Schwarzian derivatives of rational functions

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1. **Properties of the Schwarzian derivative,**

$$\{y,z\} = \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2, \quad \text{where} \quad ' = \frac{d}{dz}.$$

1.1 Consider the second order linear differential equation

$$w'' + Gw = 0. \tag{1}$$

If w and w_1 are two linearly independent solutions of (1), then $y = w_1/w$ satisfies

$$\{y,z\} = 2G. \tag{2}$$

To verify this, we recall that the Wronskian $w_1'w - w_1w' = c$ is constant. So we have

$$y = \frac{w_1}{w}, \quad y' = \frac{c}{w^2}, \quad y'' = -2c\frac{w'}{w^3},$$

and

$$y''' = -2c\frac{w''w - 3w'^2}{w^4},$$

so

$$\{y,z\} = \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2 = -2\frac{w''}{w} = 2G.$$

In the opposite direction, if y satisfies (2) then

$$w = \frac{1}{\sqrt{y'}} \quad \text{and} \quad w_1 = \frac{y}{\sqrt{y'}}$$

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satisfy (1), which is verified by direct computation.

1.2 As a corollary we obtain

$$\{y_1, z\} = \{y_2, z\} \quad \text{if and only if} \quad y_1 = L \circ y_2,$$

where L is a fractional-linear transformation.

1.3 If y is a meromorphic function of z then $\{y, z\}$ is meromorphic, moreover, it is holomorphic except at the critical points of y, where it has poles of exactly second order. This can be verified directly, or using 1.1.

2. **Regular singular points of the equation (1).** A point z_0 is called singular if G is not holomorphic at z_0. A singular point is regular (see [5, 6]) if G has a pole of order at most 2 at this point. Making the change of variable $w(z) = v(1/z),\ G(z) = H(1/z)$, and $\zeta = 1/z$, we obtain

$$\zeta^4 v'' + 2\zeta^3 v' + H(\zeta) v = 0, \tag{3}$$

and the singular point at ∞ is regular iff

$$G(z) = O(z^{-2}), \quad z \to \infty. \tag{4}$$

If one solution of (2) is meromorphic in some domain D then all solutions are meromorphic in D (because all solutions are obtained from one of them by a fractional-linear transformation), and all singularities of (1) in D are regular in this case.

2.1 Suppose that 0 is a regular singular point of the equation (1). To use the general theory of regular singular points (see, for example, [5, 6]) we write the equation in the form

$$z^2 w'' + P(z) w = 0, \quad \text{where} \quad P(z) = a_0 + a_1 z + a_2 z^2 + \ldots. \tag{5}$$

Put $F(r) = r(r-1) + a_0$, this is called the *characteristic polynomial* of (5), corresponding to the point 0. Let r_1 and r_2 be the two solutions of the *indicial equation*

$$F(r) = r(r-1) + a_0 = (r - r_1)(r - r_2) = 0. \tag{6}$$

The following cases may occur:

a) If $r_1 - r_2$ is not an integer, equation (5) has two linearly independent convergent power series solutions of the form $w_j(z) = z^{r_j} Q_j(z),\ j = 1, 2$. Here Q_j are McLauren series (containing only non-negative integral powers of z.)

b) If $r_1 - r_2$ is an integer, then r_1 and r_2 are real, because $r_1 + r_2 = 1$ by Vieta's theorem. We label them so that $r_1 \geq r_2$. Then, if $r_1 - r_2 \neq 0$, there are two linearly independent solutions of the form

$$w_1(z) = z^{r_1} Q_1(z) \quad \text{and} \quad w_2 = z^{r_2} Q_2(z) + C w_1(z) \log z,$$

where C is a constant, Q_1 and Q_2 are McLauren series. If $r_1 = r_2$ there are two linearly independent solutions of the form

$$w_1(z) = z^{r_1} Q_1(z) \quad \text{and} \quad w_2(z) = w_1(z) \log z + z^{r_1} Q_2(z),$$

so a logarithm is always present in the general solution in this case.

2.2 We are interested in the case, when

$$r_1 = r_2 + 2, \quad \text{and} \quad C = 0. \tag{7}$$

(The conditions that $r_1 - r_2$ is a positive integer, and $C = 0$ are necessary and sufficient for the ratio of two linearly independent solutions to be meromorphic at 0. The additional condition $r_1 - r_2 = 2$ ensures that this meromorphic ratio has a simple critical point at 0.)

The first of these conditions (7), together with $r_1 + r_2 = 1$, imply that

$$r_2 = -1/2, \quad r_1 = 3/2 \quad \text{and thus by (6)} \tag{8}$$

$$a_0 = P(0) = -3/4. \tag{9}$$

To find the necessary and sufficient condition for $C = 0$ in (7) in terms of coefficients a_j of P in (5), we plug the power series $w(z) = z^r(c_0 + c_1 z + \ldots)$ with $r = r_2 = -1/2$ and $c_0 = 1$ into (5), where $a_0 = -3/4$, according to (8). We obtain

$$r(r-1) + a_0 = 0, \tag{10}$$

$$[(r+1)r + a_0]c_1 = -a_1 c_0, \tag{11}$$

$$[(r+2)(r+1) + a_0]c_2 = -a_2 c_0 - a_1 c_1, \tag{12}$$

$$\ldots,$$

and in general

$$F(r+n)c_n = \text{polynomial in } c_0,\dots,c_{n-1}, \quad n=0,1,2,\dots, \tag{13}$$

where F is defined in (6). Equation (10) is satisfied because $r=-1/2$, and $a_0=-3/4$. Equation (11) (with $r=-1/2$, $a_0=-3/4$ and $c_0=1$) then implies $c_1=a_1$, and equation (12), whose left hand side is 0, implies

$$a_1^2+a_2=0. \tag{14}$$

Thus if (5) has a power series solutions with properties (7), then (9) and (14) are satisfied. The converse is also true: if these two conditions are satisfied, than all coefficients c_j can be successively found from (13), because $F(r+n)\neq 0$ for $n\geq 3$. Thus we have

Proposition. *In order that the ratio of two linearly independent solutions of* (5) *be meromorphic at* 0*, and have a simple critical point there, it is necessary and sufficient that conditions* (9) *and* (14) *be satisfied.*

3. **Schwarzian derivatives of rational functions.** We say that a finite critical point z_0 of a rational function f is simple if $f''(z_0)\neq 0$. (If $f(z_0)=\infty$, this has to be modified to $(1/f)''(z_0)\neq 0$.)

Theorem 1. *Suppose that f is a rational function whose finite critical points $z_1,\dots z_n$ are simple. Then*

$$\frac{1}{2}\{f,z\} = -\frac{3}{4}\sum_{k=1}^{n}\frac{1}{(z-z_k)^2}+\sum_{k=1}^{n}\frac{x_k}{z-z_k}, \tag{15}$$

where

$$x_m^2+\sum_{k\neq m}\frac{x_k}{z_m-z_k}=\frac{3}{4}\sum_{k\neq m}\frac{1}{(z_m-z_k)^2}, \quad m=1,\dots,n. \tag{16}$$

Remark. Substitution

$$x_m = y_m-\frac{1}{2}\sum_{k\neq m}\frac{1}{z_m-z_k}$$

simplifies equations (16) to

$$y_m^2=\sum_{k\neq m}\frac{y_k-y_m}{z_k-z_m}.$$

This equivalent form of equations (16) was obtained in [3].

Proof. Let f be a rational function with simple finite critical points $z_1, \ldots, z_n$. Put $G = (1/2)\{f, z\}$. Then G is a rational function with only poles at z_k, all these poles are double by 1.4, and G satisfies (4). In particular, G has the form

$$G(z) = \sum_{k=1}^{n} \frac{b_k}{(z - z_k)^2} + \frac{x_k}{z - z_k}.$$

Now the ratio of two linearly independent solutions of (1) with this G has to be a rational function. An inspection of the cases a) and b) in section 2.1 shows that this will be the case only if at each singular point z_k we have $r_1 - r_2$ an integer and $C = 0$ (so that there are no logarithms). From the additional condition that each z_m is a simple critical point of f we deduce that $r_1 - r_2 = 2$. Thus we have (9) with $a_0 = b_m$ and (14) with $a_0 = -3/4,\ a_1 = x_m$ for each singular point z_m. This gives $b_m = -3/4$ and (16).

Theorem 2. *Let $z_1, \ldots, z_n$ be distinct complex numbers, $(x_1, \ldots, x_n)$ a solution of* (16) *and $G(z)$ the rational function defined by the right hand side of* (15). *Then:*

$$\sum_{k=1}^{n} x_k = 0, \tag{17}$$

and

$$\sum_{k=1}^{n} x_k z_k - \frac{3}{4} n = \frac{1 - q^2}{4}, \tag{18}$$

where q is a positive integer.

Furthermore, the general solution of equation (11) *with this G is a rational function of degree* $(n+q+1)/2$ *having simple critical points at $z_1, \ldots, z_n$ and a critical point of multiplicity $q - 1$ at infinity.*

Proof. If (16) is satisfied then the differential equation (2) defines a meromorphic function y is $\mathbf{C}$. All critical points of y in $\mathbf{C}$ are simple and occur exactly at $z_1, \ldots, z_n$. By definition of G, we have

$$G(z) \sim c/z,\ z \to \infty.$$

Suppose first that $c \neq 0$. Then, by the well-known asymptotic analysis of the equation (1) (see, for example, [5]) we conclude that y is a meromorphic function of order $1/2$ and has one asymptotic value. In addition, it has finitely many critical points. It is clear that such function cannot exist. The conclusion is that $c = 0$, which is equivalent to (4). Computing this c from (15) we obtain (17).

Remark. We proved that (16) implies (17). Can one prove this fact in a more direct way?

As infinity is now a regular singular point of (1), we conclude that our meromorphic function y cannot have an essential singularity, so it is rational. This implies for the equation (3), that the difference between the exponents r_1 and r_2 is a positive integer, and there are no logarithms in the formal solutions. Let $\lim_{z\to\infty} z^2 G(z) = a$. The indicial equation of (3) at infinity is

$$r^2 + r + a = 0,$$

and its solutions are

$$r_1 = \frac{-1 + \sqrt{1-4a}}{2} \quad \text{and} \quad r_2 = \frac{-1 - \sqrt{1-4a}}{2}.$$

Now $q = r_1 - r_2$ is a positive integer and we conclude that

$$a = (1 - q^2)/4.$$

Computing a from (15) we obtain(18).

The critical point of y at infinity has order $q - 1$, so the total number of critical points on the Riemann sphere is $n+q-1$, so y has degree $(n+q+1)/2$.

4. Let us call two rational functions f_1 and f_2 equivalent if $f_1 = L \circ f_2$ for some fractional-linear L by 1.2. Two rational functions are equivalent if they have the same Schwarzian derivative. An equivalence class contains a real function if and only if the Schwarzian derivative of functions of this class is real. Indeed, if there is a real function in a class than its Schwarzian derivative is real. In the opposite direction, suppose that the Schwarzian derivative $G/2$ of a class is real. Then the differential equation

$$\{y, z\} = G/2$$

has at least one real solution y_0 (take any real initial conditions to solve the Cauchy problem). This means that there is a real function in the class, namely y_0.

5. Suppose that $q=1$ in (18). Then $a=0$, and the condition of absence of logarithms in the formal solution at infinity gives

$$\sum_{k=1}^{n} x_k z_k^2 = \frac{3}{2}\sum_{k=1}^{n} z_k.$$

It is clear that $q \leq n+1$, as a rational function cannot have more than half of its critical points at infinity. In the extremal case, $q=n+1$, y is a "polynomial" (up to a fractional-linear transformation) of degree $n+1$, and such solution of (16) is unique. The number of solutions with any fixed q can be counted using the method of [2], for example, it is the Catalan number for $q=1$. In general, for a fixed ≥ 2 and n, it is the number of chord diagrams (degenerate nets, using the terminology of [2]) with $n+1$ vertices on the unit circle, each vertex except one is the endpoint of exactly one chord, and the exceptional vertex is the endpoint of $q-2$ chords. The sum of all these numbers, for $1 \leq q \leq n+1$ gives the total number of solutions of (16). If all z_k are real, all these solutions are real by the result of [1].

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