LOWER ESTIMATE IN LITTLEWOOD'S CONJECTURE
ON THE MEAN SPHERICAL DERIVATIVE
OF A POLYNOMIAL AND ITERATION THEORY

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ABSTRACT. Lewis and Wu have proved J. E. Littlewood's conjecture of the growth of the spherical derivative of polynomials. We apply the theory of complex iteration to show that this conjecture is sharp in a qualitative sense.

Let f be a complex polynomial. Denoted by $\rho(f) = |f'/1(1 + |f^2|)$ its spherical derivative, let D be the unit disc, and set

$$\Phi(f) = \iint_D \rho(f(z)) \, dx \, dy.$$

Littlewood [L] remarked that $\Phi(f) \leq \pi\sqrt{n}$ for all polynomials of degree n. He asked for good upper and lower bounds for the quantity $\phi(n) = \sup\{\Phi(f): \deg f = n\}$ and conjectured that

$$(1) \qquad\qquad \phi(n) \leq cn^\alpha$$

for some $\alpha < \frac{1}{2}$. It was proved in [ES1] that $\phi(n) = o(\sqrt{n})$, $n \to \infty$ and later J. Lewis and J. M. Wu [LeW] obtained the estimate (1) with $\alpha = 2^{-1} - 2^{-264}$ by improving the potential–theoretic method of [ES1, ES2]. Upper bounds for $\phi(n)$ have very interesting consequences in value distribution theory of entire functions [L, ES1, ES2].

The lower estimate in [L] was $\phi(n) \geq c(\delta)(\log n)^{1/2-\delta}$ for every $\delta > 0$. Then W. Hayman [H] improved this to $\phi(n) \geq c \log n$.

We shall prove in this note that $\phi(n) \geq cn^\kappa$ for some $\kappa > 0$. To this end we use complex iteration theory (for a brief account of this theory see for example [B, EL]). Denote by f^n the nth iterate of a polynomial f. The Julia set $J(f)$ is the closure of repelling periodic points of f. A polynomial f is called the hyperbolic if $f'(z) \neq 0$, $z \in J(f)$. In this case the closure of the orbit of critical points of f does not intersect $J(f)$. For example $f(z) = z^2 + \lambda$ is

hyperbolic if λ is in the interior of the cardioid:

$$C \leq \left\{ \frac{e^{i\theta}}{2} - \frac{e^{2i\theta}}{4} : 0 < \theta \leq 2\pi \right\}.$$

The orbit of the unique critical point of such a polynomial tends to the attracting fixed point.

Let f be a hyperbolic polynomial. Consider a covering of $J = J(f)$ by closed squares $\mathscr{U}_j$ of diameter $r > 0$ centered at the points $z_j \in J$ and satisfying int $\mathscr{U}_j \cap$ int $\mathscr{U}_i = \varnothing$, $j \neq i$. Denote by $\mathscr{U}_j^1$ the squares with the centers z_j having diameter $2r$ and sides parallel to those of $\mathscr{U}_j$. Choose the diameter r so small that $\mathscr{U}^1 = \bigcup_j \mathscr{U}_j^1$ contains no critical values of iterates. This is possible because critical values of f^n are contained in the orbit of critical points of f and f is hyperbolic. All branches of f^{-n} are single valued and univalent in $\mathscr{U}_j^1$. They map $\mathscr{U}_j$ onto some regions V_k which cover J and int $V_k \cap$ int $V_l = \varnothing$, $k \neq l$. Choose a_k in V^k such that $f^n(a_k) = z_j$ for some j. On applying the Koebe distortion theorem to each branch of f^{-n} in $\mathscr{U}_j^1$, we obtain

$$(2) \qquad\qquad \text{area } V_k \asymp |(f^n)'(a_k)|^{-2}$$

and

$$(3) \qquad\qquad |(f^n)'(z)| \asymp |(f^n)'(a_k)|, \qquad z \in V_k,$$

($x \asymp y$ means $c_1 x \leq y \leq c_2 x$ with constants c_1 and c_2 depending only on $\{\mathscr{U}_j\}$). If $z \in V = \bigcup_k V_k$ then $f^n(z) \in \mathscr{U}^1$ and thus f^n is bounded on V uniformly in n. Futhermore we can suppose that $J \subset \mathscr{D} = \{z : |z| < 1\}$ and consequently $V \subset \mathscr{D}$ for large n. Using (2) and (3) we obtain

$$(4) \quad \Phi(f^n) \geq c \iint_V |(f^n)'| = c \sum_k \iint_{V_k} |(f^n)'| \asymp \sum_j \sum_{f^n(z)=z_j} |(f^n)'(z)|^{-1}.$$

It is known that for every hyperbolic polynomial f and every $t \in \mathbb{R}$, the following limit exists:

$$P(t) = P_f(t) = \lim_{n \to \infty} \tfrac{1}{n} \log \sum_{f^n(z)=a} |(f^n)'(z)|^{-t}.$$

This limit is called *pressure* and it does not depend on a. (See [Bo, R].) It follows from (4) that

$$\Phi(f^n) \geq c \exp(nP_f(1)) = c(\deg f^n)^{\kappa},$$

where $\kappa = P_f(1)/\log \deg f$. D. Ruelle [R] proved that $t \mapsto P_f(t)$ is a decreasing convex function. Moreover $P_f(\delta) = 0$, where δ is the Hausdorff dimension of J. The derivative dP_f/dt is negative at $t = \delta$. We conclude that $P_f(1) > 0$ if $\delta > 1$. This is the case for example if $f(z) = z^2 + \lambda$ and λ is in the interior of the above-mentioned cardioid C.

Note that a slightly more accurate computation shows that $\Phi(f^n) \asymp (\deg \, f^n)^\kappa$ where $\kappa = P_f(1)/\log \deg f$.

Two natural questions arise:

(1) What is $\sup\{P_f(1): f(z) = z^2 + \lambda\}$?

(2) Define the Littlewood order of a sequence of polynomials P_n by

$$\varlimsup_{n\to\infty} \log \Phi(P_n)/\log \deg \, P_n.$$

Is it true that the largest possible order occurs if P_n are iterates of a polynomial f?

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