On a conjecture of Danikas and Ruscheweyh

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Abstract

We construct a holomorphic function f in the unit disc, whose derivative belongs to the Hardy class H^1, and the image of the unit circle under

$$z \mapsto \int_1^z f'(\zeta)\frac{d\zeta}{\zeta}$$

is a simple curve, but f is not univalent.

1. Introduction. In [1] Danikas and Nestoridis proved an interesting mean value theorem for the space H^1. They showed that if $f \in H^1$ and z_0 is any point in $\Delta := \{z : |z| < 1\}$ then there exist a, b, such that $a < b < a + 2\pi$ and

$$\Phi_{a,b}(f) := \int_a^b f(e^{it})dt = (b-a)f(z_0).$$

In particular if $f' \in H^1$ and

$$\Phi_{a,b}(f') \neq 0, \quad a < b < a + 2\pi, \tag{1}$$

then $f'(z) \neq 0$ in Δ, so this f is locally univalent in Δ. This lead Danikas and Ruscheweyh [2] to ask whether (1) actually implies

$$f \text{ is univalent in } \Delta. \tag{2}$$

In this paper we provide a negative answer to this question by showing that (1) and (2) are independent, i. e. neither implies the other.

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2. We first provide a very simple example where (2) holds but (1) is false.

We set

$$f(z) = \frac{z}{1-\rho z}, \quad z \in \Delta,\ 0 < \rho < 1. \tag{3}$$

Then f maps Δ onto the disk which has the points $-1/(1+\rho),\ 1/(1-\rho)$ as ends of a diameter. Thus f is certainly univalent. On the other hand if ρ is close to 1, f does not satisfy (1).

To see this we write $0 < a < \pi,\ b = 2\pi - a$. Then, since $f'(e^{it}) = \overline{f'(e^{i(2\pi - t)})}$,

$$\Phi_{a,\pi}(f') = \overline{\Phi_{\pi,b}(f')},$$

so that $\Phi_{a,b}(f')$ is real and

$$\Phi_{a,b}(f') = 2\mathrm{Re}\left\{\Phi_{a,\pi}(f')\right\} = 2\mathrm{Im}\int_a^{-1} f'(z)\frac{dz}{z},$$

where integration is along $|z| = 1$ in the anticlockwise direction. Since

$$\frac{f'(z)}{z} = \frac{1}{z(1-\rho z)^2} = \frac{1}{z} + \frac{\rho}{1-\rho z} + \frac{\rho}{(1-\rho z)^2},$$

we obtain

$$\begin{aligned}\Phi_{a,b}(f') &= 2\mathrm{Im}\left\{\log\frac{z}{1-\rho z} + \frac{1}{1-\rho z}\right\}_{e^{ia}}^{-1} \\ &= 2\left[\arg\frac{z}{1-\rho z}\right]_{e^{ia}}^{-1} - \frac{2\rho\sin a}{(\rho-\cos a)^2 + \sin^2 a}.\end{aligned}$$

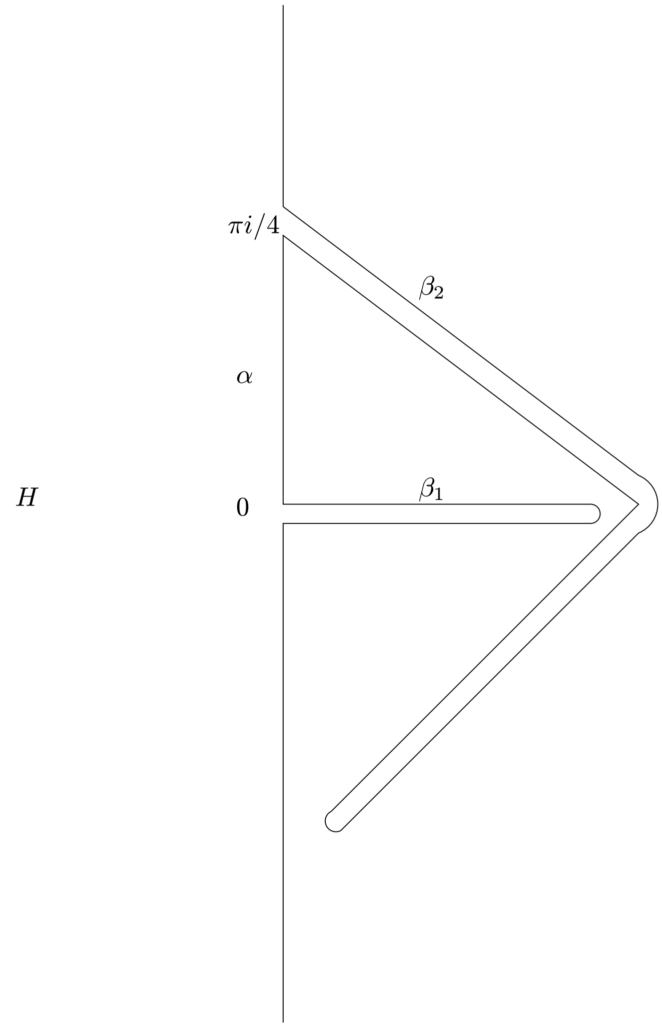
This is a continuous function of a in $[0,\pi]$ and is equal to 2π when $a = 0$. Also if $\cos a = \rho$, and $\rho \to 1$

$$\Phi_{a,b}(f') = -\frac{2+o(1)}{\sin a} \to -\infty.$$

Thus if ρ is close to 1 there exist a, b such that $\Phi_{a,b}(f') = 0$, and $0 < a < \cos^{-1}\rho,\ b = 2\pi - a$. Thus f given by (3) satisfies (2) but not (1).

3. Next we construct a function which satisfies (1) but not (2).

We fix $\epsilon \in (0,\ 0.001)$ and consider the region D_0 shown in the picture.



This region can be described as the union of the following three sets:

the left half-plane H,
the ϵ-neighborhood of the segment $[-\epsilon i,\, 1-\epsilon i]$ and
the ϵ-neighborhood of the broken line

$$[\pi i/4,\, 1.01] \cup [1.01,\, 0.01 - i].$$

Our ϵ is sufficiently small for D_0 to be a Jordan region. The set $\partial D_0 \backslash i\mathbf{R}$ consists of two simple arcs which we call β_1 and β_2. The component of the set $\partial D_0 \cap i\mathbf{R}$ between β_1 and β_2 will be called α. We notice that $\alpha \cap \overline{\beta_1} = \{0\}$.

The segments

$$\gamma_1 = [0,1] \quad \text{and} \quad \gamma_2 = [0.01 - i + \epsilon\sqrt{2}i,\ 1.01 - \epsilon\sqrt{2}] \tag{4}$$

also belong to the boundary ∂D_0.

Now we define the region

$$D = \bigcup_{n \in \mathbf{Z}} (D_0 + 2\pi n i).$$

This D is a Jordan region in the extended complex plane, invariant under the transformation $z \mapsto z + 2\pi i$.

Let $h : H \to D$ be the conformal map, which leaves three boundary points $0, 2\pi i$ and ∞ fixed. Then we have

$$h'(z) \to k \quad \text{as} \quad z \to \infty, \quad |\mathrm{Im}\, z| < -\mathrm{Re}\, z, \tag{5}$$

where k is a positive constant. Notice that both functions h and $h_0(z) := h(z+2\pi i) - 2\pi i$ fix 0 and ∞ and satisfy (5) with the same constant k. So $h = h_0$ that is

$$h(z + 2\pi i) = h(z) + 2\pi i, \quad \text{for} \quad z \in H. \tag{6}$$

It is clear that h can be continuously extended to the boundary ∂H. Moreover, h' has continuous extension to ∂H minus a discrete set of points, and this extension is locally integrable.

We define P by $h([0,\, iP]) = \alpha$, and the intervals b_1 and b_2 on the imaginary axix by $h(b_1) = \beta_1$ and $h(b_2) = \beta_2$. Then

$$|b_1| \to 0 \quad \text{and} \quad |b_2| \to 0, \quad \text{as} \quad \epsilon \to 0 \tag{7}$$

by an easy harmonic measure argument.

For every positive δ we have

$$|h(z) - z| < \delta \quad \text{for} \quad z \in [0,\, iP], \tag{8}$$

if ϵ is small enough. Indeed, the Caratheodory convergence theorem together with the Schwarz reflection principle imply that $h \to \mathrm{id}$ as $\epsilon \to 0$ uniformly on every compact subset of $(0,\, iP)$. In addition, the map $h : [0,\, iP] \to \alpha$ is monotone, so we obtain (8).

It is also easy to see that

$$P \to \pi/4 \quad \text{as} \quad \epsilon \to 0. \tag{9}$$

We define a multiply-valued function $g(z) = h(\log z)$, $0 < |z| < 1$, and notice that $g(ze^{2\pi i}) = g(z) + 2\pi i$, which follows from (6). Thus the derivative g' is a single-valued function in $0 < |z| < 1$, and the asymptotics (5) implies that g' has a simple pole at the origin. We conclude that the function $zg'(z)$ is analytic in the unit disk. It has integrable boundary values on the circle $|z| = 1$, so we can define

$$f(z) = \int_1^z \zeta g'(\zeta)\, d\zeta, \quad |z| \leq 1,$$

where the path of integration belongs to the closed unit disk. This function f satisfies the assumptions of the Danikas and Ruscheweyh conjecture (1), because

$$g(z) = \int_1^z f'(\zeta)\frac{d\zeta}{\zeta}$$

maps the circle $|z| = 1$ onto a simple curve ∂D. So it remains to show that f is not univalent.

Lemma 1 *For $t \in b_1$ we have*

$$f(e^{it}) - g(e^{it}) \to 0 \quad \textit{as} \quad \epsilon \to 0,$$

uniformly with respect to t.

Proof.

$$\begin{aligned} |f(e^{it}) - g(e^{it})| &= \left| \int_1^{e^{it}} (\zeta - 1) g'(\zeta)\, d\zeta \right| = \left| \int_0^t (e^{i\tau} - 1) h'(i\tau) i d\tau \right| \\ &\leq |b_1| \int_0^t |h'(i\tau)|\, d\tau = |b_1||\beta_1| = o(1) \end{aligned}$$

where we used (7) and $|\beta_1| \leq 2.5$. □

We put

$$b = \int_0^{\pi/4} (e^{i\tau} - e^{i\pi/4}) i d\tau = e^{i\pi/4} - 1 - \frac{i\pi}{4} e^{i\pi/4} \approx 0.262 + 0.152i,$$

and $a = e^{i\pi/4}$.

Lemma 2 *For* $t \in b_2$

$$f(e^{it}) - ag(e^{it}) - b \to 0 \quad as \quad \epsilon \to 0,$$

uniformly with respect to t.

Proof.

$$f(e^{it}) - ag(e^{it}) = \int_0^t (e^{i\tau} - e^{i\pi/4}) h'(i\tau) i d\tau = \int_0^P + \int_P^t .$$

The second integral is estimated as in Lemma 1:

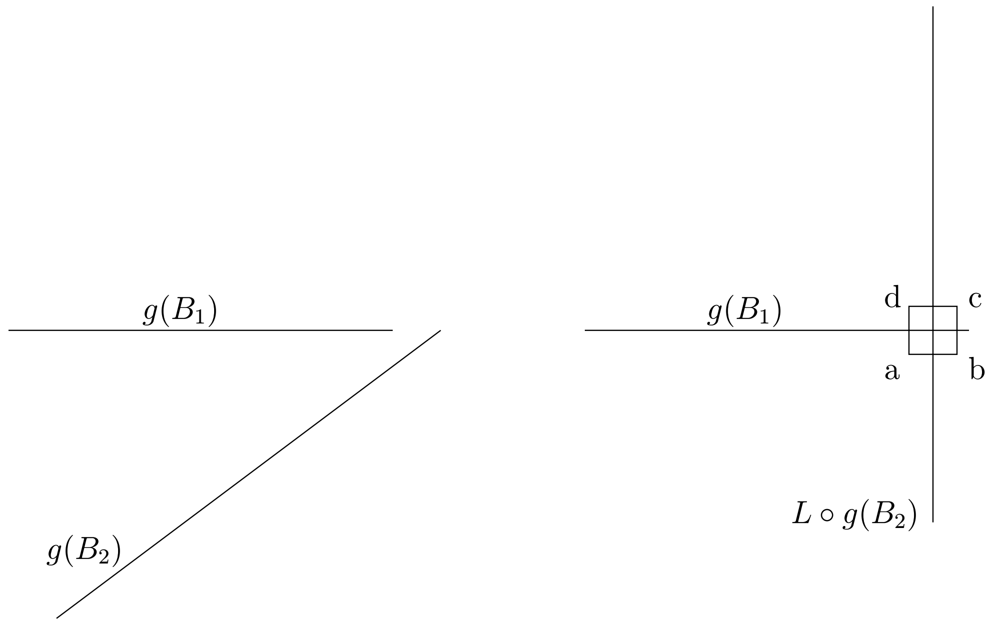
$$\left| \int_P^t (e^{i\tau} - e^{i\pi/4}) h'(i\tau) i d\tau \right| \leq |b_2| \int_P^t |h'(i\tau)|\, |d\tau| = o(1).$$

The first integral tends to b:

$$\begin{aligned} & \int_0^P (e^{i\tau} - e^{i\pi/4}) h'(i\tau) i d\tau - b \\ = \; & \int_0^P (e^{i\tau} - e^{i\pi/4})(h'(i\tau) - 1) i d\tau - \int_P^{\pi/4} (e^{i\tau} - e^{i\pi/4}) i d\tau \\ = \; & (h(i\tau) - i\tau)(e^{i\tau} - e^{i\pi/4})\Big|_0^P - \int_0^P ie^{i\tau}(h(i\tau) - i\tau) d\tau + o(1) = o(1), \end{aligned}$$

where we used (8) and (9). □

Now we consider the segments γ_1 and γ_2 on the boundary ∂D, defined in (4). The g-preimages of these segments, which we call B_1 and B_2 are mapped by the principal branch of the logarithm into b_1 and b_2, respectively. We define $L(z) = az + b$. Then $L(\gamma_2)$ is a vertical segment, which crosses the horizontal segment γ_1 at the point $q := 1.01/\sqrt{2} - \epsilon + \operatorname{Re} b \approx 0.9762$. Consider a closed square $S = [a, b, c, d]$ centered at q, with sides parallel to the coordinate axes, and having length 0.01:



From lemmas 1 and 2, and the fact that f is continuous in the closed unit disk, we conclude that for every positive δ there exist ϵ in $(0, 0.001)$ and r in $(0, 1)$, such that

$$|f(rz) - g(z)| < \delta \quad \text{for} \quad z \in B_1$$

and

$$|f(rz) - ag(z) - b| < \delta \quad \text{for} \quad z \in B_2.$$

In other words, the parametric curves $f(B_1)$ and $f(B_2)$ are uniformly close to $g(B_1) = \gamma_1$ and $L \circ g(B_2) = L(\gamma_2)$, respectively. It follows from the Bolzano–Weierstrass theorem that the arcs $f(rB_1)$ and $f(rB_2)$ have subarcs l_1 and l_2 in Q respectively, such that l_1 connects the sides $[d, a]$ and $[b, c]$ of Q, and l_2 connects the sides $[a, b]$ and $[c, d]$. We extend the arc l_1 by two horizontal rays outside Q to obtain a closed curve L_1 in $\bar{\mathbf{C}}$ and the arc l_2 by two vertical rays to obtain a closed curve L_2 in $\bar{\mathbf{C}}$. These two closed curves L_1 and L_2 on the Riemann sphere intersect transversally at infinity. So they have to intersect at least at one more point, because the intersection number of two closed curves in the sphere is always zero [3]. Thus l_1 and l_2 intersect. We conclude that $f(rB_1)$ and $f(rB_2)$ intersect, so that f is not univalent. □

We note finally that the Danikas–Nestoridis Mean Value Theorem remains true, if we replace dt by dz. In fact if $f' \in H^1$ and for every pair of distinct points a_1 and z_2 on $T := \{z : |z| = 1\}$ we have

$$\int_{z_1}^{z_2} f'(z)\, dz \neq 0,$$

then f maps T onto a simple closed curve Γ. Thus f maps Δ onto the interior of Γ, so that $f'(z) \neq 0$ in Δ. We deduce that, if $F \in H^1$ and $z_0 \in \Delta$, then there exist distinct points z_1 and z_2 on T, such that

$$(z_2 - z_1)F(z_0) = \int_{z_1}^{z_2} F(z)\, dz.$$

For otherwise we may write $f'(z) = F(z) - F(z_0)$ and apply the above remark to obtain a contradiction.

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References

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