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ON THE ZERO SETS OF CERTAIN ENTIRE FUNCTIONS

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ABSTRACT. We consider the class **B** of entire functions of the form

$$f = \sum p_j \exp g_j,$$

where p_j are polynomials and g_j are entire functions. We prove that the zero-set of such an f, if infinite, cannot be contained in a ray. But for every region containing the positive ray there is an example of $f \in \mathbf{B}$ with infinite zero-set which is contained in this region.

Let **B** be Borel's class of entire functions of one complex variable that are finite sums of entire functions with only finitely many zeros (possibly none). Clearly $f \in \mathbf{B}$ if and only if

$$f = \sum_{j=0}^{n} p_j \exp g_j, \tag{1}$$

where the p_j are polynomials and the g_j are entire functions. This class is called B_1 in [HRS].

Theorem 1. *No function in* **B** *can have as its zero set an infinite set of positive real numbers.*

Theorem 2. *Given any open set Ω in the complex plane that contains the positive real axis, there is a function f in* **B** *whose zero set is an infinite subset of Ω.*

Proof of Theorem 1. We will use H. Cartan's theory of holomorphic curves [C, L]. An $n+1$-vector of entire functions $(f_0, \dots, f_n)$ without zeros common to all f_j defines a holomorphic curve F which is a holomorphic map of the complex plane $\mathbf{C}$ into the complex projective space $\mathbf{P}^n$. The characteristic $T(r, F)$ is defined in the following way:

$$T(r,F) = \frac{1}{2\pi}\int_0^{2\pi} \max\left(\log|f_0|, \dots, \log|f_n|\right)(re^{i\theta})\,d\theta.$$

For any vector $\mathbf{a} = (a_0, \dots a_n) \in \mathbf{C}^{n+1}\backslash\{0\}$ define

$$N(r,\mathbf{a},F) = \frac{1}{2\pi}\int_0^{2\pi} \log|a_0 f_0 + \dots + a_n f_n|(re^{i\theta})\,d\theta.$$

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Such a vector $\mathbf{a}$ defines a hyperplane in $\mathbf{P}^n$ by the equation $a_0x_0+\ldots+a_nx_n=0$. If we denote by $n(r,\mathbf{a},F)$ the number of preimages of this hyperplane under F which are contained in the disk $\{z:|z|\le r\}$, then by the Jensen formula

$$N(r,\mathbf{a},F)=\int_0^r\{n(t,\mathbf{a},F)-n(0,\mathbf{a},f)\}\frac{dt}{t}+n(0,\mathbf{a},F)\log r+\text{const.}$$

If $n=1$, the Cartan characteristic $T(r,F)$ coincides (up to an additive constant) with the usual Nevanlinna characteristic of the meromorphic function f_1/f_0. We will use the Second Main Theorem of Cartan, which (in a simplified form) states the following: *Let* $\mathbf{a}_1,\dots,\mathbf{a}_q$ *be an admissible system of vectors; that is, any* $n+1$ *of them are linearly independent. If the components* $f_0,\dots,f_n$ *of a curve* F *are linearly independent, then*

$$\sum_{j=1}^{q}N(r,\mathbf{a}_j,F)\ge(q-n-1+o(1))T(r,F),\quad r\in\mathbf{R}^+\backslash E, \tag{2}$$

where E *is an exceptional set of finite length.*

The following theorem due to E. Borel (see, for example [L, p. 186]) is a simple corollary of the Second Main Theorem of Cartan. *Let* $f_j=p_j\exp g_j$, *where* $p_j\ne 0$ *are polynomials and* g_j *are entire functions. If* $\{f_0,\dots,f_n\}$ *are linearly dependent, then there are two functions* $\exp g_j$ *and* $\exp g_k$, *which are proportional (with constant coefficients).*

It follows from Borel's theorem that every function of the class $\mathbf{B}$ can be written in *reduced form*, namely the functions $f_j=p_j\exp g_j$ in (1) are linearly independent. Furthermore in the proof of Theorem 1 we may assume without loss of generality that f is transcendental, the polynomials p_j have no zeros common to all p_j and that $g_0=0$.

With these assumptions we introduce the holomorphic curve F with coordinates $f_j=p_j\exp g_j$, $0\le j\le n$, and show first that

$$r=O(T(r,F)),\quad r\to\infty. \tag{3}$$

Because f in (1) is assumed to be transcendental, at least one of g_j is not constant. Assume that $g_n\ne$ const. Then by the definition of characteristic and by our assumption that $g_0=0$ we have

$$\begin{aligned}2\pi T(r,F)&\ge\int_0^{2\pi}\max\{\log|f_0|,\log|f_n|\}d\theta\\&\ge\int_0^{2\pi}\max\{0,\mathit{Re}\,g_n\}d\theta+O(\log r)\ge cr+O(\log r),\end{aligned}$$

for some $c>0$, which proves (3).

We need the following estimate

$$T(r,f)\le T(r,F)+O(\log r),\quad r\to\infty. \tag{4}$$

To prove this we use first the inequality $\log|a+b| \le \max\{\log|a|, \log|b|\} + \log 2$ and then our assumption that $g_0 = 0$ (so $\log|f_0| = \log|p_0| = O(\log r)$):

$$\begin{aligned} 2\pi T(r,f) &= \int_0^{2\pi} \log^+ |f| d\theta \\ &\le \int_0^{2\pi} \max\{\log|f_0|, \dots, \log|f_n|\}^+ d\theta + O(1) \\ &= \int_0^{2\pi} \max\{0, \log|f_1|, \dots, \log|f_n|\} d\theta + O(\log r) \\ &\le \int_0^{2\pi} \max\{\log|f_0|, \dots, \log|f_n|\} d\theta + O(\log r) \\ &= 2\pi T(r,F) + O(\log r). \end{aligned}$$

Now we apply the Second Main Theorem of Cartan with $q = n+2$, and the following vectors: $\mathbf{a}_j$ for $1 \le j \le n+1$ is the j-th row of the $(n+1)\times(n+1)$ unit matrix and $\mathbf{a}_{n+2} = (1, \dots, 1)$ is the row of 1's. Then we have $N(r, \mathbf{a}_j, F) = O(\log r)$, $r \to \infty$, and $N(r, \mathbf{a}_{n+2}, F) = N(r, 0, f)$, the usual Nevanlinna counting function of zeros of the entire function f. From (2) it follows that

$$N(r,0,f) \ge (1+o(1))T(r,F), \quad r \in \mathbf{R}^+ \backslash E. \tag{5}$$

Combined with (4) this implies

$$N(r,0,f) \sim T(r,f), \quad r \to \infty,\ r \in \mathbf{R}^+ \backslash E. \tag{6}$$

In particular, this asymptotic equality combined with (3) implies that the genus of f is at least 1 (maybe infinite).

Finally we use the following result of A. Edrei and W. Fuchs [EF] and J. Miles [M]: *If f is an entire function of genus at least* 1, *with positive zeros, then there is a set E_1 of zero logarithmic density and a constant $\epsilon > 0$ such that*

$$N(r,0,f) \le (1-\epsilon)T(r,f), \quad r \in \mathbf{R} \backslash E_1.$$

Since this estimate is incompatible with (6), Theorem 1 must hold.

Proof of Theorem 2. By taking a smaller region if necessary (but still including the positive real axis), we may assume that Ω is connected and simply connected, and is bounded by a single smooth simple curve $\gamma : [-1,1] \to \mathbf{C}$ such that $\gamma(t) \to \infty$ as $t \to \pm 1$ and γ intersects the real axis once (this intersection happens on the negative ray). The complement T of Ω is an Arakelyan set, i.e. Ω is connected and locally connected at ∞ (see [GAI]). Using the Arakelyan approximation theorem [GAI] we find a non-constant entire function g with the property $|g(z) - 1/2| < 1/4$, $z \in T$. Thus $g^{-1}(\mathbf{Z}) \subset \Omega$ and $f(z) = \exp[2\pi i g(z)] - 1$ gives the required example.

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