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The Arithmetic of Entire Functions under Composition

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1. INTRODUCTION

In this paper, we prove, among other things, that any family of nonconstant entire functions of one complex variable has a greatest common right factor under composition. We prove a corresponding result for any family of pairwise dependent entire functions of N complex variables. Since f and $af+b$, where $a, b \in \mathbb{C}$ and $a \neq 0$ have all the same properties from the point of view of factoring under composition, we once and for all identify them, but continue (perhaps a little improperly) to talk about the equivalence classes as "functions." Moreover we identify f and $f \circ A$ where A is a biholomorphic one-to-one map of $\mathbb{C}^N$ onto $\mathbb{C}^N$. Here are some definitions.

Let $f = f(z_1, ..., z_n)$ be a nonconstant entire function of N complex variables, and let $g = g(w)$ be a nonconstant holomorphic function in the region consisting of the complex plane minus any possible value omitted by f. Then the composition $h = g \circ f$, $h(z_1, ..., z_N) = g(f(z_1, ..., z_N))$ is a well-defined entire function, and in this case we call f a right factor of h. If we have a family of the form $\{h_\alpha\} = \{g_\alpha \circ f\}$, then we say that f is common right factor of $\{h_\alpha\}$. If we have $h = g_\alpha \circ f_\alpha$, where α runs over an index set A, then we call h a common left multiple of the f_α.

For entire functions f and g of N complex variables, define $f \leqslant g$ if $f(z) = f(w)$ implies $g(z) = g(w)$, $z, w \in \mathbb{C}^N$. Then we remark that $f \leqslant g$ if and only if f is a right factor of g. The "if" part is trivial. For the "only

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if" part, just define $h = g(f^{-1})$ as a germ at some point and then continue analytically along any curve in the image of f. We may ignore the singularities of f^{-1} because of Iversen's theorem [NEV], which says that for any curve there exist arbitrarily small deformations of this curve such that f^{-1} can be analytically continued along these deformed curves. If we describe a closed path, f^{-1} takes the values a and b at the beginning and end of the path respectively. We have $f(a) = f(b)$. Thus $g(a) = g(b)$ by assumption. So h is single-valued and $g = h \circ f$.

Thus we have made the nonconstant entire functions of N complex variables, for each fixed N, into a lattice. It is natural to ask about glb's (greatest lower bounds) and lub's (least upper bounds) within this lattice.

Given a family $\{f_\alpha\}$ of nonconstant entire functions of N complex variables, we say that a nonconstat entire function g of N complex variables is a weak greatest common right factor of $\{f_\alpha\}$ if (writing in terms of the ordering) $g \leqslant f_\alpha$ for all α and if $h \leqslant f_\alpha$ for all α and $g \leqslant h$ implies $g = h$. Thus a weak greatest common right factor is a glb in the weak sense. There is no prior guarantee that there are not several different glb's.

In the same context, we say that g is a strong greatest common right factor of $\{f_\alpha\}$ if $g \leqslant f_\alpha$ for all α and if $h \leqslant f_\alpha$ for all α implies $h \leqslant g$.

Thus g is a greatest common right factor of $\{f_\alpha\}$ if, first of all, it is a common right factor of $\{f_\alpha\}$ and, secondly, if every other common right factor of $\{f_\alpha\}$ is a right factor of g.

Similarly, we define weak (and strong) least common left multiples of $\{f_\alpha\}$. We remark that even in the simplest situations, there may be no common left multiple at all. For example e^z and e^{iz} have no common left multiple. That is, the equation $F(e^z) = G(e^{iz})$ has no nonconstant entire solutions, for then $F(e^z)$ would be a doubly periodic entire function.

Given two nonconstant entire functions f and g on $\mathbb{C}^N$, we say that f and g are dependent if

$$\begin{vmatrix} \dfrac{\partial f}{\partial z_i} & \dfrac{\partial f}{\partial z_j} \\ \dfrac{\partial g}{\partial z_i} & \dfrac{\partial g}{\partial z_j} \end{vmatrix} \equiv 0 \qquad \text{for all} \quad i, j = 1, ..., N.$$

This is the same as saying that the rank of the Jacobian matrix of f and g is 1. We note that any family of nonconstant entire functions of *one* complex variable is dependent.

Here is an instructive example suggested by C. C. Yang. Let

$$F(z) = ze^z \circ e^z = e^z \circ (z + e^z). \tag{1.1}$$

Suppose

$$f(z) = A(z) \circ e^z = B(z) \circ (z + e^z), \tag{1.2}$$

i.e., that f has both e^z and $z + e^z$ as right factors. We claim then that

$$f(z) = C(z) \circ F(z). \tag{1.3}$$

In other words, $F(z)$ is the least common left multiple of e^z and $(z + e^z)$. To prove the claim, we have from (1.2)

$$B(z) \circ (z + 2\pi i + e^z) = B(z + e^z), \tag{1.4}$$

and on letting $w = z + e^z$, we get

$$B(w + 2\pi i) = B(w). \tag{1.5}$$

Hence, for some $C(z)$,

$$B(z) = C(z) \circ e^z \tag{1.6}$$

and so from (1.2) we get

$$f(z) = C(z) \circ e^z \circ (z + e^z) = C(z) \circ F(z),$$

as claimed.

Our first two theorems are special cases of the succeeding two theorems, but they permit shorter and more direct proofs, given in Section 2.

THEOREM 1.1. *Any family of nonconstant entire functions of one complex variable has a (unique) strong greatest common right factor.*

THEOREM 1.2. *Any family of nonconstant entire functions of one complex variable that has a common left multiple has a (unique) strong least common left multiple.*

THEOREM 1.3. *Any family of pairwise dependent nonconstant entire functions of N complex variables has a (unique) strong greatest common right factor.*

THEOREM 1.4. *Any family of nonconstant entire functions of N complex variables that has a common left multiple has a (unique) strong least common left multiple.*

In Section 4, we will state and prove Theorem 1.3′ and 1.4′, which are the weak versions of Theorem 1.3 and Theorem 1.4, in the sense that we replace the word "strong" with the word "weak" in the conclusions. We will use these theorems to prove Theorems 1.3 and 1.4.

We say that a nonconstant entire function f of N variables is *prime* to mean that every right factor of f has the form $af+b$ for some $a, b \in \mathbb{C}$, $a \neq 0$. (In case $N=1$, of course, we also permit $az+b$ as a right factor.) For example, it was proved in [*RUY*] that if $A(z)$ and $B(w)$ are nonconstant entire functions of one variable, then $A(z)+B(w)$ is a prime entire function of two variables. It follows from Theorem 1.3 that if f and g are dependent entire functions of N complex variables, for $N \geqslant 2$, and if f is a prime, then $g=h \circ f$ for some entire function h of one complex variable. For f and g have a strong greatest common right factor H; $f=\varphi \circ H$ and $g=\psi \circ H$. Because f is prime, we have $H=af+b$. So $g=\psi \circ (af+b)=\eta \circ f$.

2. THE CASE $N=1$

In this section, we give short proofs of Theorems 1.1 and 1.2 that depend on a theorem of Grauert on analytic equivalence relations. These proofs seem not to extend to the case $N \geqslant 2$.

Let f be a nonconstant holomorphic map of $\mathbb{C}$ to any Riemann surface. (By Picard's theorem, the Riemann surface can only be one of $\mathbb{C}$ or $\mathbb{C}^*=\mathbb{C}\backslash\{a\}$ or the sphere or a torus.) To any such map corresponds an equivalence relation $\sim$ in $\mathbb{C}$ defined by $x \sim y$ if and only if $f(x)=f(y)$. Consider the graph G of this equivalence relation in $\mathbb{C}^2$, i.e., the set of points (x, y) in $\mathbb{C}^2$ such that $f(x)=f(y)$. This is an analytic subset of $\mathbb{C}^2$ of pure codimension one. This means that every point of G has a neighborhood such that the intersection of G with this neighborhood coincides with the zero set of some nonconstant function analytic in this neighborhood. To see this, if f maps to $\mathbb{C}$ or $\mathbb{C}\backslash\{a\}$, we can just take the function $f(x)-f(y)$ in the above statement. If f maps to the sphere and x and y are not poles, we take $f(x)-f(y)$ again. If x and y are poles, take $1/f(x)-1/f(y)$. The argument in the case of the torus is the same, using local coordinates on the torus.

It is evident that the set G contains no "vertical" or "horizontal" lines, i.e., complex lines of the form $x=a$ or $y=a$. This is because f is nonconstant. Now for the converse statement, which is Grauert's theorem (see [GRA].) (We give only a limited version.)

THEOREM G. *Let R be any equivalence relation on $\mathbb{C}$ whose graph is an analytic subset of $\mathbb{C}^2$ containing no vertical or horizontal lines. Further suppose that the graph of R is everywhere of codimension one. Then there exists a holomorphic map f from $\mathbb{C}$ to one of the four Riemann surfaces listed above, such that xRy if and only if $f(x)=f(y)$.*

This is a very particular case of the theorem stated on the first page (p. 115) of [GRA]. In our one-dimensional case it can be proved easily and directly; we do this in the Appendix.

Now we turn to the factorization considerations. Let f be an entire function. We will say that g is its right factor if g is a holomorphic map from $\mathbb{C}$ to a Riemann surface S and there exists a holomorphic map h from S to $\mathbb{C}$ such that $f = h \circ g$.

Observation. Let F be the graph of the equivalence relation defined by f and let G be the graph of the equivalence relation defined by g. Then g is a right factor of f if and only if G is a subset of F.

Proof. Let $f = h(g)$. If (x, y) belongs to G then $g(x) = g(y)$, so $f(x) = f(y)$ and so (x, y) belongs to f. In the opposite direction, suppose G is a subset of F. We have to define h. Take $w \in S$, where S is the image surface of g. Let x be any g-preimage of w, and set $h(w) = f(x)$. This does not depend on the particular choice of the preimage because, by assumption, $g(x) = g(y)$ implies $f(x) = f(y)$. So a function h from S to $\mathbb{C}$ is defined and $f = h \circ g$. It is trivial that h is holomorphic.

We now prove that if G is an analytic subset of $\mathbb{C}^2$ which is the graph of an equivalence relation that contains no horizontal or vertical lines, then the derived set (i.e. the set of limit points) G' of G is also the graph of an equivalence relation. It is trivial that for every x, the point (x, x) belongs to G'. It is also trivial that if $(x, y) \in G'$ then $(y, x) \in G'$. It remains to prove that $(x, y) \in G'$ and $(y, z) \in G'$ implies $(x, z) \in G'$. Now (x, y) is not an isolated point of G. So there is a sequence (x_n, y_n) of points in G which tends to (x, y), and all the members of the sequence are different from (x, y). Because G contains no vertical lines, we may assume that all x_n are different from x. Now (y, z) is also not isolated in G and G does not contain the vertical line given by setting the first coordinate equal to y. Denote the coordinates in $\mathbb{C}^2$ by u and v. The analytic set G has dimension 1 at the point (y, z) because this point is not isolated. By the Local Uniformization Theorem (See [CHI], p. 71) a part of G in a neighborhood of (y, z) can be given by $u = p(t)$, $v = q(t)$, where t runs through a neighborhood of 0 in $\mathbb{C}$, and the functions p and q are holomorphic in this neighborhood, $p(0) = y$, $q(0) = z$, and neither p nor q is constant, because G does not contain vertical or horizontal lines. Because a nonconstant analytic function p is an open map, its image contains a whole neighborhood of y. So there is a sequence $t_n \to 0$ such that $y_n = p(t_n)$. Set $z_n = q(t_n)$. Then $(y_n, z_n) \in G$, $(y_n, z_n) \neq (y, z)$, and $(y_n, z_n) \to (y, z)$. This means that for all y_n close to y, the set G contains points (y_n, z_n) close to (y, z), but different from (y, z). Thus, because G is the graph of an equivalence relation, it contains a sequence (x_n, z_n) tending to (x, z). These points are all different from (x, z)

because x_n is different from x. So (x, z) belongs to the derived set G', which is what we wanted to prove.

Let $\mathscr{F}$ be a family of entire functions (of one variable). For each function, consider the graph of the corresponding equivalence relation, and take the intersection of all these graphs—it forms the graph of an equivalence relation. It is also an analytic set because the intersection of analytic sets is analytic [GUN]. Now let G be the perfect part of the intersection. (G is just the derived set of the intersection. From Theorem 15 on p. 89 of [GUR], it follows that G has no isolated points, because the isolated points are the irreducible varieties of dimension zero.) Then (by Theorem G) G corresponds to some function g mapping $\mathbb{C}$ to some Riemann surface. This function g is a common right factor of the family $\mathscr{F}$. Let k be another function which is a common right factor for the whole family. The graph K of the equivalence relation induced by k is contained in each graph of our family of graphs, and so it is contained in their intersection. But K has pure dimension 1. (This is true for any graph of an equivalence relation induced by a function.) So K is perfect. Thus K is contained in the derived set of the intersection—in other words, K is contained in G. So k is a right factor of g. Thus, Theorem 1.1 is proved, once we observe that S cannot be compact, since h maps S to $\mathbb{C}$, and h is not constant (Here, we choose one $f \in \mathscr{F}$ and write $f = h \circ g$).

We sketch the proof of Theorem 1.2. It is similar to the proof just given. Let $\mathscr{F}$ be a family of entire functions which are all right factors of some entire function f. Then the equivalence-relation graphs corresponding to the functions of $\mathscr{F}$ are contained in the graph F corresponding to f. Now take the intersection of all analytic subsets of $\mathbb{C}^2$ which are (i) contained in F, (ii) are the graphs of equivalence relations, and (iii) contain all the graphs of equivalence relations induced by functions of $\mathscr{F}$. This is a non-empty family because it contains F and the intersection is an analytic set, which we call G. It is the graph of some equivalence relation and it contains all the graphs corresponding to functions in $\mathscr{F}$.

Let G' be the derived set of G. It still has all the properties (i), (ii), (iii)-property (iii) because all the graphs corresponding to functions in $\mathscr{F}$ are perfect. Thus, we may apply Theorem G to conclude that there is a function g from $\mathbb{C}$ to some Riemann surface S corresponding to G'. This function is the "strong least common left multiple" of the family g. Finally, it is entire because it is a right factor of f.

3. THE Y-PROCESS

We prove a factorization result for degenerate mappings from $\mathbb{C}^N$ to $\mathbb{C}^2$. First we need the local version.

LEMMA 3.1. *Let* $F\colon (\mathbb{C}^N, A) \to (\mathbb{C}^2, B)$ *be a nonconstant germ of a holomorphic map, which is degenerate in the sense that the rank of the Jacobian matrix is* $\leqslant 1$ *at all points. Then F can be factored into germs of holomorphic maps in the following way:* $F = h \circ g$, *where*

$$g\colon (\mathbb{C}^N, A) \to (\mathbb{C}, 0) \tag{3.1}$$

and

$$h\colon (\mathbb{C}, 0) \to (\mathbb{C}^N, B), \tag{3.2}$$

where h is injective. The germ h is uniquely defined up to a precomposition with an injective holomorphic germ $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$.

Proof. Assume without loss of generality that $A = 0$ and $B = 0$ and that F is holomorphic in some neighborhood V of 0 in $\mathbb{C}^N$. For every point $A \in V$ we define the fiber through A as $L^A(F) = \{Z \in V\colon F(Z) = F(A)\}$. We have $\operatorname{rank}_Z F \leqslant 1$, $Z \in V$ so $\operatorname{corank}_Z F = 2 - \operatorname{rank}_Z F \geqslant 1$, $Z \in V$. By Lemma 6 from [GUN], p. 137, we have $\dim_Z L_Z(F) \geqslant 1$, $Z \in V$. If $\dim_Z L_Z(F) = 2$ at some point $Z \in V$, then $F = \text{constant}$. So $\dim_Z L_Z(F) = 1$ for all $Z \in V$. Now we apply Theorem 8 of [GUN], p. 140 to conclude that there is a neighborhood $V' \subseteq V$ such that $X = F(V')$ is a holomorphic subvariety in an open neighborhood V'' of 0 in $\mathbb{C}^2$. By Lemma 11 in [GUN], p. 143, we see that X is of dimension 1 at each of its points. Now shrink, if necessary, our neighborhoods V' and V'' in such a way that X has no singular points with the possible exception of 0 in $\mathbb{C}^2$. This is possible to do because the singular points of one-dimensional varieties are isolated [GUN]. Then it follows from the local uniformization theorem (see [CHI], p. 70) that there is a holomorphic bijection h of some neighborhood of zero in $\mathbb{C}$ onto X. So the conclusion of the lemma holds with $g = h^{-1} \circ F$. The uniqueness statement is evident.

To formulate the next result we need the following

DEFINITION. A map $H\colon \mathbb{C} \to \mathbb{C}^2$ satisfies condition Y if, λ being holomorphic in a neighborhood of 0 in $\mathbb{C}$, $\lambda(0) = 0$, $\lambda'(0) \neq 0$ and

$$H(z + u) = H(\lambda(z) + v)$$

near 0 implies that $u = v$.

THEOREM 3.2. *Let* $F\colon \mathbb{C}^N \to \mathbb{C}^2$ *be a nonconstant mapping such that the rank of the Jacobian matrix of F does not exceed* 1. *Then there exist entire mappings* $G\colon \mathbb{C}^N \to \mathbb{C}$ *and* $H\colon \mathbb{C} \to \mathbb{C}^2$ *such that* $F = H \circ G$, *or else* $G\colon \mathbb{C}^N \to \mathbb{C} \backslash \{a\}$ *and* $H\colon \mathbb{C} \backslash \{a\} \to \mathbb{C}^2$ *such that* $F = H \circ G$. *Furthermore, H is locally injective and satisfies the Y-condition.*

The factorization $F = H \circ G$ given by this theorem will be called the *Y-process*. To be more precise the *input* of the Y-process is a degenerate map $F = (\phi, \psi)$, where ϕ and ψ are entire functions whose Jacobian determinant is equal to 0 and the *output* is the pair (H, G). It is clear that G is a right common factor of ϕ and ψ. At the end of this section we will show that this is actually the strong greatest right common factor.

Before starting the proof, we introduce some "universal objects" namely a (non-connected) Riemann surface S and a holomorphic map $\tilde{H}: S \to \mathbb{C}^2$.

Consider the set of all pairs (D, φ), where D is a neighborhood of 0 in $\mathbb{C}$ and $\varphi: D \to \mathbb{C}^2$ is an injective holomorphic map. We say that (D_1, φ_1) is equivalent to (D_2, φ_2) if there exist neighborhoods of 0, D_3 and D_4, $D_3 \subseteq D_2$, $D_4 \subseteq D_1$ and a biholomorphic bijection $\psi: D_3 \to D_4$ such that

$$\psi(0) = 0$$

and

$$\varphi_2 = \varphi_1 \circ \psi \qquad \text{on } D_3.$$

The equivalence class of the map φ is denoted by $[\varphi]$. The set of all equivalence classes is called S.

We are going ton define an analytic structure on S such that S will become a (non-connected) Riemann surface. To do this, we have to define some injective maps from disks $B_\varepsilon = \{z: |z| < \varepsilon\}$ to S that will be the coordinate maps. The images of these maps will be called neighborhoods. We will have to show that the correspondence maps are analytic and that the topology defined by these neighborhoods is Hausdorff.

We now define the coordinate maps. Fix any $\varepsilon > 0$ and a holomorphic injective map $\varphi: B_{2\varepsilon} \to \mathbb{C}^2$. For every $a \in B_\varepsilon$ denote by $p_a \in S$ the class of the map

$$\varphi_a: B_\varepsilon \to \mathbb{C}^2, \qquad z \mapsto \varphi(z + a).$$

We call the map $a \mapsto p_a$, $B_\varepsilon \to S$ a coodinate map. It is injective because φ is injective. For if $a, b \in B_\varepsilon$ and $a \neq b$ then $\varphi_a(0) \neq \varphi_b(0)$ and thus $p_a = [\varphi_a] \neq [\varphi_b] = P_b$. The images of the coordinate maps cover S.

We now show that the correspondence maps are analytic. Let $T_1: B_\varepsilon \to S$ and $T_2: B_\varepsilon \to S$ be two coordinate maps defined with the help of two holomorphic injections φ' and φ'': $B_{2\varepsilon} \to \mathbb{C}^2$ as above. If the images of T_1 and T_2 intersect, we have a bijection $L = T_2^{-1} \circ T_1$: $U_1 \to U_2$ where U_1 and U_2 are some subsets of $B_{2\varepsilon}$. This L is called a correspondence map, and we have to show that it is analytic.

Take a point $a_0 \in U_1$ and set $b_0 = L(a_0) \in U_2$. By the definition, we have

$$[\varphi'_{a_0}] = [\varphi''_{b_0}],$$

which means that there exists a holomorphic injective map ψ, $\psi(0)=0$ such that $\varphi'_{a_0}=\varphi''_{b_0}\circ\psi$ or

$$\varphi'(z+a_0)\equiv\varphi''(\psi(z)+b_0)$$

in some neighborhood of 0. We can write this as

$$\varphi'(z+\zeta+a_0)\equiv\varphi''(\psi(z+\zeta)-\psi(\zeta)+\psi(\zeta)+b_0),$$

which holds for all small enough values of z and ζ. Putting $a=a_0+\zeta$, $b(a)=b_0+\psi(\zeta)=b_0+\psi(a-a_0)$ and $\psi_a(z)=\psi(z+\zeta)-\psi(\zeta)$, we get

$$\varphi'_a=\varphi''_{b(a)}\circ\psi_a$$

and

$$\psi_a(0)=0.$$

This means that $[\varphi'_a]=[\varphi''_{b(a)}]$ so $L(a)=b(a)$ because L is bijective. But $b(a)$ is a holomorphic function of a, so L is holomorphic.

We now prove that S is a Hausdorff space. Let p_1 and p_2 be two elements of S, $p_1\neq p_2$. We want to find disjoint neighborhoods of p_1 and p_2. Take a representative for each class p_1 and p_2,

$$\varphi_i\colon B_\varepsilon\to\mathbb{C}^2,\qquad i=1,2.$$

If $\varphi_1(0)\neq\varphi_2(0)$ then we can find an $\varepsilon>0$ such that $\varphi_1(B_\varepsilon)=\varnothing$ so then p_1 and p_2 have disjoint neighborhoods.

From now on we assume that $\varphi_1(0)=\varphi_2(0)$. Assume that p_1 and p_2 have no disjoint neighborhoods. This means that there are $a_j\to 0$ and $b_j\to 0$ such that

$$\varphi_1(z+a_j)=\varphi_2(\psi_j(z)+b_j),\qquad z\in U_j, \tag{3.3}$$

where ψ_j are holomorphic with $\psi_j(0)=0$ and U_j are some neighborhoods of 0. Let φ_i^1 and φ_i^2 be the first and second coordinates of φ_i, $i=1,2$. We assume without loss of generality that all these functions are nonconstant. We have then

$$\varphi_i^k(z)=z^{m_{i,k}}q_{i,k}(z),\qquad q_{i,k}(0)\neq 0,$$

where q_i^k are holomorphic functions in some neighborhood of 0 and $m_{i,k}$ are natural numbers. Then any branch of the analytic function $(\varphi_2^1)^{-1}\circ\varphi_1^1$, which tends to 0 when z tends to 0 can be analytically continued in some punctured neighborhood V of 0 and has the form

$$s_1(z)=z^{m_{1,1}/m_{2,1}}r_1(z), \tag{3.4}$$

where, r_1 being holomorphic at 0, $r_1(0)\neq 0$.

Thus we have

$$\varphi_1^1(z) = \varphi_2^1(s_1(z)), \qquad z \in V. \tag{3.5}$$

Now take j in (3.3) so large that $(U_j + a_j) \cap V \neq \emptyset$ and fix this value of j. Comparing (3.5) with (3.3), we conclude that $s_1(z) = \psi_j(z - a_j) + b_j - a_j$ in the sense that the left side is the analytic continuation of the right side. Applying the same argument to the second coordinate we conclude that

$$\varphi_1^2(z) = \varphi_2^2(s_2(z)), \tag{3.6}$$

where

$$s_2(z) = z^{m_{1,2}/m_{2,2}} r_2(z) \tag{3.7}$$

and again from (3.3) it follows that $s_2(z) = \psi_j(z - a_j) + b_j - a_j$. We conclude that $s_1 = s_2$. Thus from (3.4) and (3.7) follows $m_{1,1} m_{2,2} = m_{1,2} m_{2,1}$ or $m_{1,1}/m_{1,2} = m_{2,1}/m_{2,2}$. But both fractions $m_{1,1}/m_{1,2}$ and $m_{2,1}/m_{2,2}$ are irreducible because φ_1 and φ_2 are both injective. We conclude that $m_{1,1} = m_{2,1}$ and $m_{1,2} = m_{2,2}$, so the function $s = s_1 = s_2$ is actually holomorphic in a full neighborhood of 0 and its derivative at 0 does not vanish. From (3.5) and (3.6) we conclude that $\varphi_1 = \varphi_2 \circ s$ and this contradicts the assumption that φ_1 and φ_2 represent different elements of S.

Next we define the map

$$\tilde{H}: S \to \mathbb{C}^2, \qquad p \mapsto p(0).$$

Remark 3.3. It is evident from the definition of S that for the classes p which serve as points of S the value $p(0)$ is well defined that is does not depend on the germ representing the class. It is also evident from the definition of the analytic structure on S that $\tilde{H}$ is analytic.

Now we have a preliminary form of Theorem 3.2.

THEOREM 3.2*. *Let $F: \mathbb{C}^N \to \mathbb{C}^2$ be a nonconstant mapping such that the rank of the Jacobian matrix of F does not exceed* 1. *Then $F = \tilde{H} \circ \tilde{G}$ where $\tilde{G}$ is a holomorphic map from $\mathbb{C}^N$ to S and $\tilde{H}$ is the projection from S to $\mathbb{C}^2$ defined in Remark* 3.3 *($\tilde{H}$ does not depend on the choice of F). The map $\tilde{G}$ is uniquely determined by F.*

Proof of Theorem 3.2*. If we have an entire function F satisfying the condition of the Theorem, then, by Lemma 3.1, to each point $A \in \mathbb{C}^N$ an element h_A is associated. Denote the resulting map by $\tilde{G}$: $A \mapsto h_A$. Then it is clear that $F = \tilde{H} \circ \tilde{G}$, where the map $\tilde{H}$ is defined in the Remark 3.3.

Let us prove that $\tilde{G}$ is analytic. Fix $A \in \mathbb{C}^N$ and apply Lemma 3.1. We obtain a neighborhood V of A and the factorization in this neighborhood

$$F = h \circ g,$$

where $g\colon (V, A) \to (U, 0)$ and $h\colon (U, 0) \to (\mathbb{C}^2, F(A))$ where U is some neighborhood of 0 in $\mathbb{C}$. Then for $Z_0 \in V$ we can obtain the similar factorization,

$$F = h_{Z_0} \circ g_{Z_0},$$

valid in a neighborhood of the point Z_0, by putting

$$f_{Z_0}(Z) = g_A(Z) - g_A(Z_0)$$
$$h_{Z_0}(Z) = h_A(Z + g_A(Z_0)).$$

So the element $\tilde{G}(Z) \in S$ is equal to $[h(\cdot + F(Z_0))]$. Thus it depends analytically on Z_0 with respect to the analytic structure we introduced in S.

It remains to notice that $\tilde{G}$ is uniquely determined by F. This follows from the uniqueness statement in Lemma 3.1 and the definition of S.

Proof of Theorem 3.2. If we have an entire mapping F, satisfying the conditions of the Theorem, we first apply Theorem 3.2* to obtain the factorization $F = \tilde{H} \circ \tilde{G}$. Denote by S_1 the image of $\tilde{G}$. Then S_1 is a connected Riemann surface because it is the continuous image of a connected set. If we restrict $\tilde{G}$ on any complex line in $\mathbb{C}^N$ on which F is nonconstant, then we get a nonconstant holomorphic map from the complex line to the Riemann surface S_1. Thus S_1 is a parabolic Riemann surface in the sense of Nevanlinna's book [NEV], namely the universal covering surface of S is the complex plane. Now we have a nonconstant restriction of the map $\tilde{H}\colon S_1 \to \mathbb{C}^2$, so S_1 is noncompact. Thus by Picard's theorem S_1 is $\mathbb{C}$ or $\mathbb{C}^* = \mathbb{C} \backslash \{a\}$ for some $a \in \mathbb{C}$.

In the first case, we let P be a holomorphic bijection of $\mathbb{C}$ onto S_1 and we let $H = \tilde{H} \circ P^{-1}$ and $G = P \circ \tilde{G}$ to get our conclusion. A similar device works in the second case, except that P is a biholomorphic bijection of $\mathbb{C}^*$ onto S_1. (See Fig. 1.)

It remains to prove that H satisfies the Y-condition. Observe that the statement $H(z+u) \equiv H(\lambda(z)+v)$ in a neighborhood of 0 is equivalent to $[H(z+u)] = [H(z+v)]$ by the definition of the equivalence classes. Denote by P^* the map $u \mapsto [H(z+u)]$. Then the Y-condition is the same as the injectivity of P^*. By the definition of $\tilde{G}$ we have $\tilde{G} = P^* \circ G$. On the other hand $\tilde{G} = P \circ G$. Thus, because the range of G is dense in $\mathbb{C}$ we conclude that $P = P^*$. It follows that P^* is injective because P is injective. This proves that H satisfies the Y-condition.

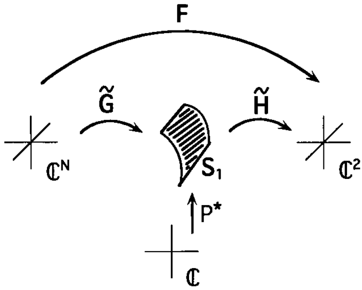

FIGURE 1

The proof of Theorem 3.2 is thus complete.

Now we show that factorization given by Theorem 3.2 is unique.

THEOREM 3.4. *In the context of Theorem* 3.2 *the factorization* $F = H \circ G$, *is unique assuming that* H *is locally injective and satisfies condition* Y.

Thus, given a factorization $F = H \circ G$ we can check whether this factorization is obtained by the Y-process by verifying two conditions: that H is locally injective and satisfies the Y-condition.

Proof of Theorem 3.4. Suppose that H_1 is locally injective and satisfies condition Y. Define P_1^* as above, so that P_1^* is injective. We check that $H_1 = H$, $G_1 = G$ and $P_1^* = P^*$, where H, G and P^* are defined in the proof of Theorem 3.2. For we may choose, in Lemma 3.1

$$g_A(z) = G_1(z) - G_1(A)$$

$$h_A(z) = H_1(z + G_1(A)),$$

which is admissible because H_1 is locally injective.

PROPOSITION 3.5. *Every finite family of pairwise dependent entire functions has a strong greatest right common factor.*

If we have two dependent entire functions f_1 and f_2 from $\mathbb{C}^N$ to $\mathbb{C}^1$, then let $F = (f_1, f_2)$ from $\mathbb{C}^N$ to $\mathbb{C}^2$, and apply Theorem 3.2*. Denote by S_1 the image of $\tilde{G}$. Then, as before, S_1 is a connected parabolic noncompact Riemann surface, so there is a one-to-one map k from S_1 to $\mathbb{C}$ or $\mathbb{C}\backslash\{a\}$. We have the decomposition $F = (\tilde{H} \circ k^{-1}) \circ (k \circ \tilde{G})$. We claim that the right factor $k \circ \tilde{G}$ in this decomposition is the greatest common right factor of f_1 and f_2. Indeed, let $F = f \circ g$ be any decomposition. Then f has rank $\leqslant 1$

and Theorem 3.2* is applicable to f. It gives $f = \tilde{H} \circ q$, so that $F = \tilde{H} \circ q \circ g$. On the other hand, $F = \tilde{H} \circ \tilde{G}$ as before, and from the uniqueness statement in Theorem 3.2* we conclude that $q \circ g = \tilde{G}$. Thus g is a right factor of $G = k \circ \tilde{G}$, because k is one-to-one. This proves that any two dependent entire functions have a strong greatest common right factor. It follows immediately that any finite set of dependent functions has a strong greatest common right factor. To get the strong greatest common right factor of an infinite family requires some new ideas, which we now go into.

Remark 3.6. Theorems 3.2, 3.2* and 3.4, as well as Lemma 3.1 remain true if we consider a holomorphic map $F: \mathbb{C}^N \to \mathbb{C}^n$, $n \geqslant 2$ whose Jacobian matrix has rank 1. The proofs also remain the same.

4. THE EXISTENCE OF WEAK GREATEST COMMON RIGHT FACTORS AND LEAST COMMON LEFT MULTIPLES

DEFINITION. A family $\mathscr{F}$ of nonconstant entire functions of N variables is linearly normal if for each $f \in \mathscr{F}$ there exist complex constants a_f and b_f, $a_f \neq 0$, such that any net $Q = \{a_f f + b_f : f \in \mathscr{F}_0 \subseteq \mathscr{F}\}$ has a subnet Q^* that is uniformly convergent on compact subsets of $\mathbb{C}^N$ to a nonconstant entire function.

EXAMPLES. The set $\{nz: n = 1, 2, 3, ...\}$ although not normal in any neighborhood of 0 is linearly normal in all of $\mathbb{C}$ since $(1/n)\, nz + 0 \to z$ uniformly on compacta. The family $\{z^n: n = 1, 2, ...\}$ in $\mathbb{C}$ is not linearly normal on $\mathbb{C}$. For suppose that $a_n z^n + b_n \to g$ uniformly on compacta. On considering $|z| > 1$, we see that a_n must approach 0. But then for $|z| < 1$, $b_n \to g$ so that $g = const$ for $|z| < 1$ and hence for all $z \in \mathbb{C}$.

PROPOSITION 4.1. *Let f_0 be a nonconstant entire function on $\mathbb{C}^N$ and let Φ be the set of nonconstant entire funtions $\leqslant f_0$ (in the ordering of the introduction.) Then Φ is a linearly normal family.*

Choose an $f_0 \in \Phi$ and suppose that $f_0(z') = 0$, $f_0(z'') = 1$ for suitable $z', z'' \in \mathbb{C}^N$. This can be achieved by a transformation $f_0 \mapsto af_0 + b$, $a, b \in \mathbb{C}$, $a \neq 0$, if necessary.

Now if $f \leqslant f_0$ then $f(z') \neq f(z'')$ so by a linear change of variables, we may suppose $f(z') = 0$ and $f(z'') = 1$ for all $f \in \Phi$.

Let

$$G_0 = \{z \in \mathbb{C}^N: f_0(z) \neq 0 \text{ and } f_0(z) \neq 1\}.$$

If $f \in \Phi$ and $z \in G_0$ then $f(z) \neq 0$ and $f(z) \neq 1$. For suppose, for example, that $z \in G_0$ and $f(z) = 0$. Then $f(z) = f(z')$ and hence $f_0(z) = f_0(z')$ which is impossible. So, on G_0, all the functions in Φ omit 0 and 1. Hence, Φ restricted to G_0 forms a normal family by Montel's Theorem (see [HIL]). Let $\{f_\alpha\}$ be a net of functions drawn from Φ that converges uniformly on compact sets in G_0 in the spherical metric on the range. If the limit function is finite, then the f_α are uniformly bounded on compact subsets of G_0.

LEMMA 4.2. *Given a family of entire functions on $\mathbb{C}^N$ that is uniformly bounded on compact subsets of the complement of a given proper holomorphic variety, the elements of the family must be uniformly bounded on compact subsets of $\mathbb{C}^N$.*

Proof. (This proof is due to C. McMullen, whom we thank.) For $N = 1$ and the subvariety $V = \{z = 0\}$ this is easy: the cricle $S^1 = \{|z| = 1\}$ is a compact subset of the complement of V, and by the maximum principle, if an entire function satisfies $|f| \leqslant M$ on S^1 then $|f| \leqslant M$ on V.

To treat the general subvariety V in $\mathbb{C}^N$, pick a point v in V and a complex line L passing through v in general position. Then v is an isolated point of $L \cap V$ so that there is an S^1 in L encircling v and avoiding V. We may translate this S^1 to nearby parallel lines L'; it remains outside V since V is closed, and so obtain a compact set $K = S^1 \times \bar{\mathbb{D}}^{N-1}$ (where $\bar{\mathbb{D}}$ is a closed disk) outside V. The sup of the absolute value of an entire function f on K bounds it on a neighborhood of v.

We apply Lemma 4.2 by taking $V = \{z \in \mathbb{C}^N\colon f_0(z)(f_0(z) - 1) = 0\}$.

So in case the limit function f is finite on G_0, we see that it extends to be an entire function on $\mathbb{C}^N$. Surely it is nonconstant, since $f(z') = 0$ and $f(z'') = 1$.

So we have handled every case except the one where the above procedure always gives the limit $\equiv \infty$, no matter what z' and z'' may be. We call this case the "big bang." Let us now show that the big bang cannot happen.

Let P_1 be the procedure where the f_α are normalized so that $f_\alpha(z') = 0$ and $f_\alpha(z'') = 1$. Let P_s be the procedure where $f_\alpha(z') = 0$, $f_\alpha(s) = 1$, and let P_t be the procedure where $f_\alpha(z') = 0$, $f_\alpha(t) = 1$. For any complex numbers s and t, let ${}_1f_\alpha$, ${}_sf_\alpha$, ${}_tf_\alpha$ be the suitably normalized f_α.

Now ${}_sf_\alpha = a_1 f_\alpha + b$. Since ${}_sf_\alpha(z') = b$, we have $b = 0$ so that ${}_sf_\alpha(z) = af(z)$. But ${}_sf_\alpha(s) = 1 = a_1 f_\alpha(s)$ so that $a = 1/{}_1f_\alpha(s)$. We conclude that ${}_sf_\alpha(t) = {}_1f_\alpha(t)/{}_1f_\alpha(s)$. Under the hypothesis of the big bang, we have ${}_1f_\alpha(t)/{}_1f_\alpha(s) \to \infty$. By symmetry (interchanging s and t) we also have ${}_1f_\alpha(s)/{}_1f_\alpha(t) \to \infty$. This is a contradiction since we have a net of numbers approaching ∞ such that the net of reciprocal numbers also approaches ∞. Proposition 4.1 is proved.

PROPOSITION 4.3. *Let $\mathscr{F}$ be a family of nonconstant entire functions on $\mathbb{C}^N$ such that any two elements of $\mathscr{F}$ are dependent. Then there exists a common right factor g of all the functions in $\mathscr{F}$, i.e. $g \leqslant f$ for all $f \in \mathscr{F}$.*

Proof. The proof is by transfinite induction (see, for example [SUP]).

Let $\{f_\alpha : \alpha \in A\}$ be a well-ordering of $\mathscr{F}$. If α is not a limit ordinal in A then we let $g_{\alpha-1}$, by induction, be such that $g_{\alpha-1} \leqslant f_\beta$ for all $\beta \leqslant \alpha - 1$. Apply the Y-process to $F = (g_{\alpha-1}, f_\alpha)$ to produce $G = g_\alpha$ with $g_\alpha \leqslant g_{\alpha-1}$ and $g_\alpha \leqslant f_\alpha$. Then $g_\alpha \leqslant f_\beta$ for all $\beta \leqslant \alpha$.

If α is a limit ordinal in A, then for every $\beta < \alpha$ there is an entire function g_β with the properties $g_\beta \leqslant f_\alpha$ and $g_\beta \leqslant f_\gamma$ for all $\gamma \leqslant \beta$. This is our induction hypothesis. Then $\{g_\beta : \beta < \alpha\}$ is a decreasing chain of entire functions (we can choose $g_0 = f_0$) and so we can apply Proposition 4.1 to conclude that $\{g_\beta\}$ is a linearly normal family. Let g be a nonconstant finite limit. This g works. To see this, fix $f = f_\delta \in \mathscr{F}$, $\delta \leqslant \alpha$. We may write $f = \varphi \circ g_\delta$. We have to prove $g \leqslant f$. To this end, suppose we have $z', z'' \in \mathbb{C}^N$ with $z' \neq z''$ but $g(z') = g(z'')$. We must prove that $f(z') = f(z'')$. By Hurwitz's theorem, we can choose $z'(\beta)$ and $z''(\beta)$ with $z'(\beta) \to z'$, $z''(\beta) \to z''$ and $g_\beta(z'(\beta)) = g_\beta(z''(\beta))$, $\beta < \alpha$. Now $f(z'(\beta)) \to f(z')$, $f(z''(\beta)) \to f(z'')$. But $f(z'(\beta)) = \varphi(g_\delta(z'(\beta))$, $f(z''(\beta)) = \varphi(g_\delta(z''(\beta))$. Furthermore, $g_\delta(z'(\beta)) = g_\delta(z''(\beta))$ because $g_\beta \leqslant g_\delta$ for β close enough to α. Hence $f(z'(\beta)) = f(z''(\beta))$. Consequently $f(z') = f('')$.

THEOREM 1.3′. *Same as Theorem* 1.3 *except that the greatest common right factor in the conclusion is asserted only to be weak and not necessarily strong.*

THEOREM 1.4′. *Same as Theorem* 1.4 *except that the least common left multiple in the conclusion is asserted only to be weak and not necessarily strong.*

Proof of Theorem 1.3′. By Proposition 4.3 there exist non trivial ascending chains $\{g_\alpha\}$ of common right factors of $\mathscr{F}$. By the Hausdorff maximal theorem [SUP], there is a maximal such chain Γ. Every $\gamma \in \Gamma$ satisfies $\gamma \leqslant f_0$. So we may apply the method of the preceding paragraphs to find a top element g of Γ. This g is clearly a weak greatest common right factor of $\mathscr{F}$.

The proof of Theorem 1.4′ is along the same lines except that we take descending chains, and we omit it.

5. FROM WEAK TO STRONG

It is easy to prove that if the family $\mathscr{F}$ has a *unique* weak greatest common right factor, then it is strong. For suppose that ρ is the unique weak greatest common right factor of $\mathscr{F}$. Further, suppose ρ' divides (right) every function in $\mathscr{F}$. Then, going back to the proof of Theorem 1.3′, there is a maximal ascending chain of terms containing ρ', and the top element ρ'' of this chain is a weak common right factor for $\mathscr{F}$. But by the supposed uniqueness, $\rho = \rho''$, so $\rho' \leqslant \rho$ since $\rho' \leqslant \rho'' = \rho$.

Similarly, if $\mathscr{F}$ has a unique weak least common left factor, then it is strong.

PROPOSITION 5.1. *If two nonconstant entire functions of N complex variables have a common left multiple, then they have a strong least common left multiple.*

Proof. By Theorem 1.4′, if we call the entire functions A and B, they must have at least one *weak* common left multiple. It is enough, by the previous remark to prove that it is *unique*. Suppose they have two, say M and M'. We prove that $M = M'$. Let K be the output of the Y-process applied to M and M'. We will prove that K is a common left multiple of A and B. This leads to the following segment of the ordering (see Fig. 2), which violates M and M' being weak least common left multiples unless $M = K = M'$. (Of course it follows from Theorem 3.2 that K is a common right divisor of M and M'.) Write $M = m \circ A$ and $M' = m' \circ A$ and to the Y-process on $F = (m, m')$ to get $m = \varphi \circ \eta$, $m' = \psi \circ \eta$. By Theorem 3.2, $H = (\varphi, \psi)$ is locally injective and satisfies condition Y. Thus we have

$$M = \varphi \circ (\eta \circ A), \qquad M' = \psi \circ (\eta \circ A),$$

so that $H = (\varphi, \psi)$ and $G = \eta \circ A$ must, by Theorem 3.4, be the output of the Y-Process applied to M, M'. (By Theorem 3.4, we have only to check that (φ, ψ) is locally injective and satisfies the Y-condition. But we just concluded this from Theorem 3.2.) And it is evident that $\eta \circ A$ is right-divisible by A. So $K = \eta \circ A$. Similarly $K = \tilde{\eta} \circ B$ so that K is a common left multiple of A and B, as claimed.

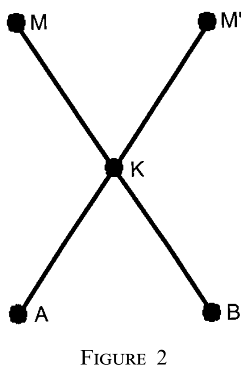

FIGURE 2

Proof of Theorem 1.3. Let G and G' be two greatest common right factors of $\mathscr{F}$. We need only prove that $G = G'$. Let L be the strong least common left multiple of G and G'. We claim that L is a common right factor of $\mathscr{F}$. For choose $A \in \mathscr{F}$. Well, A is a common left multiple of G and G'. So $L \leqslant A$. But this property of L violates the supposed properties of G and G' unless $G = L = G'$. The proof is complete.

Proof of Theorem 1.4. Theorem 1.4 is proved in essentially the same way as Theorem 1.3, and we only sketch the proof. Let M be a common left multiple of $\mathscr{F}$, and let L and L' be two weak common left multiples of $\mathscr{F}$. We must prove $L = L'$, let G be the strong greatest common right factor of L and L'. Show that G is a common left multiple of $\mathscr{F}$, etc.

APPENDIX

In this Appendix, we give a proof of Theorem G of Grauert that was used in our second section. We first clarify our notations.

Notations: $z, w, \ldots$ with or without subscripts-points in the plane; (z, w)-points in $\mathbb{C}^2$. Upper case letters like $Z = \{z_k\}$-(unordered) countable sets of points in the plane with no finite limit points. We denote by $B(z, r)$ the disk in the plane with the center at z and radius r and $B(z_1, z_2, r)$ the bidisk $B(z_1, r) \times B(z_2, r)$. Finally D will always stand for the diagonal $\{(z, z): z \in \mathbb{C}\} \subset \mathbb{C}^2$.

An analytic set $\Gamma \subset \mathbb{C}^2$ of pure dimension 1 is called an *analytic equivalence relation* if it has the following properties:

(a) $(z, z) \in \Gamma$, $z \in \mathbb{C}$;

(b) $(z, w) \in \Gamma \Rightarrow (w, z) \in \Gamma$;

(c) $(z_1, z_2) \in \Gamma$ and $(z_2, z_3) \in \Gamma \Rightarrow (z_1, z_3) \in \Gamma$;

(d) No vertical line $\{(z_0, w): w \in \mathbb{C}\}$ lies in Γ.

(Now it follows from b that no horizontal line is contained in Γ.

THEOREM G. *If Γ is an analytic equivalence relation then there exists a Riemann surface S and holomorphic map $f: \mathbb{C} \to S$ such that $f(z) = f(w)$ iff $(z, w) \in \Gamma$.*

Proof. Let S be the set of equivalence classes and $f(z)$ be the class of the point $z \in \mathbb{C}$. We will introduce first a topology on S and then define an analytic structure in such a way that S will become a Riemann surface and f will be analytic.

First it is evident that f is not constant. It is surjective by the definition and the number of classes is more than one.

For every $z_0 \in \mathbb{C}$ and $r > 0$ we define

$$N(z_0, r) = \{f(z): |z - z_0| < r\} \subset S. \tag{1}$$

We will call these sets *basic neighborhoods* and we will verify now that they form a base of a topology on S. More precisely, if $X \in S$ and $X = f(z_0)$ we call $N(z_0, r)$ a basic neighborhood of X.

1. First we have $X = f(z_0) \in N(z_0, r)$. It remains to check that the intersection of any two basic neighborhoods of a point $X = f(z_0)$ contains another basic neighborhood of X. Let $N(z_0, r_0)$ and $N(z_1, r_1)$ be two basic neighborhoods of X. This means that $f(z_0) = f(z_1) = X$. In other words $(z_0, z_1) \in \Gamma \subset \mathbb{C}^2$ and there is a finite set of pairs of functions (p_k, q_k), $k = 1, ..., n$, analytic in a neighborhood V of 0 in $\mathbb{C}$ such that

$$p_k(0) = z_0, \qquad q_k(0) = z_1$$

and

$$\Gamma \cap U = \bigcup_{k=1}^{n} \{(p_k(z), q_k(z)): z \in V\}. \tag{2}$$

None of these functions p_k and q_k is constant because Γ contains no vertical and no horizontal lines. Choose $\delta > 0$ so small that

$$B(0, \delta) \subset V$$

and

$$|p_k(w) - z_0| < r_0, \quad |q_k(w) - z_1| < r_1 \quad \text{if } w \in B(0, \delta) \text{ and } k = 1, ..., n. \tag{3}$$

Now choose $r_2 > 0$ such that

$$r_2 < r_0 \tag{4}$$

and

$$B(z_0, r_2) \subset p_k(B(0, \delta)). \tag{5}$$

We claim that $N(z_0, r_2) \subset N(z_0, r_0) \cap N(z_1, r_1)$. The first inclusion is evident because of (4). Let us prove the second inclusion,

$$N(z_0, r_2) \subset N(z_1, r_1). \tag{6}$$

Let $X \in N(z_0, r_2)$. Then by the definition of N we have $X = f(z)$ for some $z \in B(z_0, r_2)$. By (5) there is a $w \in B(0, \delta)$ and k such that $z = p_k(w)$. Then

$z' = q_k(w)$ has by (3) the property $|z_1 - z'| < r_1$ and by (2) we have $(z, z') \in \Gamma$. So $f(z) = f(z')$, thus $X = f(z')$ and $z' \in B(z_1, r_1)$, in other words $X \in N(z_1, r_1)$, which proves (6).

We proved that basic neighborhoods actually define a topology on S. (The open sets are unions of basic neighborhoods).

2. Now we prove that the topology on S is Hausdorff. Let $X_0 \neq X_1$ be two points in S. Pick z_0 and z_1 such that $X_1 = f(z_1)$ and $X_0 = f(z_0)$. Then $(z_0, z_1) \notin \Gamma$ because $X_0 \neq X_1$. As Γ is closed in $\mathbb{C}^2$ we can find a bidisc

$$B(z_0, z_1, r) \cap \Gamma = \varnothing \tag{7}$$

We claim that $N(z_1, r) \cap N(z_0, r) = \varnothing$, where $r > 0$ is defined by (7). Assume the opposte. Then by (1) there are points $z' \in B(z_0, r)$ and $z'' \in B(z_1, r)$ such that $(z', z'') \in \Gamma$, which contradicts (7).

3. From the definition of f and the topology on S it follows that $f: \mathbb{C} \to S$ is open and continuous. Let us show that f is locally one to one, except at some isolated points in $\mathbb{C}$. We call a point $z_0 \in \mathbb{C}$ ordinary if (z_0, z_0) is a non-singular point of Γ. Singular points of a one-dimensional analytic set Γ are isolated, so the set $E \subset \mathbb{C}$ of non-ordinary points is isolated. Let z_0 be an odinary point. Then as the diagonal D belongs to Γ, there exists $r > 0$ such that

$$B(z_0, z_0, r) \cap \Gamma = B(z_0, z_0, r) \cap D. \tag{8}$$

So f restricted to $B(z_0, r)$ is one-to-one.

4. Let $M \subset S$ be the set of points which have no ordinary preimages under f. Then M is isolated in S. We will introduce an analytic structure on $S \backslash M$ such that f will become holomorphic on $\mathbb{C} \backslash E$. Let $X_0 \in S \backslash M$ and z_0 be an ordinary point such that $X_0 = f(z_0)$. Choose $r > 0$ such that the restriction of f on $B(z_0, r)$ is one-to-one. We call this restriction a coordinate map. The only thing to prove now is that the coordinate maps are consistent (their compositions are holomorphic). Let $N(z_1, r_1)$ and $N(z_2, r_2)$ be two basic neighborhoods generated by ordinary points z_1 and z_2, and assume that these neighborhoods intersect. Denote the restrictions of f on $N(z_1, r_1)$ and $N(z_2, r_2)$ by f_1 and f_2 respectively. Let $X \in N(z_1, r_1) \cap N(z_2, r_2)$. This means that there are points $z' \in B(z_1, r_1)$ and $z'' \in B(z_2, r_2)$ such that $X = f(z') = f(z'')$. It is enough to prove that $f_2^{-1} \circ f_1$ is holomorphic at z'.

We will show first that the point $(z', z'') \in \Gamma$ is non-singular. It is enough to show that for some $r > 0$ the set $\Gamma \cap B(z', z'')$ is the graph of a holomorphic function ϕ. (Then we can interchange z' and z'' to see that ϕ has a holomorphic inverse.)

Assume that this is not the case. Then there are sequences

$$z_k \to z', \qquad u_k \to z'', \qquad v_k \to z'' \tag{9}$$

such that $u_k \neq v_k$ and $(z_k, u_k) \in \Gamma$, $(z_k, v_k) \in \Gamma$. Then by transitivity $(u_k, v_k) \in \Gamma$ and (9) contradicts the assumption that the point z'' is ordinary:

$$\Gamma \cap B(z'', z'', r_0) = D \cap B(z'', z'', r_0) = D \cap B(z'', z'', r_0).$$

Now we have $f_2^{-1} \circ f_1 = \phi$ in a neighborhood of z', so the composition is holomorphic.

5. We have defined the analytic structure on $S \backslash M$ such that the restriction $f\colon C \backslash E \to S \backslash M$ is holomorphic. Recall that $f\colon \mathbb{C} \to S$ is defined as a continuous and open map. Still we cannot conclude the proof with the classical removability theorem, because we don't know yet that S is a surface! So we use the following result of M. Ohtuska [OHT].

THEOREM O. *Let f be a holomorphic map from a punctured disk $B \backslash z_0$ to some Riemann surface S^*. Then one of the following holds*:

(i) *The function f has a limit $a \in S^*$ as $z \to z_0$ and the singularity at z_0 is removable.*

(ii) *The Riemann surface S^* can be extended by adding one point (puncture) a to S^* such that f extended by $f(z_0) = a$ will become holomorphic in B, or*

(iii) *The range of values of f at any punctured neighborhood of z_0 is conformally equivalent to a parabolic Riemann surface.*

In our situation the alternative (iii) is certainly excluded because f is continuous. So we can extend the Riemann surface $S^* = S \backslash M$ by adding a set of isolated points N and obtain the new Riemann surface $S' = S^* \cup N$ and a new function $g\colon \mathbb{C} \to S'$ which is holomorphic and coincides with f on $\mathbb{C} \backslash E$. We have the natural map $h\colon S' \to S$ (such that $f = h \circ g$), which is the identity on S^* and sends N to M. It remains to prove that h is one-to-one. We have $f = h \circ g$. Evidently h is continuous. It is also open because f and g are open. If there are $X_1 \neq X_2$ in S' such that $X_0 = h(X_1) = h(X_2) \in S$, then take disjoint neighborhoods V_1 of X_1 and V_2 of X_2 on S', not containing other points of N and consider a sequence $Y_k \to X_2$, $Y_k \in V_2$. Then by continuity $h(Y_k) \to X_0$ but on other hand $h(V_1)$ does not contain any of the points $h(Y_k)$, because h is the identity on $(V_1 \backslash X_1) \cup (V_2 \backslash X_2)$. So $h(V_1)$ is not open. The contradiction proves that h is one-to-one.

This is the end of the proof.

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