdots \wedge v_{m-1}] \\
&\quad \wedge \mu[v_1 \wedge \cdots \wedge v_{m-2} \wedge v_m] \wedge \mu[v_{m-1} \wedge v_m],
\end{aligned}$$

the last expression is not less than $\bigwedge_{1\le i<j\le m}\mu[v_i\wedge v_j]$, by the inductive hypothesis. This completes the proof of the lemma.

*Remark* 1. We notice the following corollary of Lemma 4. Let $v_1, \dots, v_m$ be functions that are continuous and $\delta$-subharmonic in $\Omega$ and have the property that all the functions $v_{ij} = v_i \wedge v_j$ are subharmonic. Then the function $\bigwedge_{1\le i\le m} v_i$ is also subharmonic in $\Omega$. The following questions naturally arise:

1. Is Lemma 4 valid for all $\delta$-subharmonic functions (without the continuity hypothesis)?
2. Is Lemma 4 valid for $\delta$-subharmonic functions in the space $\mathbb{R}^m$, $m \ge 3$? This question also arises for Theorem 2.

**Lemma 5** (A. F. Grishin [25]). *If $v \ge 0$ is $\delta$-subharmonic in $\Omega$ and $v(z) = 0$ on some Borel set $X$, the restriction to $X$ of the Riesz charge of $v$ is a nonnegative measure.*

*Proof of Theorem* 2 As we noticed at the beginning of §2, it is enough to prove Theorem 2 for continuous functions.

We proceed by induction on $n$. If $n = 0$, $q \in \mathbb{N}$, the theorem is evidently valid. Let us suppose that the theorem is valid for $n = N - 1$ with any $q > 2n$. Let us prove it for $n = N$, $q > 2n$.

Let $\xi \in \Omega$ be a point for which $u_j(\xi) < u(\xi)$ for at least one index $j$. Without loss of generality, we may suppose that $u_q(\xi) < u(\xi)$. By continuity, this inequality is valid in some neighborhood $V \subset \Omega$ of $\xi$. In this neighborhood we have

$$w = u_q + u + \left\{\sum_{k=1}^{q-1} u_k - (q - 1 - 2(N-1))u\right\}. \tag{2.6}$$

The functions $u, u_1, \dots, u_{q-1}$ satisfy hypothesis (1.8) of Theorem 2 with $n = N - 1$. Therefore, by the inductive hypothesis, the expression in braces in (2.6) is a subharmonic function. Consequently $w$ is subharmonic in $V$.

Now let $X = \{z \in \Omega : u(z) = u_1(z) = \cdots = u_q(z)\}$. Let us show that the restriction of the Riesz charge of $w$ to $X$ is a nonnegative measure. (Here we do not need to use the inductive hypothesis.) For any $I \in I_N$ we set

$$u_I = \sum_{k\in I} u_k + Nu.$$

For any $I$, $J \in I_N$ we have

$$u_I \wedge u_J \ge \sum_{k\in I} u_k + \sum_{k\in J} u_k \stackrel{\text{def}}{=} u_{IJ} \quad \text{in } \Omega$$

and

$$u_I \wedge u_J = u_{IJ} = 2Nu \quad \text{on } X.$$

If we apply Lemma 5 to $X$ and to $u_I \wedge u_J - u_{IJ}$, we find that $\mu[u_I \wedge u_J]|_X$ is a nonnegative measure. Since

$$w = \bigwedge_{I\in I_n} u_I,$$

it follows from Lemma 4 that $\mu[w]|_X \ge 0$.

This completes the proof.

*Supplement.* After this paper had been prepared for publication, B. Fuglede kindly informed the authors that the methods of fine potential theory ([27], [28]) make it possible to eliminate the hypothesis of the continuity of $v$ in Lemma 2. However, the idea of the preceding proof remains valid. It also eliminates the requirement of the continuity of $v_1, \dots, v_m$ in Lemma 4.

## §3. The "stable" version of Theorem 2

Let $\|\psi\| = \int_{\mathscr{D}(1)} |\psi|\, dx\, dy$. We denote by $B(L)$ the set of continuous functions $\psi$ such that $0 \le \psi \le 1$, $\operatorname{supp}(\psi) \subset \mathscr{D}(1)$, and $|\operatorname{grad}\psi| \le L$.

**Theorem** $2'$. *Let $L, M > 0$ and $q, n \in \mathbb{N}$, $q > 2n$. Then for every $\delta > 0$ there is a number $\alpha = \alpha(\delta, L, M, q, n) > 0$ with the following property. If $U$ is subharmonic, and $U_1, \dots, U_q$ are $\delta$-subharmonic in $\mathscr{D}(2)$ with the Riesz charges $\nu \ge 0$ and $\nu_1, \dots, \nu_q$, together with*

$$\left(\nu + \sum_{k=1}^{q} \nu_k\right)(\overline{\mathscr{D}(1)}) \le M,$$

$$\left\|\bigvee_{k\in I} U_k - U\right\| \le \alpha \quad \forall I \in I_{n+1}, \tag{3.1}$$

$$\sum_{k=1}^{q} \nu_k^-(\overline{\mathscr{D}(1)}) \le \alpha,$$

*the charge $\kappa = \sum_{k=1}^{q} \nu_k - (q-2n)\nu$ satisfies*

$$\int \psi\, d\kappa \ge -\delta \quad \forall \psi \in B(L).$$

*Proof*. Suppose that the theorem is false. Then there are a number $\delta > 0$, a sequence $(U^j, U_1^j, \dots, U_q^j)$, $j \in \mathbb{N}$, of vectors with the properties (3.1) and

$$\left\|\bigvee_{k\in I} U_k^j - U^j\right\| \to 0, \qquad j \to \infty,\ I \in I_{n+1}, \tag{3.2}$$

$$\sum_{k=1}^{q} (\nu_k^j)^-(\overline{\mathscr{D}(1)}) \to 0, \qquad j \to \infty, \tag{3.3}$$

and a sequence of functions $\psi^j \in B(L)$, for which

$$\int \psi^j\, d\kappa^j \le -\delta, \qquad j \in \mathbb{N}. \tag{3.4}$$

The class $B(L)$ is equicontinuous and therefore, by passing to a subsequence if necessary, we may assume that $\psi^j \rightrightarrows \psi$, $0 \le \psi \le 1$, $\psi \in C(\overline{\mathscr{D}(1)})$. Then, by (3.1), we have

$$\left|\int \psi\, d\kappa^j - \int \psi^j\, d\kappa^j\right| \le |\kappa^j|(\overline{\mathscr{D}(1)}) \max_{\overline{\mathscr{D}(1)}} |\psi - \psi^j| \to 0,$$

as $j \to \infty$, and it follows from (3.4) that

$$\int \psi\, d\kappa^j \le -\delta/2, \qquad j \ge j_0. \tag{3.5}$$

If we use (3.1) and (3.3), and choose, if necessary, a subsequence, we may suppose that, as $j \to \infty$,

$$\nu^j \to \nu^0 \ge 0, \qquad \nu_k^j \to \nu_k^0 \ge 0,$$

$$\kappa^j \to \kappa^0 = \sum_{k=1}^{q} \nu_k^0 - (q-2n)\nu^0. \tag{3.6}$$

The convergence is supposed to be in the weak topology of the charge space dual to the space $C(\overline{\mathscr{D}(1)})$. It follows from (3.5) and (3.6) that

$$\int \psi \, d\kappa^0 \leq -\delta/3. \tag{3.7}$$

Let $G * \lambda$ be the Green potential of the charge $\lambda$ in the disk $\mathscr{D}(1)$. The weak convergence of measures implies the convergence of potentials in $L_1$, and hence it follows from (3.6) that

$$\begin{gathered} G * \nu^j \to G * \nu^0, \qquad G * \nu_k^j \to G * \nu_k^0, \\ G * \kappa^j \to G * \kappa^0 \quad \text{in } L_1(\mathscr{D}(1)). \end{gathered} \tag{3.8}$$

We rewrite (3.2) in the form

$$\left\| \bigvee_{k \in I} (U_k^j - U^j) \right\| \to 0, \qquad j \to \infty, \; I \in I_{n+1}. \tag{3.9}$$

Let

$$U_k^j - U^j = H_k^j - G * (\nu_k^j - \nu^j),$$

where $H_k^j$ is harmonic in $\mathscr{D}(1)$. It follows from (3.8) and (3.9) that

$$\left\| \bigvee_{k \in I} H_k^j \right\| \leq \text{const}, \qquad j \in \mathbb{N}, \; I \in I_{n+1}, \tag{3.10}$$

and consequently the harmonic functions $H_k^j$ are uniformly bounded above on compact subsets of $\mathscr{D}(1)$. If we select a subsequence, we may suppose that $H_k^j \to H_k$ as $j \to \infty$, uniformly on compact sets, and that some of the functions $H_k$ can be identically $-\infty$. Let us suppose that $H_k \not\equiv -\infty$ for $1 \leq k \leq q' \leq q$ and $H_k \equiv -\infty$ for $q' < k \leq q$. It follows from (3.10) that $q - q' \leq n$. Let us set $n' = n - (q - q') \geq 0$. Then $q' - 2n' \geq 0$. Let us show that the subharmonic functions $u = -G * \nu^0$ and $u_k = H_k - G * \nu_k^0$, $1 \leq k \leq q'$, satisfy the hypotheses of Theorem 2 with $q$ and $n$ replaced by $q'$ and $n'$. In fact,

$$\|(U_k^j - U^j) - (u_k - u)\| \to 0, \quad j \to \infty, \quad 1 \leq k \leq q'. \tag{3.11}$$

We observe that the mapping $(w_1, w_2) \mapsto w_1 \vee w_2$ is continuous in $L_1$. Hence it follows from (3.9) and (3.11) that

$$\left\| \bigvee_{k \in I} (u_k - u) \right\| = 0, \quad \text{i.e., } u = \bigvee_{k \in I} u_k,$$

for every $I \in I_{n'+1} \cap \{1, 2, \ldots, q'\}$. If we apply Theorem 2 to the functions $u, u_1, \ldots, u_{q'}$, we obtain

$$\sum_{k=1}^{q'} \nu_k^0 - (q' - 2n')\nu^0 \geq 0,$$

and consequently

$$\kappa^0 = \sum_{k=1}^{q} \nu_k^0 - (q - 2n)\nu^0 = \sum_{k=1}^{q'} \nu_k^0 - (q' - 2n')\nu^0 + \sum_{k=q'+1}^{q} \nu_k^0 + \{(q' - 2n') - (q - 2n)\}\nu^0.$$

The expression in braces is nonnegative:

$$q' - 2n' - q + 2n = q' - 2n + 2q - 2q' - q + 2n = q - q' \geq 0.$$

Therefore $\kappa \geq 0$, contrary to (3.7). This completes the proof of the theorem.

## §4. A PARTITION OF UNITY

We use a construction, originated by Whitney and Dieudonné (see [4], §1.4), for a continuous partition of unity.

We use $c$ and $c_j$ to denote absolute constants; the notation $s \asymp t$ means that there are positive constants $c_1$ and $c_2$ such that $c_1 t \leq s \leq c_2 t$.

Up to the end of this section we fix the numbers $r > 0$ and $\Delta$, $0 < \Delta < r/2$, $r' = r + \Delta$. For each point $z \in \mathscr{D}(r')$ we set

$$|\xi|_z = \frac{2|\xi|}{r' - |z|}, \qquad \xi \in \mathbb{C}. \tag{4.1}$$

Then, in the terminology of [4], §1.4, $|\cdot|_z$ stands for a slowly varying metric on $\mathscr{D}(r')$; specifically, $z \in \mathscr{D}(r')$, $|z - z_1|_z < 1$ implies that $z_1 \in \mathscr{D}(r')$ and

$$\tfrac{1}{2}|\xi|_z \leq |\xi|_{z_1} \leq 2|\xi|_z, \qquad \xi \in \mathbb{C} \tag{4.2}$$

(see [4], p. 29).

**Lemma 6.** *There is a finite set of points $z_j \in \mathscr{D}(r')$, $1 \leq j \leq p$, with the following properties:*

1) $p \leq cr/\Delta$;
2) *the disks* $\mathscr{D}_j = \{z : |z - z_j|_{z_j} \leq 1\} \overset{\text{def}}{=} \mathscr{D}(z_j, \rho_j)$ *are contained in* $\mathscr{D}(r')$;
3) *every point* $z \in \mathscr{D}(r')$ *belongs to at most* $c_1$ *disks* $\mathscr{D}_j$;
4) *if, for some* $m \in \mathbb{N}$, *we have* $\mathscr{D}_j \cap \mathscr{D}(r' - 2^{m-1}\Delta) \neq \varnothing$, *then* $\rho_j \geq c_2 2^m \Delta$;
5) *there are nonnegative functions* $\varphi_j \in C_0^\infty(\mathscr{D}_j)$ *such that*

$$\sum \varphi_j = 1 \tag{4.3}$$

*on* $\mathscr{D}(r)$, *and the functions* $\psi_j(z) = \psi_i(\rho_j z + z_j) \in C_0^\infty(\mathscr{D}(1))$ *satisfy*

$$|\operatorname{grad} \psi_j(z)| \leq c_3. \tag{4.4}$$

*Proof.* By Theorem 1.4.10 of [4], there is a sequence of $z_j \in \mathscr{D}(r')$ such that the disks $\mathscr{D}_j = \mathscr{D}(z_j, \rho_j) = \{z : |z - z_j|_{z_j} < 1\}$ form a covering of $\mathscr{D}(r')$ of multiplicity not exceeding an absolute constant $c_1$. In addition, by the same theorem we can choose nonnegative functions $\varphi_j \in C_0^\infty(\mathscr{D}_j)$ that satisfy (4.3) on $\mathscr{D}(r')$ and have the property that

$$|\operatorname{grad} \varphi_j(z)| \leq c_3 |1|_{z_j} \asymp \rho_j.$$

We retain only the indices $j$ for which $\mathscr{D}_j \cap \mathscr{D}(r) \neq \varnothing$. Then (4.3) is, as before, satisfied on $\mathscr{D}(r)$. For some $m \in \mathbb{N}$, let $\mathscr{D}_j \cap \mathscr{D}(r' - 2^{m-1}\Delta) \neq \varnothing$. Then, if we choose a point $z \in \mathscr{D}_j \cap \mathscr{D}(r' - 2^{m-1}\Delta)$, we obtain

$$|z_j| - (r' - 2^{m-1}\Delta) \leq |z - z_j| \leq \frac{r' - |z_j|}{2},$$
$$|z_j| \leq r - 2^{m-2}\Delta,$$

from which

$$\rho_j = \frac{r' - |z_j|}{2} \geq 2^{m-3}\Delta,$$

and property 4) is established.

We now estimate the number $p$ of the indices for which $\mathscr{D}_j \cap \mathscr{D}(r' - 2^{m-1}\Delta) \neq \varnothing$. Consider the annulus

$$K_m = \{z : r' - 2^m\Delta \leq |z| < r' - 2^{m-1}\Delta\},$$
$$m = 1, 2, \ldots, l, \qquad l = [\log_2(r'/\Delta)] + 1;$$

let $J_m = \{j : \mathscr{D}_j \cap K_m \neq \varnothing\}$. We estimate $\operatorname{card}(J_m)$ by comparing areas. If $j \in J_m$, it follows from property 4) that $\rho_j \geq c_2 2^m\Delta$, and consequently $\operatorname{area}(\mathscr{D}_j) \geq c_2 2^{2m}\Delta^2$. On the other hand, $\operatorname{area}(K_m) \leq c_2 2^m \Delta r$. Therefore $\operatorname{card}(J_m) \leq cr/2^m\Delta$, and consequently

$$p \leq \sum_{m=1}^{l} \operatorname{card}(J_m) \leq c\frac{r}{\Delta}\sum_{m=1}^{\infty} 2^{-m} = c\frac{r}{\Delta}.$$

This completes the proof of the lemma.

We choose a finite nonnegative measure $\mu$ on $\mathscr{D}(r')$ (in what follows, $\mu$ will be the Cartan measure of the curve $f$), and set

$$a_j = \int \varphi_j \, d\mu. \tag{4.5}$$

**Lemma 7.** *In the notation introduced above, the following propositions are valid.*

1) *Let* $K > 0$ *and* $J_K = \{j : 1 \leq j \leq p,\ a_j \leq K\}$. *Then*

$$\sum_{j \in J_K} a_j \leq c\frac{Kr}{\Delta}.$$

2) *Let* $\nu$ *be a nonnegative measure on* $\mathscr{D}(r')$, *and* $J_{\nu,\beta} = \{j : 1 \leq j \leq p,\ \nu(\mathscr{D}_j) \geq \beta a_j\}$. *Then*

$$\sum_{j \in J_{\nu,\beta}} a_j \leq \frac{c}{\beta}\nu(\mathscr{D}(r')).$$

*Proof.* By properties 1)–3) of the partition of (4.3), we have

$$\sum_{j \in J_{\nu,\beta}} a_j \leq \frac{c}{\beta}\sum_{j \in J_{\nu,\beta}} \nu(\mathscr{D}_j) \leq \frac{c}{\beta}\nu(\mathscr{D}(r')),$$
$$\sum_{j \in J_K} a_j \leq cK \operatorname{card}(J_K) \leq cKp = c\frac{Kr}{\Delta}.$$

This completes the proof of the lemma.

## §5. Proof of Theorem 1

Let

$$v = \bigvee_{j=0}^{n} \log|f_j|,$$

let $\mu$ be the Riesz measure of $v$ (Cartan measure of the curve $f$), and let $A(r, f) = \mu(\overline{\mathscr{D}(r)})$. Then

$$T(r, f) = \int_{r_0}^{r} \frac{A(t, f)}{t}\, dt = \frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta})\, d\theta.$$

Let $\{Q_k\}_{k=1}^{q}$ be an admissible system of homogeneous polynomials of degree $d$. We set $v_k = \log|Q_k(f_0, \ldots, f_n)|$ and denote by $\mu_k$ the Riesz measure of $v_k$, $1 \leq k \leq q$, $n(r, Q_k) = \mu_k(\overline{\mathscr{D}(r)})$,

$$N(r, Q_k) = \int_{r_0}^{r} \frac{n(t, Q_k)}{t}\, dt.$$

Since the system $S = \{Q_k\}$ is admissible, it follows that, for every $I \in I_{n+1}$,

$$c_1 \bigvee_{j=0}^{n} |f_j|^d \le \bigvee_{k\in I} |Q_k(f_0, \dots, f_n)| \le c_2 \bigvee_{j=0}^{n} |f_j|^d,$$

where $0 < c_1 < c_2 < \infty$ are constants that depend only on $S$. Therefore

$$\left|\bigvee_{k\in I} v_k - dv\right| \le c(S), \qquad I \in I_{n+1}. \tag{5.1}$$

We first consider the "trivial" case when $f$ is a rational curve of degree $L$, i.e., $\lim_{r\to\infty} A(r, f) = L$. In this case

$$T(r, f) = L\log r + O(1), \qquad r \to \infty.$$

Let us show that we can choose a representation $f = (f_0, \dots, f_n)$, in which the $f_j$, $0 \le j \le n$, are polynomials, and $\max_{0\le j\le n} \deg f_j = L$.

In fact, let $f = (\tilde{f}_0, \dots, \tilde{f}_n)$ be any representation of the curve $f$,

$$\tilde{v} = \bigvee_{j=0}^{n} \log|\tilde{f}_j|, \qquad L = \mu(\mathbb{C}) < \infty.$$

Then $\tilde{v} = v + H$, where $H$ is harmonic and

$$v(z) = \int_{\mathscr{D}(1)} \log|z-\xi|\, d\mu_\xi + \int_{\{|\xi|\ge 1\}} \log\left|1 - \frac{z}{\xi}\right| d\mu_\xi.$$

Let $g = H + i\widetilde{H}$, where $\widetilde{H}$ is the harmonic conjugate of $H$, and $f_j = \tilde{f}_j e^{-g}$. Then

$$\bigvee_{j=0}^{n} \log|f_j| = v.$$

In addition, when $r \to \infty$

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f_j(re^{i\theta})|\, d\theta \le \frac{1}{2\pi}\int_v^{2\pi} v(re^{i\theta})\, d\theta = L\log r + O(1),$$

and therefore $f_j$ are polynomials of degree at most $L$. Evidently, $\max_{0\le j\le n} \deg f_j = L$.

It follows from (5.1) that among the polynomials $Q_k(f_0, \dots, f_n)$, $1 \le k \le q$, at most $n$ have degree less than $dL$. Let the degrees of the polynomials $Q_1(f_0, \dots, f_n)$, $\dots$, $Q_{q-n}(f_0, \dots, f_n)$ be equal to $dL$. Then, in the first place, for $r \ge r_0$ we have

$$\begin{aligned}\sum_{k=1}^{q} n(r, Q_k) \ge \sum_{k=1}^{q-n} n(r, Q_k) &= d(q-n)L \\ &= d(q-n)\lim_{\rho\to\infty} A(\rho, f) \ge d(q-n)A(r, f),\end{aligned}$$

and, in the second place,

$$\begin{aligned} d(q-n)T(r, f) &= d(q-n)L\log r + O(1) \\ &= \sum_{k=1}^{q-n} N(r, Q_k) + O(1) \le \sum_{k=1}^{q} N(r, Q_k) + O(1), \qquad r \to \infty,\end{aligned}$$

as required.

We now turn to the fundamental case when $A(r, f) \to \infty$ as $r \to \infty$.

Let us show that (5.1) implies the unintegrated second fundamental theorem,

$$(5.2)\qquad (q-2n)dA(r,f) \le \sum_{k=1}^{q} n(r,Q_k) + o(A(r,f)), \quad \|$$

and then, using the elementary Lemma 9, we integrate (5.2).

We specify numbers $\eta > 1$ and $\varepsilon > 0$. We set $A(r) = A(r,f)$, $r' = r+\Delta$, where $\Delta = \Delta(r)$ is defined by

$$(5.3)\qquad \Delta = r/\log^{\eta} A(r).$$

We say that $r$ is unexceptional if

$$(5.4)\qquad A(r') \le (1+\varepsilon)A(r).$$

**Lemma 8.** *When $r$ runs through the unexceptional values, the corresponding $r'$ runs through $\mathbb{R}\backslash E$, where $E$ is a set of finite logarithmic measure.*

*Proof.* We use the following theorem of Borel and Nevanlinna ([1], Chapter III, Theorem 1.2). Let $h$ and $g$ be continuous functions that increase unboundedly on $[r_0, \infty)$, and let $\int^{\infty} g(x)\,dx < \infty$. Then

$$h(t+g(h(t))) \le h(t)+1,$$

outside a set of pairwise disjoint intervals with finite total length. If we set $h(t) = H(e^t)$, $r = e^t$, and use the inequality

$$\exp(t+g(h(t))) \ge e^t(1+g(h(t))),$$

we obtain

$$H(r+rg(H(r))) \le H(r)+1. \quad \|$$

We now take $g(x) = x^{-\tau}$, and $H(r) = \log^{\tau} A(r)$, where $\tau = \sqrt{\eta} > 1$. We obtain

$$(5.5)\qquad A\left(r+\frac{r}{\log^{\eta} A(r)}\right) \le (1+\varepsilon)A(r). \quad \|$$

We now take $r' = \gamma(r)$. The function $\gamma(r)$ is not necessarily monotonic.

Let $I' = [r_1', r_2']$, $r_1' < r_2'$, be an interval whose $\gamma$-preimage consists of excluded values. One of the connected components of the preimage $\gamma^{-1}(I')$ is an interval $I = [r_1, r_2]$, $r_1 < r_2$, such that $\gamma(r_i) = r_i'$, $i = 1, 2$. We have

$$\int_{I'} d\log t = \log\frac{r_2'}{r_1'} = \log\frac{r_2}{r_1} + \eta\log\frac{\log A(r_1)}{\log A(r_2)} < \log\frac{r_2}{r_1} = \int_I d\log t.$$

This completes the proof of the lemma.

We continue the proof of Theorem 1. Recall that we are considering subharmonic functions $v, v_1, \dots, v_q$ with Riesz measures $\mu, \mu_1, \dots, \mu_q$, and (5.1) satisfied. The numbers $\eta > 1$ and $\varepsilon > 0$ are fixed. We consider an unexceptional value $r$. To this there correspond $r'$ and $\Delta$ that satisfy (5.3) and (5.4). By applying Lemma 6, we obtain finite sequences of points $z_j \in \mathscr{D}(r')$, numbers $\rho_j > 0$, and compactly supported functions $\varphi_j$, $1 \le j \le p$, that satisfy hypotheses 1)–5) of Lemma 6. By 1) and (5.3), we have

$$(5.6)\qquad p \le c\frac{r}{\Delta} \le c\log^{\eta} A(r).$$

Let $J = \{1, \dots, p\}$ and denote by $J^* \subset J$ the set of indices $j$ for which

$$a_j = \int \varphi_j \, d\mu \geq A(r)/\log^\tau A(r), \qquad \tau = \eta^2 > \eta, \tag{5.7}$$

$$\left(\mu + \sum_{k=1}^{q} \mu_k\right)(\overline{\mathscr{D}_j}) \leq \frac{1}{\varepsilon} a_j. \tag{5.8}$$

We now apply Lemma 7 with $\beta = 1/\varepsilon$, $K = A(r)/\log^\tau A(r)$, and $\nu = \mu + \sum_{k=1}^{q} \mu_k$. We find that for the exceptional indices $J \backslash J^*$

$$\sum_{j \in J \backslash J^*} a_j \leq C\varepsilon \left(A(r', f) + \sum_{k=1}^{q} \mu_k(\mathscr{D}(r'))\right) \tag{5.9}$$

for $r \geq r_0$.

We transfer $v$ and $v_k$ from the disk $\mathscr{D}_j = \mathscr{D}(z_j, \rho_j)$ to the standard disk $\mathscr{D}(1)$ after normalizing them on $a_j$. That is, for $j \in J^*$ we consider the following functions, which are subharmonic in $\mathscr{D}(2)$:

$$U_k^j = \frac{1}{a_j} v_k(\rho_j z + z_j), \qquad 1 \leq k \leq q,$$

$$U^j = \frac{d}{a_j} v(\rho_j z + z_j).$$

Let us show that these functions satisfy the hypotheses of Theorem 2′ for $r \geq r_1$, where $r_1$ depends on $S$ and $A(r)$.

In fact, (3.1), with $M = 1/\varepsilon$, follows from (5.8). By (4.4), the functions $\psi_j(z) = \varphi_j(\rho_j z + z_j)$ belong to $B(L)$, where $L$ is an absolute constant, and by (5.1) and (5.7) we have

$$\begin{aligned} \left\| \bigvee_{k \in I} U_k^j - U^j \right\| &\leq \frac{c}{a_j} \max_{z \in \overline{D}_j} \left| \bigvee_{k \in I} v_k - dv \right| \leq \frac{c(S)}{a_j} \\ &\leq \frac{c_1(S)}{A(r, f)} \log^\tau A(r, f) \leq \alpha(\varepsilon, L, 1/\varepsilon, q, n), \qquad r \geq r_1. \end{aligned}$$

Consequently, by Theorem 2′ we have

$$\int \psi_j \, d\kappa^j \geq -\varepsilon, \qquad j \in J^*, \tag{5.10}$$

where $\kappa^j$ is the Riesz charge of the function

$$\sum_{k=1}^{q} U_k^j - (q - 2n) U^j,$$

or, after the substitutions $z \mapsto (z - z_j)/\rho_j$ and $a_j = \int \varphi_j \, d\mu$,

$$d(q - 2n) \int \varphi_j \, d\mu \leq \sum_{k=1}^{q} \int \varphi_j \, d\mu_k + \varepsilon \int \varphi_j \, d\mu,$$

$j \in J^*$. Summing on $j$, $1 \leq j \leq p$, and using (5.9) and (4.5), we obtain

$$d(q - 2n) \int \Phi \, d\mu \leq \sum_{k=1}^{q} \int \Phi \, d\mu_k + \varepsilon \int \Phi \, d\mu + c\varepsilon \left(A(r') + \sum_{k=1}^{q} \mu_k(\mathscr{D}(r'))\right),$$

where

$$\Phi = \sum_{j=1}^{p} \varphi_j .$$

If we use the facts that $\Phi(z) \equiv 1$ in $\mathscr{D}(r)$ and $\Phi(z) \equiv 0$ outside $\mathscr{D}(r')$ by Lemma 6, we obtain

$$d(q-2n)A(r) \leq \sum_{k=1}^{g} \mu_k(\mathscr{D}(r')) + c\varepsilon \left( A(r') + \sum_{k=1}^{q} \mu_k(\mathscr{D}(r')) \right) .$$

Applying (5.4), we obtain

$$d(q-2n)A(r') \leq (1+c\varepsilon) \sum_{k=1}^{q} \mu_k(\mathscr{D}(r')) . \tag{5.11}$$

This inequality is satisfied outside a set of values $r'$ of finite logarithmic measure (by Lemma 8).

Now let $\varepsilon_j \to 0$, and let $E_j$ be the set on which (5.11) is not satisfied with $\varepsilon = \varepsilon_j$. We choose $r_j$ so that

$$\int_{E_j \cap [r_j, \infty)} d\log t < 2^{-j}, \qquad r_1 < r_2 < \cdots . \tag{5.12}$$

Consider the function $\varepsilon(r) = \varepsilon_j$ for $r_j < r \leq r_{j+1}$. Then (5.11) is satisfied with $\varepsilon = \varepsilon(r)$ outside the set

$$\bigcup_{j=1}^{\infty} (E_j \cap [r_j, r_{j+1}]) \cup [0, r_1],$$

which, by (5.12), has finite logarithmic measure.

Consequently (5.2), or equivalently (1.5), is established.

To deduce (1.6) from (1.5), we use the following lemma. Looking ahead to the proofs of Theorems 3 and 4, we state it in a form more general than we need just to deduce (1.6) from (1.5). For a set $e \subset \mathbb{R}_+$ we set $e(x) = e \cap [0, x]$ and $e(\rho, x) = e \cap [\rho, x]$.

**Lemma 9.** *Let $e \subset \mathbb{R}_+$ be a measurable set and $\varepsilon > 0$. Then there exists a set $e^* \subset \mathbb{R}_+$ such that for every $s > 0$ we have*

$$|e^*(s)| \leq \frac{4}{\varepsilon} |e(s)| , \tag{5.13}$$

*and, for any nonnegative nondecreasing function $a(x)$ and all numbers $r \in \mathbb{R}_+ \backslash e^*$, $\rho < r$, we have*

$$\int_{e(\rho, r)} a(x)\, dx < \varepsilon \int_{[\rho, r]} a(x)\, dx . \tag{5.14}$$

*Proof* (S. Yu. Favorov). Let us set

$$e^* = \{ r \in \mathbb{R}_+ : \exists x = x(r) < r \text{ such that } |e(x, r)| > \varepsilon(r-x) \} .$$

We show that (5.13) is satisfied. The set $e^*(s)$ is "independent" of $e \backslash e(s)$, since when $r \leq s$ we have $e(x, r) = e(s) \cap [x, r]$. Consider the covering

$$e^*(s) \subset \bigcup_{r \in e^*(s)} (x(r), 2r - x(r)) .$$

The length of the intervals in this covering is bounded:

$$2r - x(r) - x(r) = 2(r - x(r)) \le \frac{2|e(s)|}{\varepsilon}.$$

Therefore we can choose a subcovering whose multiplicity is at most two ([1], Chapter VI, proof of Theorem 3.2). Let $r_n$ be the centers of the intervals in this subcovering. We have

$$|e^*(s)| \le 2\sum_n (r_n - x(r_n)) \le \frac{2}{\varepsilon}\sum_n |e(x(r_n), r_n)| \le \frac{4}{\varepsilon}|e(s)|.$$

We now prove (5.14). Let $\lambda(x) = |e(x, r)|$ for a given $r \in \mathbb{R}_+ \backslash e^*$. Then $\lambda(x) \le \varepsilon(r - x)$ and

$$\begin{aligned}\varepsilon\int_{[\rho, r]} a(x)\,dx - \int_{e(\rho, r)} a(x)\,dx &= \int_\rho^r a(x)\,d(\lambda(x) - \varepsilon(r - x))\\ &= -\int_\rho^r (\lambda(x) - \varepsilon(r - x))\,da(x) - a(\rho)(\lambda(\rho) - \varepsilon(r - \rho))\\ &= \int_\rho^r (\varepsilon(r - x) - \lambda(x))\,da(x) + a(\rho)(\varepsilon(r - \rho) - \lambda(\rho)) \ge 0,\end{aligned}$$

as required. This completes the proof of the lemma.

In Lemma 9, we take $\rho = 0$, $a(x) = A(\exp x)$, and $e = \{x \in (0, \infty) : \exp x \in E\}$, where $E$ is an exceptional set of finite logarithmic length in the unintegrated second fundamental theorem (5.2). Then $e$ is a set of finite length; the set $e^*$ constructed in Lemma 9 also has finite length; and $E^* = \{r \in [1, \infty) : \log r \in e^*\}$ is a set of finite logarithmic length. Then by (5.14)

$$\int_{E(r)} \frac{A(t)}{t}\,dt \le \varepsilon\int_1^r \frac{A(t)}{1}\,dt \le \varepsilon T(r, f), \qquad r \notin E^*. \tag{5.15}$$

If we divide (5.2) by $r$ and integrate, using (5.15), we obtain

$$d(q - 2n)T(r, f) \le \sum_{k=1}^q N(r, Q_k) + \varepsilon T(r, f). \quad \|$$

It remains only to replace the number $\varepsilon > 0$ by a function $\varepsilon(r) \to 0$ in the same way as in the proof of (5.2). This completes the proof of the theorem.

## §6. The case of variable coefficients

Let us recall some definitions. Let $\sigma$ be a function that increases to $+\infty$ on $[0, \infty)$, and $M_\sigma$ a field of meromorphic functions $a(z)$ such that

$$T(r, a) = O(\sigma(r)), \qquad r \to \infty.$$

Let $B_\sigma$ be a set of continuous functions $h : \mathbb{C} \to \mathbb{R} \cup \{\pm\infty\}$, such that

$$\int_0^{2\pi} |h(re^{i\theta})|\,d\theta = O(\sigma(r)), \qquad r \to \infty.$$

If $g \in M_\sigma$, we have $\log|g| \in B_\sigma$. For a field $K$ we denote by $K[x_0, \dots, x_n]$ the ring of polynomials in $n + 1$ variables over $K$. For a polynomial $F \in M_\sigma[x_0, \dots, x_n]$ we denote by $F(z) \in \mathbb{C}[x_0, \dots, x_n]$ the polynomial over $\mathbb{C}$ obtained by substituting a specific value $z \in \mathbb{C}$ into the coefficients of $F$. In addition, for any polynomial $F \in M_\sigma[x_0, \dots, x_n]$ of degree $d$ we have

$$|F(z)(w_0, \dots, w_n)| \le H(z)\|w\|^d, \qquad \log H \in B_\sigma, \tag{6.1}$$

where $w = (w_0, \dots, w_n) \in \mathbb{C}^{n+1}$ and $\|w\| = \max_{0\le j\le n} |w_j|$.

Let $S = \{Q_k\}_{k=1}^q$, $q > 2n$, be a system of homogeneous forms in $M_\sigma[x_0, \dots, x_n]$, of degree $d$. A system $S$ is said to be admissible if for every set $I \in I_{n+1}$ there is a $z \in \mathbb{C}$ such that the system of equations

$$Q_k(z)(w) = 0, \qquad k \in I, \tag{6.2}$$

has only the trivial solution $w = 0$ in $\mathbb{C}^{n+1}$.

We specify $I \in I_{n+1}$ and consider homogeneous forms $Q_k(z) \in \mathbb{C}[x_0, \dots, x_n]$, $k \in I$, in the system $S$. We need the concept of the resultant of a system of $n+1$ forms in $n+1$ variables (see, for example, [26], Chapter XI). The resultant $R_I(z)$ is an integral-valued polynomial in the coefficients of the forms $Q_k(z)$, $k \in I$, such that the condition $R_I(z) = 0$ is necessary and sufficient for the existence of a nontrivial solution $w \in \mathbb{C}^{n+1}$, $w \neq 0$, for the system (6.2); the existence of such a polynomial is established in [26], Chapter XI. From the definition of the resultant and the properties of the Nevanlinna characteristic ([1], Chapter I) it follows that $R_I \in M_\sigma$. Consequently the admissibility of $S$ means that $R_I(z) \not\equiv 0$ for every set $I \in I_{n+1}$; consequently, if $S$ is admissible, for all $z$ except for a discrete set the system (6.2) of equations has no nontrivial solutions.

It is proved in [26], Chapter XI, that there is a number $s \in \mathbb{N}$ such that, for all $j$, $0 \le j \le n$, the identity

$$x_j^s R_I(z) = \sum_{k\in I} b_{kj}(z)(x_0, \dots, x_n) Q_k(z)(x_0, \dots, x_n) \tag{6.3}$$

is satisfied, where the $b_{kj}(z) \in \mathbb{C}[x_0, \dots, x_n]$ are polynomials whose coefficients are integral-valued polynomials in the coefficients of $Q_k$, $k \in I$. Therefore $b_{kj} \in M_\sigma[x_0, \dots, x_n]$.

Let $w \in \mathbb{C}^{n+1}$, $\|w\| = 1$. From (6.1) and (6.3) we obtain

$$\begin{aligned} |w_j|^s |R_I(z)| &\le \sum_{k\in I} |b_{kj}(z)(w_0, \dots, w_n)|\,|Q_k(z)(w_0, \dots, w_n)| \\ &\le H(z) \sum_{k\in I} |Q_k(z)(w_0, \dots, w_n)| \\ &\le H(z) \max_{k\in I} |Q_k(z)(w_0, \dots, w_n)|, \qquad \log H \in B_\sigma, \quad 0 \le j \le n. \end{aligned}$$

If we sum the resulting inequalities over $j$, $0 \le j \le n$, and use the inequality

$$\sum_{j=0}^n |w_j|^s \ge c, \qquad \|w\| = 1,$$

we obtain

$$|R_I(z)| \le cH(z) \max_{k\in I} |Q_k(z)(w_0, \dots, w_n)|.$$

Since $R_I \in M_\sigma$, we then obtain, for $\|w\| = 1$,

$$H_1(z) \le \max_{k\in I} |Q_k(z)(w_0, \dots, w_n)|, \qquad \log H_1 \in B_\sigma. \tag{6.4}$$

If we use the homogeneity of the forms $Q_k$ and the inequalities (6.1) and (6.4), we finally obtain

$$\begin{aligned} H_1(z)\|w\|^d \le \max_{k\in I} |Q_k(z)(w_0, \dots, w_n)| \le H_2(z)\|w\|^d, \\ w \in \mathbb{C}^{n+1}\setminus\{0\}, \quad \log H_i \in B_\sigma, \quad I \in I_{n+1}. \end{aligned} \tag{6.5}$$

*Proof of Theorem* 3. For a meromorphic curve $f$ that has the representation $f = (f_0, \dots, f_n)$ in homogeneous coordinates, we set

$$v = \bigvee_{j=0}^{n} \log|f_j|,$$

$$v_k(z) = \log|Q_k(z)(f_0(z), \dots, f_n(z))|, \qquad 1 \le k \le q, \ q > 2n,$$

where the forms $Q_k \in M_\sigma[x_0, \dots, x_n]$ with

$$\sigma(r) = \frac{T(r, f)}{\log^\tau T(r, f)}, \qquad \tau > 1,$$

form an admissible system $S$. As before, $\mu$ denotes the Riesz measure of the subharmonic function $v$; then $A(r) = A(r, f) = \mu(\mathscr{D}(r))$.

If we substitute the coordinates of $f$ in (6.5), we obtain an analog of (3.2): for every set $I \in I_{n+1}$ we have

$$h(z) = \left|\bigvee_{k\in I} v_k - dv\right| \in B_\sigma. \tag{6.6}$$

We specify numbers $\varepsilon > 0$ and $\eta$, $1 < \eta < \tau$. We apply Lemma 8 of §5 and select unexceptional values $r$ and $r'$ such that

$$r' = r + \frac{r}{\log^\eta A(r)} = r + \Delta, \tag{6.7}$$

$$A(r') \le (1+\varepsilon)A(r), \tag{6.8}$$

and $E_0$ is the set of exceptional values $r'$ for which (6.8) is not satisfied; its logarithmic length is finite. Applying Lemma 6 from §4, we construct a partition of unity corresponding to the selected values of $r$ and $\Delta$, and let

$$a_j = \int \varphi_j \, d\mu,$$

where the $\varphi_j$ form the partition of unity in $\mathscr{D}(r)$.

As in §5, we transfer the $\delta$-subharmonic functions $v_k$, $1 \le k \le q$, and the subharmonic function $v$, from the disk $\mathscr{D}_j = \mathscr{D}(z_j, \rho_j)$ to $\mathscr{D}(1)$, normalizing these functions by $a_j$. We let

$$U_k^j(z) = \frac{1}{a_j} v_k(\rho_j z + z_j), \qquad U^j(z) = \frac{d}{a_j} v(\rho_j z + z_j).$$

As in the proof of Theorem 1, we need to select a set of indices $J^* \subset J = \{1, 2, \dots, p\}$ so that when $j \in J^*$ the functions $u_k^j$ and $U^j$ satisfy the hypotheses of Theorem 2 and

$$\sum_{j\in J\setminus J^*} a_j \le \varepsilon A(r). \tag{6.9}$$

The main technical difficulty is in estimating the numbers

$$\alpha_{I,j} = \left\|\bigvee_{k\in I} U_k^j - U^j\right\|, \qquad I \in I_{n+1}, \ j \in J^*,$$

where, as before,

$$\|\psi\| = \iint_{\mathscr{D}(1)} |\psi| \, dx \, dy.$$

We set

$$\sum(r) = \sum_{r=1}^{p} a_j \alpha_{I,j}.$$

From the definition of $\alpha_{I,j}$, and (6.6), we obtain

$$\sum(r) \le c \sum_{j=1}^{p} \frac{1}{\rho_j^2} \iint_{\mathscr{D}_j} h(x+iy)\, dx\, dy, \qquad h \in B_\sigma.$$

Let us set

$$K_m = \mathscr{D}(r' - 2^{m-1}\Delta) \backslash \mathscr{D}(r' - 2^m \Delta), \qquad 1 \le m \le l = 1 + \left[\log_2 \frac{r'}{\Delta}\right].$$

Then by properties 2)–4) of Lemma 6, we obtain

$$\begin{aligned} \sum(r) &\le c \sum_{m=1}^{l} \sum_{j:\mathscr{D}_j \cap K_m \ne \varnothing} \frac{1}{\rho_j^2} \iint_{\mathscr{D}_j} h(x+iy)\, dx\, dy \\ &\le c \sum_{m=1}^{l} \frac{1}{2^{2m}\Delta^2} \int_0^{2\pi} \int_{r'-2^m\Delta}^{r'-2^{m-1}\Delta} h(te^{i\theta}) t\, dt\, d\theta \qquad (6.10) \\ &\le c \sum_{m=1}^{l} \frac{\sigma(r') 2^m \Delta r}{2^{2m}\Delta^2} \le c \frac{\sigma(r') r}{\Delta} \le c \frac{T(r') \log^\eta A(r)}{\log^\tau T(r')}. \end{aligned}$$

For a given $\alpha > 0$, we set

$$J(\alpha) = \{ j \in J : \alpha_{I,j} \le \alpha \}. \tag{6.11}$$

Then, by (6.10), we have the following estimate:

$$\sum_{j \in J \backslash J(\alpha)} a_j = \sum_{j \in J \backslash J(\alpha)} \frac{a_j \alpha_{I,j}}{\alpha_{I,j}} \le \frac{1}{\alpha} \sum(r) \le \frac{c}{\alpha} \frac{T(r')}{\log^\tau T(r')} \log^\eta A(r). \tag{6.12}$$

If $A(r)$ increases slowly, the characteristic $T(r)$ cannot be estimated above by $A(r)$; consequently, generally speaking, we cannot deduce the required inequality (6.9) from (6.12). Therefore we consider separately the values of $r$ for which $T(r)$ is not bounded above by $A(r)$; for these values we directly prove the integrated second fundamental theorem.

We set

$$F_1 = \{ r' : T(r') \le A(r') \log^\gamma A(r') \} \backslash E_0,$$
$$F_2 = \{ r' : T(r') > A(r') \log^\gamma A(r') \},$$

where $E_0$ is the exceptional set from Lemma 8, on which (6.8) is not satisfied, and $\gamma = (\tau - \eta)/2$.

We consider two cases.

*Case* 1. $r \in F_1$.

Let $\nu^j$ denote the Riesz measure of the function $U^j$ and $\nu_k^j$ the Riesz charges of $U_k^j$. To apply Theorem 2, we need to select the values of $j \in J$ for which $U_k^j$ has a "nearly nonnegative" Riesz charge; more precisely, we need to estimate the numbers

$$\sum_{k=1}^{q} (\nu_k^j)^- (\overline{\mathscr{D}(1)}) \overset{\text{def}}{=} b_j.$$

We denote by $n(r, S)$ the total number of poles (counting multiplicity) of the coefficients of the forms $Q_k$, $1 \le k \le q$,

$$N(r, s) = \int_{r_0}^{r} \frac{n(t, S)}{t}\, dt.$$

From the hypotheses of our Theorem 3 and Nevanlinna's first fundamental theorem, it follows that

$$N(r, S) = O(\sigma(r)) = O\left(\frac{T(r)}{\log^{\tau} T(r)}\right), \qquad r \to \infty.$$

Let us consider the set

$$E_1 = \{r' : n(r', S) \ge N(r', S) \log^{\eta} N(r, S)\}.$$

We have

$$\int_{E_1} d\log t \le \int_{E_1} \frac{n(t, S)}{N(t, S)\log^{\eta} N(t, S)}\, d\log t = \int_{E_1} \frac{dN(t, S)}{N(t, S)\log^{\eta} N(t, S)} < \infty,$$

so that the set $E_1$ has finite logarithmic length. However, if $r \in F_1 \backslash E_1$, we have

$$\begin{aligned} n(r', S) &\le N(r', S)\log^{\eta} N(r', S) \\ &\le T(r')\log^{\eta-\tau} T(r') \le A(r')\log^{-\gamma} A(r'). \end{aligned} \tag{6.13}$$

If we now use properties 2) and 3) of the partition in Lemma 6, and (6.13), we obtain

$$\begin{aligned} \sum_{j=1}^{p} a_j b_j &= \sum_{j=1}^{p} a_j \left(\frac{1}{a_j}\sum_{k=1}^{q} \mu_k^-(\overline{D}_j)\right) \\ &\le c\sum_{k=1}^{q} \mu_k^-(\mathscr{D}(r')) \le cn(r', S) \le cA(r')\log^{-\gamma} A(r'). \end{aligned} \tag{6.14}$$

We denote by $J^* \subset J = \{1, 2, \dots, p\}$ the set of indices $j$ for which we have

$$\left(\mu + \sum_{k=1}^{q} \mu_k\right)(\overline{\mathscr{D}_j}) \le \frac{1}{\varepsilon} a_j, \tag{6.15}$$

$$\alpha_{I,j} \le \log^{-\gamma/2} A(r'), \tag{6.16}$$

$$b_j \le \log^{-\gamma/2} A(r'). \tag{6.17}$$

Let $J_1$, $J_2$, $J_3$ be the respective sets of indices for which (6.15)–(6.17) are not satisfied. By Lemma 7 (conclusion 2)) and properties 2) and 3) of the partition in Lemma 6,

$$\sum_{j\in J_1} a_j \le \varepsilon \sum_{j\in J}\left(\mu + \sum_{k=1}^{q}\mu_k\right)(\overline{\mathscr{D}_j}) \le c\varepsilon\left(A(r') + \sum_{k=1}^{q}\mu_k(\mathscr{D}(r'))\right). \tag{6.18}$$

In addition, if we use (6.11) and (6.12), $\alpha = \log^{-\gamma/2} A(r')$, and the monotonicity of $\psi(T) = T\log^{-\tau} T$, we obtain

$$\begin{aligned} \sum_{j\in J_2} a_j &\le c\log^{\gamma/2} A(r')\frac{A(r')\log^{\gamma} A(r')}{\log^{\tau} A(r')}\log^{\eta} A(r) \\ &\le cA(r')\log^{-\gamma/2} A(r'). \end{aligned} \tag{6.19}$$

Finally, by (6.14) and the definition of $J_3$ we obtain

$$\sum_{j\in J_3} a_j = \sum_{j\in J_3} \frac{a_j b_j}{b_j} \le \log^{\gamma/2} A(r') \sum_{j\in J} a_j b_j \le cA(r') \log^{-\gamma/2} A(r'). \tag{6.20}$$

Combining (6.18)–(6.20), we obtain

$$\sum_{j\in J\setminus J^*} \int \varphi_j \, d\mu = \sum_{j\in J\setminus J^*} a_j \le c\varepsilon \left( A(r') + \sum_{k=1}^{q} \mu_k(\mathscr{D}(r')) \right), \qquad r' \in F_1\setminus E_1. \tag{6.21}$$

If $j \in J^*$, we can apply Theorem 2′ to the functions $U^j$ and $U_k^j$. For $r' \ge r_1$ and $r' \in F_1\setminus E_1$ its hypotheses are satisfied by (6.15)–(6.17). We find that, for $r' \ge r_1$, $r' \in F_1\setminus E_1$, and $j \in J^*$, we have

$$\int \left\{ \sum_{j\in J^*} \varphi_j \right\} d\left( \sum_{k=1}^{q} \mu_k - d(q-2n)\mu \right) \ge -\varepsilon \int \left( \sum_{j\in J^*} \varphi_j \right) d\mu. \tag{6.22}$$

If we combine (6.21) and (6.22), as in §5, we obtain

$$d(q-2n)A(r') \le (1+c\varepsilon) \sum_{k=1}^{q} \mu_k(\mathscr{D}(r')), \qquad r' \in F_1\setminus E_1. \tag{6.23}$$

*Case* 2. $r \in F_2$. Then

$$T(r) - T(r/2) = \int_{r/2}^{r} \frac{A(t)}{t}\, dt \le A(r)\log 2 \le T(r)\log 2/\log^{\gamma} A(r) = o(T(r)),$$

i.e.,

$$T(r/2) \sim T(r), \qquad r \in F_2,\ r \to \infty. \tag{6.24}$$

Let us show that

$$d(q-n)T(r/2) \le \sum_{k=1}^{q} N(r/2, Q_k) + o(T(r)) \tag{6.25}$$

when $r \to \infty$, $r \in F_2$. (By (6.24), this is stronger than (1.6).) Let (6.25) not be satisfied. Then there is a sequence $r_j \to \infty$, $r_j \in F_2$, such that

$$d(q-n)T(r_j/2) \ge \sum_{k=1}^{q} N(r_j/2, Q_k) + \varepsilon T(r_j), \qquad \varepsilon > 0. \tag{6.26}$$

We consider the subharmonic function

$$v = \bigvee_{j=0}^{n} \log|f_j|$$

with Riesz measure $\mu$ and $\delta$-subharmonic functions

$$v_k(z) = \log|Q_k(z, f_0(z), \dots, f_n(z))|$$

with charges $\mu_k$. By the first fundamental theorem, we have

$$\left( \sum_{k=1}^{q} \mu_k^+ \right) \left( \mathscr{D}\left( \frac{2}{3} r \right) \right) = O(T(r)), \qquad r \to \infty, \tag{6.27}$$

$$\left( \sum_{k=1}^{q} \mu_k^- \right) \left( \mathscr{D}\left( \frac{2}{3} r \right) \right) = o(T(r)), \qquad r \to \infty, \tag{6.28}$$

$$\mu(\overline{\mathscr{D}(r)}) = A(r) = o(T(r)), \qquad r \to \infty,\ r \in F_2. \tag{6.29}$$

Let $H_r$ be the least harmonic majorant of $v$ in $\mathscr{D}(\frac{2}{3}r)$. We set

$$P^j(z) = d(T(r_j))^{-1}(v(r_j z) - H_{r_j}(r_j z)),$$
$$P_k^j(z) = (T(r_j))^{-1}(v_k(r_j z) - H_{r_j}(r_j z)), \qquad z \in \mathscr{D}(\tfrac{2}{3}).$$

Then the $P^j$ are Green potentials that form a compact family. By (6.6), we have

$$\int_0^{2\pi} \left| \bigvee_{k\in I} P_k^j(re^{i\theta}) - P^j(re^{i\theta}) \right| d\theta \to 0, \tag{6.30}$$

$I \in I_{n+1}$, $j \to \infty$, for every $r$, $0 < r < 1$.

If we argue as in the proof of Theorem 2′, and use (6.27)–(6.29), we can select a subsequence of indices $j$ so that $P^j \to u$ and $P_k^j \to u_k$. It follows from (6.29) that $u = 0$.

The functions $u_k$ are subharmonic or else identically $-\infty$. By formula (6.30), we have

$$\bigvee_{k\in I} u_k = 0, \qquad I \in I_{n+1}, \text{ in } \mathscr{D}(\tfrac{2}{3}).$$

This means that the functions $u_k$, except, perhaps, for $n$ of them, are identically 0. For definiteness, let $u_1 = u_2 = \cdots = u_{q-n} = 0$. Then

$$\int_0^{2\pi} \left( \sum_{k=1}^{q-n} P_k^j - (q-n)P^j \right) \left( \frac{1}{2} e^{i\theta} \right) d\theta \to 0, \qquad j \to \infty,$$

and consequently

$$\int_0^{2\pi} \left( \sum_{k=1}^{q-n} v_k(r_j e^{i\theta}/2) - d(q-n)v(r_j e^{i\theta}/2) \right) d\theta = o(T(r_j)), \qquad j \to \infty. \tag{6.31}$$

Now we obtain from (6.31), taking account of the definitions of $T(r)$ and $N(r, Q_k)$,

$$\begin{aligned} d(q-n)T(r_j/2) &\le \sum_{k=1}^{q-n} N(r_j/2, Q_k) + o(T(r_j)) \\ &\le \sum_{k=1}^{q} N(r_j/2, Q_k) + o(T(r_j)), \qquad j \to \infty, \end{aligned}$$

which contradicts (6.26). Therefore (6.25) is proved; if we take account of (6.24), it leads to

$$d(q-2n)T(r) \le \sum_{k=1}^{q} N(r, Q_k) + o(T(r)), \qquad r \to \infty, \ r \in F_2. \tag{6.32}$$

To complete the proof, it remains only to deduce (1.6) from (6.23) and (6.32).

**Lemma 10.** *Let* $F_1 \cup F_2 \cup E = [1, \infty)$, *let* $A(t)$ *tend monotonically to* $\infty$, *and let* $\eta(t) \le A(t)$. *Let us suppose that*

$$\int_E d\log t < \infty,$$
$$\eta(t) < \varepsilon A(t), \qquad t \in F_1,$$
$$\int_1^r \frac{\eta(t)}{t}\, dt < \varepsilon \int_1^r \frac{A(t)}{t}\, dt, \qquad r \in F_2.$$

*Then there is a set* $E^*$ *with the properties*

$$\int_{E^*} d\log t < \infty, \tag{6.33}$$

$$\int_1^r \frac{\eta(t)}{t}\,dt \le 2\varepsilon \int_1^r \frac{A(t)}{t}\,dt, \qquad r \notin E^*.$$

*Proof.* We apply Lemma 9 with $e = \{x \in (0, \infty) : \exp x \in E\}$. Let $e^*$ be the exceptional set constructed in Lemma 9, and $E^* = \{r \in [1, \infty) : \log r \in e^*\}$. The set $e$, and therefore $e^*$, have finite length; consequently (6.33) is satisfied.

Let $R \in F_1 \backslash E^*$, and set $\rho(r) = \sup\{\rho \in F_2 : \rho < r\}$. If $\{\rho \in F_2 : \rho < r\} = \varnothing$, we set $\rho(r) = 1$. We have, by Lemma 9 and the definitions of $F_1$ and $F_2$,

$$\begin{aligned}\int_1^r \frac{\eta(t)}{t}\,dt &= \int_1^{\rho(r)} \eta(t)\,d\log t + \int_{F_1\cap[\rho(r),r]} \eta(t)\,d\log t + \int_{E\cap[\rho(r),r]} \eta(t)\,d\log t \\ &\le \varepsilon \int_1^{\rho(r)} A(t)\,d\log t + \varepsilon \int_{\rho(r)}^r A(t)\,d\log t + \varepsilon \int_{\rho(r)}^r A(t)\,d\log t \\ &\le 2\varepsilon \int_1^r \frac{A(t)}{t}\,dt.\end{aligned}$$

This completes the proof of the lemma.

If we take, in this lemma,

$$\eta(r) = d(q-2n)A(r) - \sum_{k=1}^q \mu_k(\mathscr{D}(r'))$$

and apply (6.23) and (6.32), we obtain

$$d(q-2n)T(r) \le \sum_{k=1}^q N(r, Q_k) + 2\varepsilon T(r). \quad \|$$

From this, inequality (1.6) follows by the standard method used in the proof of Theorem 1.

*Remark* 2. In the second case considered in Theorem 3, we have proved, in fact, a stronger conclusion than was required. Let us state this.

*On some unbounded set* $F$ *let*

$$A(r) = o(T(r)), \qquad r \in F, \; r \to \infty. \tag{6.34}$$

(*This implies* (6.24).) *Let us choose a function* $\sigma(r) = o(T(r))$, $r \to \infty$, *and consider an admissible system of forms of degree* $d$: $Q_k \in M_\sigma[x_0, \dots, x_n]$, $1 \le k \le q$, $q > n$. *Then*

$$d(q-n)T(r) \le \sum_{k=1}^q N(r, Q_k) + o(T(r)), \qquad r \in F, \; r \to \infty,$$

*and, in particular,*

$$\sum_{k=1}^q \delta(Q_k, f) \le n.$$

*Remark* 3. An analysis of the preceding proof shows that if

$$\varliminf_{r\to\infty} \frac{A(r,f)}{T(r,f)} > 0, \tag{6.35}$$

then under the hypotheses of Theorem 3 the unintegrated inequality (1.5) is valid. In fact, (6.35) makes it possible not to consider two cases separately, but to repeat the analysis of the first case, using (1.5).

For the proof of Theorem 4, we require an additional lemma on monotonic functions.

We recall that $E(r) = E \cap [0, r]$, $E \subset \mathbb{R}_+$.

**Lemma 11.** *Let* $\psi(x)$ *have the property that*

$$\psi(x)/x \uparrow +\infty, \qquad x \to +\infty. \tag{6.36}$$

*We consider the set*

$$G = \{r \geq e : T(r) \leq \log^2 r\}. \tag{6.37}$$

*Then there is a set* $E \subset G$ *for which*

$$A(r) = o(T(r)), \qquad r \to \infty, \; r \in G \backslash E, \tag{6.38}$$

$$\lim_{r\to\infty} \frac{1}{\psi(\log\log r)} \int_{E(r)} \frac{d\rho}{\rho} = 0. \tag{6.39}$$

*Proof*. Let $E^\kappa = \{r \in G : A(r) \geq \kappa T(r)\}$, $\kappa > 0$. By (6.37), we have

$$\int_{E^\kappa(t)} d\log\rho \leq \frac{1}{\kappa}\int_{E^\kappa(t)} \frac{A(\rho)}{\rho T(\rho)}\,d\rho \leq \frac{1}{\kappa}\int_e^t d\log T(\rho) \leq \frac{\log T(t)}{\kappa} \leq \frac{2\log\log t}{\kappa}.$$

Therefore, for a sequence $\kappa_j \to 0$, $j \to \infty$, we can find a sequence $t_j \to \infty$ of numbers for which

$$\int_{E^\kappa(t)} d\log\rho \leq 2^{-j}\psi(\log\log t), \qquad t \geq t_j.$$

Let us set

$$E = \bigcup_{j=1}^{\infty}(E^{\kappa_j} \cap [t_j, t_{j+1}]) \cup [e, t_1],$$

and show that this set is the one required.

In fact, if

$$r \in [t_j, t_{j+1}] \cap (G\backslash E) = [t_j, t_{j+1}] \cap (G\backslash E^{\kappa_j}),$$

we have

$$A(r) \leq \kappa_j T(r), \tag{6.40}$$

$$\frac{1}{\psi(\log\log t)}\int_{E(t)} d\log\rho \leq 2^{-j}. \tag{6.41}$$

The estimates (6.38) and (6.39) follow from (6.40) and (6.41). This completes the proof of the lemma.

*Proof of Theorem* 4. Let

$$G_1 = \{r \geq e : T(r) \geq \log^2 r\}, \qquad G_2 = \{r \geq e : T(r) < \log^2 r\}.$$

The coefficients of the forms $Q_k$ are rational functions, and their characteristics are

$$O(\log r) = O(\sqrt{T(r)}), \qquad r \in G_1, \; r \to \infty.$$

Therefore when $r \in G_1$ we may repeat the reasoning in the proof of Theorem 3.

Let

$$F_1 = \{r \in G_1 : T(r) \leq A(r) \log A(r)\} \backslash (E_0 \cup E_1),$$

where $E_0$ and $E_1$ are the exceptional sets of finite logarithmic length constructed for the proof of Theorem 3. Then the unintegrated inequality (1.5) is satisfied on $F_1$, and so is the integrated inequality (1.6) on $G_1 \backslash F_1$.

Using Lemma 11, we extract from $G_2$ a subset $E_2$ with properties (6.38) and (6.39). Then, by Remark 2, the integrated inequality (1.6) is satisfied on $G_2 \backslash E_2$.

We set $E_0 \cup E_1 \cup E_2$; then (6.39) is satisfied for this exceptional set. Let

$$F_2 = (G_1 \backslash (F_1 \cup E_0 \cup E_1)) \cup (G_2 / E_2),$$

$$\eta(r) = d(q-2n)A(r) - \sum_{k=1}^{q} \mu_k(\mathscr{D}(r')) \leq (q-2n)\, d A(r).$$

We have $[e, \infty) = F_1 \cup F_2 \cup E$, and

$$\eta(r) \leq \varepsilon A(r), \qquad r \in F_1,$$

$$\int_e^r \frac{\eta(t)}{t}\, dt \leq \varepsilon \int_e^r \frac{A(t)}{t}\, dt, \qquad r \in F_2.$$

Hence, as in the proof of Theorem 3, we conclude with the help of Lemma 9 that (1.6) is satisfied for $r \notin E^*$, where the exceptional set $E^*$, like $E$, satisfies (6.39). Thus Theorem 4 is established.

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Physico-Technical Institute of Low Temperatures
Academy of Sciences of the Ukrainian SSR
Kharkov

Received 27/NOV/89

Translated by R. P. BOAS