Meromorphic Functions with Small Ramification

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Abstract. Let f be a meromorphic function in the plane and $N_1(r)$ stands for the Nevanlinna counting function of critical points of f. We obtain a complete description of functions f of finite lower order with the property

$$N_1(r) = o(T(r)),\ r \to \infty.$$

In particular, we show that this property is equivalent to

$$\sum_{a \in \bar{\mathbf{C}}} \delta(a, f) = 2.$$

1. Introduction. For a function f meromorphic in the complex plane $\mathbf{C}$ we use the standard notations of Nevanlinna theory: $T(r,f)$, $N(r.f)$, $m(r,f)$, $\delta(a)$, etc. (See [20], [14].) Recall that the "ramification term"

$$N_1(r) = N(r, 1/f') + 2N(r,f) - N(r,f')$$

counts the multiple points of f. The second main theorem of Nevanlinna states that for arbitrary collection of distinct $a_1, \ldots, a_q \in \bar{\mathbf{C}}$ we have

$$\sum_{k=1}^{q} m(r, a_k) + N_1(r) \leq 2T(r) + S(r), \tag{1}$$

where S is a small error term. Particularly for functions f of finite order $S(r) = O(\log r)$, $r \to \infty$, and from (1) it follows that

$$\sum_{k=1}^{q} \delta(a_k) + \vartheta \leq 2, \tag{2}$$

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where

$$\delta(a) = \liminf_{r\to\infty} \frac{m(r,a)}{T(r)}, \quad \vartheta = \limsup_{r\to\infty} \frac{N_1(r)}{T(r)}.$$

While the first member of (2) has been studied extensively, very little is known about the behavior of the ramification term.

Suppose that f has finite order and no multiple points,

$$N_1(r) \equiv 0. \tag{3}$$

It follows that the Schwarzian derivative

$$F = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \tag{4}$$

is an entire function because, as a direct computation shows, poles of F may occur only at multiple points of f. Taking into account that f is of finite order and using the lemma on logarithmic derivative, we conclude that

$$m(r,F) = O(\log r), \qquad r \to \infty,$$

so F is a polynomial. Now (4) can be considered as an algebraic differential equation with respect to f. The general solution of this equation can be represented as the ratio of two linearly independent solutions of the linear equation

$$y'' + \frac{1}{2}Fy = 0.$$

Using asymptotic integration of this equation, F. Nevanlinna [18] (see also [19] and [20], Section 262) has studied meromorphic functions of finite order without multiple points. These functions have the following properties:

(a) $T(r,f) \sim cr^{n/2}$, where $c > 0$ and $n \geq 2$ is a natural number.

(b) The plane is partitioned into n equal angular domains:

$$D_j = \{z : \phi_{j-1} < \arg z < \phi_j\}, \quad 1 \leq j \leq n, \quad \phi_n = \phi_0$$

so that for some $b_j \in \bar{\mathbf{C}}$ we have

$$\log\frac{1}{|f(re^{i\theta}) - b_j|} = \pi c r^{n/2} \sin\frac{n}{2}(\theta - \phi_{j-1}) + o(r^{n/2}), \quad r \to \infty,$$

uniformly with respect to θ in any angle that lies strictly inside D_j. If $b_j = \infty$, then the left-hand side has to be replaced by $\log|f(re^{i\theta})|$.

Thus, if a number $a \in \bar{\mathbf{C}}$ occurs $p(a)$ times among the $\{b_j\}$, then $\delta(a) = 2p(a)/n$; in particular, all deficiencies are rational and

$$\sum_{a \in \bar{\mathbf{C}}} \delta(a) = 2. \tag{5}$$

Another approach for obtaining this result, due to L. Ahlfors [2], consists in the investigation of the Riemann surface onto which the function f maps the plane. It can be shown that this Riemann surface has a finite number of logarithmic branch points and does not have any algebraic branch points. Such Riemann surfaces admit a full description, and the statements (a) and (b) are obtained by explicitly constructing a map of the Riemann surface onto the plane, close to a conformal one.

In this paper we study meromorphic functions of finite (lower) order that satisfy

$$N_1(r) = o(T(r, f)), \qquad r \to \infty. \tag{6}$$

Our main result is that (1) implies (5), which supports the general principle, stated by R. Nevanlinna ([20], Section 228) as follows: "It is natural for various reasons to hypothesize that under normal circumstances *equality* holds in the inequalities which we have called the second main theorem, provided that all of the deficient values $a_1, a_2, \ldots$ are considered". Some other results illustrating this principle are in [23], Ch. IV. On the other hand, as R. Nevanlinna observes on the same page, there are entire functions of *infinite order* with very regular behavior which satisfy (3) and have no finite deficient values.

Observe that for a function f of finite order the assumption (5) together with the second main theorem (1) implies (1). The arguments described above led F. Nevanlinna [18] to the following conjecture. Let f be a meromorphic function of finite order ρ, satisfying (5). Then:

(i) 2ρ is a natural number ≥ 2.
(ii) If $\delta(a) > 0$, then $\delta(a) = p(a)/\rho$, where $p(a)$ is a natural number.
(iii) All deficient values are asymptotic.

After some partial results by A. Pfluger and A. Weitsman, a complete proof of F. Nevanlinna's conjecture was obtained by D. Drasin [8]. His proof may be considered as one of the most complicated proofs in function theory. It involves tools such as Ahlfors' theory of covering surfaces, quasiconformal deformations etc. A shorter proof based mainly on classical potential theory was given in [10]. Some further improvements and generalizations were given in [11].

A conjecture that the relation (1) for functions of finite order implies (i), (ii), (iii) and even (5) seems very natural. A partial supporting result was obtained recently by D. Shea [22]: If a meromorphic function satisfies (1), then its lower order is at least $\frac{1}{2}$.

In this paper we give a complete description of meromorphic functions of finite lower order with the property (1), from which it follows that the conjecture is true.

Theorem *Let f be a meromorphic function of finite lower order satisfying* (1). *Then* (i), (ii), (iii), *and* (5) *hold. If we normalize such that $\delta(\infty) = 0$, then*

$$\log \frac{1}{|f'(re^{i\theta})|} = \pi r^\rho \ell_1(r) \big| \cos \rho\big(\theta - \ell_2(r)\big) \big| + o\big(r^\rho \ell_1(r)\big), \quad r \to \infty, \tag{7}$$

uniformly with respect to θ when $re^{i\theta} \notin C_0$. Here C_0 is the union of the disks $D(z_k, r_k)$ such that

$$\sum_{\{k:|z_k|<R\}} r_k = o(R), \qquad r \to \infty,$$

while ℓ_j are continuous functions with the properties $\ell_1(ct) \sim \ell_1(t)$ and $\ell_2(ct) = \ell_2(t) + o(1)$ as $t \to \infty$ uniformly with respect to $c \in [1, 2]$. Moreover,

$$T(r, f) \sim r^\rho \ell_1(r), \qquad r \to \infty. \tag{8}$$

REMARK 1. The assumptions of the Theorem can be slightly relaxed by replacing the lower order by so called lower Pólya order, which is defined in (19), Section 3.

REMARK 2. Our result contains D. Drasin's theorem [8]. We could not prove that (1) implies (5) directly, without proving first the properties (i), (ii), (iii).

The method of the proof is of purely potential-theoretic nature. In Section 2 we recall some facts from potential theory (including the definition and properties of fine topology). All necessary properties of fine topology are stated explicitly in Section 2. The proof of the theorem begins in Section 3, which concludes with the proof of the most difficult part of the theorem, namely, that the asymptotics (7) holds in some annuli $\varepsilon_j r_j < |z| < \varepsilon_j^{-1} r_j$, $\varepsilon_j \to 0$, $r_j \to \infty$. We finish the proof in Section 4. Section 5 contains the detailed proof of a crucial lemma from potential theory, which is closely related to the main lemma in [10]. To make the paper self-contained we include its proof in all detail.

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2 Preliminaries

2.1. δ-subharmonic functions. The general reference for subharmonic functions is [15]. Fix a domain $\Omega \in \mathbf{C}$. Denote by ${L^1}_{\text{loc}}$ the space of functions summable on each compact in Ω. Subharmonic functions belong to ${L^1}_{\text{loc}}$. Let v_1 and v_2 be subharmonic functions. The element $v = v_1 - v_2 \in {L^1}_{\text{loc}}$ is called a δ-subharmonic function. The "function" v may be undefined in those points z where $v_1(z) = v_2(z) = -\infty$. The set of such z's has capacity zero, so v is defined almost everywhere. We will say that a δ-subharmonic function v is defined at the point $z \in \Omega$ if there exists a finite or infinite limit

$$\lim_{r \to 0} \frac{1}{2\pi r} \int_0^{2\pi} v(z + re^{i\theta})\, d\theta,$$

and we shall denote this limit by $v(z)$. This definition is correct because for a subharmonic function v the indicated limit coincides with $v(z)$. Obviously if a δ-subharmonic function $v \geq 0$ a.e., then $v(z) \geq 0$ at all points z at which it is defined. In this case we write simply $v \geq 0$.

2.2. Order relations. The space of δ-subharmonic functions has a natural partial order: $v_1 \geq v_2$ means $v_1 - v_2 \geq 0$. The least upper bound and greatest lower bound of *finite* families exist with respect to this order. We denote them by $\vee\, v_n$ and $\wedge\, v_n$ respectively. Thus for example $v_1 \vee v_2 = (v_1 - v_2)^+ + v_2$ or $(v_1 \vee v_2)(z) = \max\{v_1(z), v_2(z)\}$.

The generalized laplacian of a δ-subharmonic function v is a charge (or signed measure), which is called the Riesz charge and denoted by $\mu[v]$. We will call v subharmonic if it coincides with a subharmonic function at all points of definition. This happens if and only if the corresponding Riesz charge is a (positive) measure, which will be denoted by $\mu[v] \geq 0$. The relation $\mu \geq 0$ defines a partial order on the set of all charges in Ω: we write $\mu_1 \geq \mu_2$ iff $\mu_1 - \mu_2 \geq 0$. The least upper bound and greatest lower bound with respect to this order are defined for finite families. For example $\mu_1 \vee \mu_2 = (\mu_1 - \mu_2)^+ + \mu_2$, where $(\cdot)^+$ is the positive part in the Jordan decomposition of the charge.

2.3. Fine topology. It is well known that subharmonic functions may be discontinuous. The smallest topology in the plane in which they are continuous is called fine topology (H. Cartan), a general reference is [6]. All expressions "finely closed", "finely open" etc. are related to fine topology. For example a "fine domain" is a "finely open, finely connected set". We will use the following properties of the fine topology:

- **(i)** If D is a finely open set and $z_0 \in D$, then for a set of r's of positive linear measure, the circles $\{z : |z - z_0| = r\}$ are contained in D. This is called Lebesgue-Beurling property, and is due to M. Brelot [5], p. 334–335 (see also [6], Proposition IX.2 and Theorem IX.10).
- **(ii)** If a set A has capacity zero, then it is finely closed and all its points are finely isolated. This follows immediately from the definition of fine topology.
- **(iii)** Fine domains are polygonally connected (which means that every two points of a fine domain may be connected in this domain by a polygonal curve). This property was proved by B. Fuglede [12].

It follows from (i) that fine domains have positive Lebesgue measure. So the set of fine components of a fine open set is at most countable.

Note that δ-subharmonic functions need not be finely continuous; however $\{z : v_1(z) > v_2(z)\}$ is a well defined finely open set.

Lemma 1 ([7], p. 186). *Let v_1 and v_2 be δ-subharmonic functions in Ω. If they coincide on some fine open set E, then the restrictions of their Riesz charges to E coincide.*

2.4. Normal convergence of δ-subharmonic functions. A sequence $\{v_j\}$ of δ-subharmonic functions in Ω is called normal if for every compact $K \subset \Omega$, the sequences $||v_j||_{L^1(K)}$ and $(\mu[v_j])^-(K)$ are bounded. From every normal sequence one can select a subsequence that converges in ${L^1}_{\text{loc}}$ to a δ-subharmonic function v. Furthermore, we have:

(i) $\mu[v_j] \to \mu[v]$ weakly, i. e., for every continuous function ϕ with compact support in Ω we have

$$\int \phi \, d\mu[v_j] \to \int \phi \, d\mu[v].$$

(ii) $v_j \to v$ in L^1 with respect to linear measure on any circle in Ω. (This also is true relative to any rectifiable arc, but we do not use this generalization.)

(iii) $\text{meas}_1\{z \in K : |v_j(z) - v(z)| > \varepsilon\} \to 0$ for every $\varepsilon > 0$ and every compact $K \subset \Omega$.

The symbol meas_1 in (iii) stands for Carleson 1-measure, which is defined as follows: $\text{meas}_1(E)$ is the infimum of sums of radii of discs covering a set E. For (i) and (iii) see [3], [4], [16], for (ii)–[1].

It follows from (iii), that for every compact K and every $\varepsilon > 0$, there exists a set E, $\text{meas}_1(E) < \varepsilon$, such that the convergence of some subsequence is uniform on $K \backslash E$.

Lemma 2 *Let $\{g_j\}$ be a sequence of meromorphic functions in Ω and $t_j \to 0$ be a sequence of positive numbers. Suppose that the sequences of δ-subharmonic functions $t_j \log |g_j|$ and $t_j \log |g_j{}'|$ are normal and converge to v_1 and v_2 respectively. Then*

$$v_2 \leq v_1 \tag{9}$$

and on each fine component D of the set $\{z : v_2(z) < v_1(z)\}$ the function v_1 is identically equal to some constant t. Moreover D is a fine component of the set $\{z : v_2(z) < t\}$.

From our point of view (9) is an analog of the lemma on logarithmic derivative.

Proof. First prove (9). Without loss of generality we may suppose that $\Omega = D(0, R')$, $R' > 0$. Represent the logarithmic derivative by the differentiated Poisson–Jensen formula ([14], p. 36, equation (2.2)):

$$\frac{g_j{}'(z)}{g_j(z)} = \frac{1}{2\pi}\int_0^{2\pi} \log|g_j(\mathrm{Re}^{i\theta})| \frac{2\mathrm{Re}^{i\theta}}{(\mathrm{Re}^{i\theta}-z)^2}\,d\theta + \sum\left(\frac{\bar{a}_m}{R^2-\bar{a}_m z} - \frac{1}{a_m - z}\right) + \sum\left(\frac{1}{b_m - z} - \frac{\bar{b}_m}{R^2 - \bar{b}_m z}\right),$$

where $|z| < r < R < R'$ and sums run over the zeros a_m and poles b_m of g_j in $D(0, R)$. A routine estimate gives

$$\frac{|g_j'(z)|}{|g_j(z)|} \le \frac{R}{\pi(R-r)^2}\int_0^{2\pi} |\log\ |g_j(\mathrm{Re}^{i\theta})|\,|\,d\theta + \sum \frac{1}{|c_m - z|} + \frac{n_j}{R-r},$$

where the sum runs over all zeros and poles and n_j is their total number in $D(0, R)$. Now we integrate with respect to the area over $D(0, r)$ and use the estimates:

$$(\pi r^2)^{-1}\iint \log^+\left|\frac{g_j'}{g_j}\right| dx\,dy \le \log^+\left[(\pi r^2)^{-1}\iint \left|\frac{g_j'}{g_j}\right| dx\,dy\right] + \log 2,$$

and

$$\int_0^{2\pi} |\ \log|g_j(\mathrm{Re}^{i\theta})|\,|\,d\theta = O\left(\frac{1}{t_j}\right), \quad n_j = O\left(\frac{1}{t_j}\right), j \to \infty,$$

which follow from the normality of the sequence $t_j \log|g_j|$. After multiplying by t_j and passing to the limit as $j \to \infty$ (so that $t_j \to 0$), we get

$$\int_{x^2+y^2<r^2} (v_2 - v_1)^+\,dx\,dy \le 0$$

for every $r < R$. This proves that $v_2 \le v_1$.

Now prove that v_1 is constant on every fine component D of the set $\{z : v_2(z) < v_1(z)\}$. Fix $z_0 \in D$. For a set of r's of positive linear measure, the circle $C_r = \{z : |z - z_0| = r\}$ is contained in D and we have

$$v_2(z) < N - \varepsilon < N < v_1(z), \qquad z \in C_r, \tag{10}$$

for some real N and positive ε. So we can fix an arbitrary small r such that $C_r \subset D$, (10) is satisfied and convergence

$$t_j \log|g_j{}'| \to v_2; \quad t_j \log|g_j| \to v_1 \tag{11}$$

is uniform on C_r. Fix a point $z_1 \in C_r$. We have

$$|g_j(z_1)| = \exp \frac{N+\tau_j}{t_j}$$

for some $\tau_j > 0$ and

$$|g_j{}'(z)| < \exp \frac{N-\varepsilon}{t_j}, \qquad z \in C_r.$$

After integrating $g_j{}'$ along C_r and dividing by $|g_j(z_1)|$, we get

$$\left| \left| \frac{g_j(z)}{g_j(z_1)} \right| - 1 \right| < C \exp \frac{-\varepsilon - \tau_j}{t_j},$$

where C is the length of C_r. So

$$\left| \ \log |g_j(z)| - \log |g_j(z_1)| \ \right| < C \exp \frac{-\varepsilon}{t_j},$$

and, after multiplying by t_j and passing to the limit as $j \to \infty$, we get $v_1(z) = v_1(z_1)$, $z \in C_r$; thus v_1 is constant on C_r. We have proved that every point $z_0 \in D$ can be surrounded by arbitrarily small circles on which v_1 is constant.

Now take two arbitrary points a and b in D. Connect them by a simple polygonal curve $\Gamma \subset D$. Choose small circles C^1 and C^2 centered at a and b, respectively, such that v_1 is constant on these circles and

$$|v_1(z) - v_1(a)| < \varepsilon, \quad z \in C^1,$$

$$|v_1(z) - v_1(b)| < \varepsilon, \quad z \in C^2,$$

where $\varepsilon > 0$ is a fixed arbitrary small number. The circles can be chosen so small that each of them intersects Γ only once, say at the points a_1 and b_1, respectively. Let Γ_1 be the part of Γ between a_1 and b_1. Each point of Γ_1 is the center of some arbitrarily small circles on which v_1 is constant. Choose a finite collection of these circles such that their interiors cover Γ_1. We can assume that the union of these circles together with C^1 and C^2 is connected. So v_1 is constant on this union of circles and $|v_1(a) - v_1(b)| < 2\varepsilon$. This proves that $v_1(a) = v_1(b)$ and v_1 is constant on D.

Denote this constant by t and consider the fine component D' of the set $\{z \in \Omega : v_2(z) < t\}$ containing D. We are going to prove that $D = D'$. Suppose that $a \in D' \setminus D$. It is enough to prove that $v_1(a) = t$. Choose a point $b \in D$ and connect a and b by a simple polygonal curve $\Gamma \subset D'$. Repeating the covering-by-circles argument of the preceding paragraph, we can find a simple rectifiable curve Γ' close to Γ with the following properties:

- Γ' has the ends a' and b' close to a and b and $v_1(b) = v_1(b')$, $|v_1(a) - v_1(a')| < \varepsilon$, where $\varepsilon > 0$ is arbitrary small;
- the convergence in (11) is uniform on Γ';
- $v_2(z) < t - \tau$, $z \in \Gamma'$.

We have

$$|g_j(b')| = \exp \frac{t + o(1)}{t_j},$$

and

$$|g_j{}'(z)| < \exp \frac{t - \tau}{t_j}, \qquad z \in \Gamma'.$$

Integration along Γ' gives

$$\left| \left| \frac{g_j(a')}{g_j(b')} \right| - 1 \right| \leq C \exp \frac{-\tau + o(1)}{t_j}, \qquad j \to \infty,$$

where C is the length of Γ'. Thus,

$$\big| \log |g_j(a')| - \log |g_j(b')| \big| \leq C \exp \frac{-\tau + o(1)}{t_j}$$

and after multiplying by t_j and passing to the limit as $j \to \infty$ we have $v_1(a') = v_1(b')$. Thus $|v_1(a) - v_1(b)| < \varepsilon$ and this proves the last statement of the lemma. □

2.5. Tracts of subharmonic functions and the standard decomposition. Let u be a subharmonic function in $\mathbf{C}$. Consider a function $t \mapsto V(t)$, which assigns to each $t \in \mathbf{R}$ a fine component V of the set $\{z : u(z) > t\}$. Such a function is called tract. Given any fine component V_0 of the set $\{z : u(z) > t_0\}$, there is a tract $V(t)$ such that $V(t_0) = V_0$. This follows from the fact that the function u is unbounded in every fine component of any set of the form $\{z : u(z) > t_0\}$ [12].

Lemma 3 (Denjoy–Carleman–Ahlfors Theorem for subharmonic functions) *Suppose that a subharmonic function satisfies* $u(z) \leq c|z|^\lambda$, $z \in \mathbf{C}$ *for some* $c > 0$ *and* $\lambda > 0$. *Then the number of different tracts of* u *is at most* $\max\{2\lambda, 1\}$.

This lemma is proved in [15]. (The definition of tract is slightly different in [15], but it is easy to see that the lemma is equivalent to Theorem 4.16 in [15]).

Suppose now that $u \geq 0$ is a subharmonic function in a simply connected domain $D \subset \mathbf{C}$. Denote by E a fine component of the set $\{z \in D : u(z) > 0\}$.

Then the function defined in the following way:

$$u_E(z) = u(z), \qquad z \in E,$$
$$u_E(z) = 0, \qquad z \in \mathbf{C}\backslash E$$

is subharmonic. Indeed, denote by E' the union of all discs whose boundaries are in E. It is clear that E' is open (in the standard topology). Furthermore, it follows from the property (i) of the fine topology that $E \subset E'$. The domain E' does not intersect the fine components $F \neq E$ of $\{z \in D : u(z) > 0\}$, because no such F is relatively compact in D and F is polygonally connected. So $u_E(z) = u(z)$ in the (standard) domain E' and $u(z) = 0$ in the complement of E'. Thus u_E is subharmonic.

We define the *support* of a δ-subharmonic function as the set of points where this function is defined and distinct from 0. The functions u_E defined above have disjoint supports for different E's. So we have

$$u = \sum u_E, \tag{12}$$

where the sum runs over all fine components of $\{z : u(z) > 0\}$. We call this representation the *standard decomposition* of u.

Lemma 4 *Let u be a subharmonic function in $\mathbf{C}$. Let E be a component of the set $\{z : u(z) > 0\}$ and D be a component of the set $\{z : u(z) < t\}$, $t > 0$, intersecting E. Then the set $X = E \cap D$ is connected. (All topological terms here are related to standard topology.)*

Proof. Suppose X is not connected. Then there exist simply connected domains G_1 and G_2 with the following properties:

$$G_1 \cap G_2 = \emptyset,\ G_i \cap X \neq \emptyset,\ \partial G_i \cap X = \emptyset, \qquad i = 1, 2.$$

We cannot have $u(z) \leq 0,\ z \in \partial G_1$ or $u(z) \geq t,\ z \in \partial G_1$ because otherwise E or D would be disconnected. So in each point $z \in \partial G_1$ we have one of the two inequalities: $u(z) \leq 0$ or $u(z) \geq t$ and both inequalities occur. Now we may find circles C_n with the following properties:

- the union of the circles is connected;
- on each circle we have either $u(z) \leq t/4$ or $u(z) \geq 3t/4$ and
- both inequalities occur.

These properties are inconsistent and this proves the lemma. □

3. Reduction of the Theorem to a problem of Potential Theory. We begin the proof of the Theorem. Recall that a point $a \in \bar{\mathbf{C}}$ is called non-exceptional in the sense of Valiron if $m\big(r, (f-a)^{-1}\big) = o\big(T(r,f)\big)$, $r \to \infty$. The set of Valiron exceptional values has zero capacity [20], Section 233. So, after performing a fractional-linear transformation on f we can assume that the poles of the function f are simple and that

$$m(r,f) = o\big(T(r,f)\big), \qquad r \to \infty, \tag{13}$$

so

$$\bar{N}(r,f) = N(r,f) \sim T(r,f), \qquad r \to \infty. \tag{14}$$

Moreover, choose a sequence $b_j \in \mathbf{C}$ such that b_j are not exceptional in the sense of Valiron and all b_j-points of f are simple. Then

$$m\left(r, \frac{1}{f-b_j}\right) = o\big(T(r,f)\big), \qquad r \to \infty. \tag{15}$$

Recall that a sequence $r_j \to \infty$ is called a sequence of Pólya peaks of order λ for $T(r) = T(r,f)$ if for some sequence $\varepsilon_j \to 0$, we have

$$T(r) \leq (1+\varepsilon_j)\left(\frac{r}{r_j}\right)^{\lambda} T(r_j), \qquad \varepsilon_j r_j \leq r \leq \frac{r_j}{\varepsilon_j}.$$

By some reasons which will be clear later we prefer to use the weaker relation:

$$T(r) \leq \left(\frac{r}{r_j}\right)^{\lambda-\delta} T(r_j), \qquad \varepsilon_j r_j < r < \varepsilon_0 r_j; \tag{16}$$

$$T(r) \leq \left(\frac{r}{r_j}\right)^{\lambda+\delta} T(r_j), \qquad {\varepsilon_0}^{-1} r_j < r < {\varepsilon_j}^{-1} r_j, \tag{17}$$

where $\varepsilon_j \to 0$, $j \to \infty$; $\delta > 0$ is fixed arbitrary small number and $\varepsilon_0 > 0$ may depend on δ. Set

$$\lambda^* = \sup\left\{p : \limsup_{x,A\to\infty} \frac{T(Ax)}{A^p T(x)} = \infty\right\}; \tag{18}$$

$$\lambda_* = \inf\left\{p : \liminf_{x,A\to\infty} \frac{T(Ax)}{A^p T(x)} = 0\right\}. \tag{19}$$

It is known [9] that Pólya peaks of order λ exist if and only if $\lambda_* \leq \lambda \leq \lambda^*$ and $[\rho_*, \rho] \subset [\lambda_*, \lambda^*]$, where ρ and ρ_* are, respectively, order and lower order of the function $T(r)$. By the theorem of D. Shea [22], mentioned in the introduction, we may suppose that $\rho > 0$. Until the end of this section, we fix a number $\lambda \in [\lambda_*, \lambda^*]$, $\lambda > 0$ and a sequence of Pólya peaks r_j of order λ for the function $T(r)$. It is assumed that $\delta < \lambda$ in (16), (17). In the course of the proof we will select several times subsequences from the sequence of Pólya peaks preserving the same notation r_j.

Consider the sequence of δ-subharmonic functions

$$U_j(z) = \frac{1}{T(r_j)} \log \frac{1}{|f'(r_j z)|}. \tag{20}$$

From the lemma on the logarithmic derivative it follows that

$$m\left(r, \frac{f'}{f}\right) = o(T(2r)), \qquad r \to \infty, \tag{21}$$

so, in view of (13) and (21) we have

$$m(r, f') \leq m(r, f) + m\left(r, \frac{f'}{f}\right) + \log 2 = o\big(T(2r)\big), \qquad r \to \infty. \tag{22}$$

As all poles are simple, $N(r, f') = 2N(r, f)$ so that

$$\big(2 + o(1)\big)T(r, f) \leq T(r, f') \leq 2T(r, f) + o\big(T(2r, f)\big), \qquad r \to \infty. \tag{23}$$

Using a theorem of J. Anderson and A. Baernstein [1], we conclude from (17) and (23) that the sequence U_j defined in (20) is normal. After selecting a subsequence of Pólya peaks, we get

$$U_j \to u, \qquad j \to \infty. \tag{24}$$

The function u has the following properties. First of all, (22) gives that

$$u \geq 0. \tag{25}$$

The key assumption (1) of our Theorem implies that u is subharmonic. Denote by 2μ the Riesz measure of u. The measure μ describes the asymptotic distribution of poles of f in the following sense. Define $\nu(E)$ to be the number of poles in a set $E \subset \mathbf{C}$ and set

$$\nu_j(E) = \frac{1}{T(r_j)} \nu(r_j E),$$

where rE denotes the homothety of the set E with center at the origin and scaling factor r. Then

$$\mu = \text{weak} \lim_{j \to \infty} \nu_j. \tag{26}$$

We have $N(r, 1/f') = o\big(T(r, f)\big)$, $r \to \infty$, so

$$m\left(r, \frac{1}{f'}\right) = T(r, f') + o\big(T(r, f)\big).$$

This asymptotic equality together with (23), (16) and (17) implies

(27) $$u(z) \leq c|z|^{\lambda-\delta}, \quad |z| < \varepsilon_0,$$

(28) $$u(z) \leq c|z|^{\lambda+\delta}, \quad |z| > {\varepsilon_0}^{-1}$$

and

(29) $$u \not\equiv 0.$$

Notice that if in (20) we replace f by $(f - b_n)^{-1}$, where b_n's are defined in the beginning of this section (15), then the new functions U_j have the same limit u. This follows from

$$\left(\frac{1}{f-b}\right)' = -\frac{f'}{(f-b)^2}$$

and

$$\lim_{j\to\infty} \frac{1}{T(r_j)} \Big| \log|f(r_j z) - b| \Big| \to 0.$$

(The last statement holds since $m(r, f - b) + m\big(r, (f - b)^{-1}\big) = o\big(T(r, f)\big)$ as b and ∞ are not exceptional in the sense of Valiron.)

By Lemma 3 the subharmonic function u satisfying (28) has a finite number of finely connected components of the set $\{z : u(z) > 0\}$. We denote these components by $E_1, \dots E_q$.

Fix an arbitrary large number $R > 0$ and denote

(30) $$B(R, u) = \max\{u(z) : |z| \leq R\}.$$

Choose some circles $C_k \subset E_k$ with the following properties:

(31) $$u(z) > B(R, u), \quad z \in C_k, \quad 1 \leq k \leq q;$$

- convergence in (24) is uniform on C_k, $1 \leq k \leq q$;
- if we replace in (20) f by $(f - b_n)^{-1}$, $1 \leq n \leq q + 1$, then the convergence in (24) will be uniform on C_k.

Choose arbitrary points $z_k \in C_k$ and select a subsequence of Pólya peaks such that

(32) $$f(r_j z_k) = a_{k,j} \to a_k, \qquad j \to \infty,$$

where a_k are some points on the Riemann sphere. We may suppose without loss of generality that all a_k are finite. Indeed if this is not the case, take a b from the collection $b_1, \dots, b_{q+1}$, which is different from any a_k, and replace our function f by $(f-b)^{-1}$. From this point we suppose that all a_k are finite and dispense with the b_n.

Consider the sequences of δ-subharmonic functions

$$U_{k,j}(z) = \frac{1}{T(r_j)} \log \frac{1}{|f(r_j z) - a_{k,j}|}, \qquad 1 \le k \le q. \tag{33}$$

Since $a_{k,j}$ are uniformly bounded in view of (32) and finiteness of a_k, we have $T(r, f - a_{k,j}) = T(r,f) + O(1)$ uniformly in j. Thus, by the theorem of Anderson–Baernstein, the sequences $U_{k,j}$ are normal. Selecting a subsequence of Pólya peaks we get

$$U_{k,j} \to w_k, \qquad j \to \infty, \quad 1 \le k \le q. \tag{34}$$

The functions w_k are δ-subharmonic and we have

$$0 \le w_k \le u, \qquad 1 \le k \le q. \tag{35}$$

The left inequality holds because $m(r, f - a_{j,k}) = o\big(T(r,f)\big)$, $r \to \infty$, uniformly in j, and the right inequality follows from (9) in Lemma 2.

The condition (31) and uniform convergence in (24) on C_k imply $|f'(z)| < \exp(-B(R,u)T(r_j))$, $z \in r_j C_k$. Integration of $f - a_{j,k}$ along $r_j C_k$ gives

$$\log|f(z) - a_{k,j}| < -B(R,u)T(r_j) + O(\log r_j), \quad z \in C_k,\ j \to \infty,$$

so that

$$w_k(z) \ge B(R,u), \qquad z \in C_k. \tag{36}$$

Denote by η_k the Riesz charge of w_k. We have the weak convergence of the Riesz charges:

$$\mu[U_{k,j}] \to \eta_k, \qquad j \to \infty.$$

So, by (33)

$$\eta_k{}^+ \le \mu, \tag{37}$$

where μ is the measure described in (26).

Define new δ-subharmonic functions as follows:

$$u_k(z) = w_k(z), \qquad z \in E_k;$$

$$u_k(z) = 0, \qquad z \in \mathbf{C}\setminus E_k.$$

To prove that the u_k are δ-subharmonic, consider the standard decomposition of u, defined in (12):

$$u = \sum u_{E_k}.$$

It is easy to verify using (35) that

$$u_k = \Big(w_k - \sum_{n \neq k} u_{E_n}\Big)^+,$$

and δ-subharmonicity follows immediately. We conclude from (35) that

$$0 \le u_k \le u, \qquad 1 \le k \le q. \tag{38}$$

Also (36) implies

$$u_k(z) \ge B(R,u), \qquad z \in C_k, \quad 1 \le k \le q. \tag{39}$$

Denote by μ_k the Riesz charge of u_k. We prove that

$$\mu_k \le {\eta_k}^+. \tag{40}$$

It follows from Lemma 1 that $\mu_k|_{E_k} = \eta_k|_{E_k}$ and $\mu_k|_{E_m} = 0$, $m \neq k$. To obtain that $\mu_k|_F \le \eta_k|_F$ on $F = \mathbf{C}\setminus \cup E_m$, we use the following result:

Lemma 5 (A. Ph. Grishin, [13]). *If $v_1 \ge v_2$ are two δ-subharmonic functions and $v_1(z) = v_2(z)$, $z \in F$, for some Borel set F, then $\mu[v_1]|_F \le \mu[v_2]|_F$.*

From this lemma we conclude that $\mu_k|_F \le \eta_k|_F$ and (40) follows.

Consider the set $\{z \in E_k : u_k(z) < u(z)\} = \{z \in E_k : w_k(z) < u(z)\}$. By Lemma 2 this set is the union of some fine components $D_{k,m,n}$ of the sets $\{z \in E_k : u(z) > t_{k,m}\}$. By Lemma 3 the number of these fine components $D_{k,m,n}$ is finite. Moreover, it follows from (39) that $t_{k,m}$ are strictly positive. So if we denote $t = \min_{k,m}\{t_{k,m}\}$, then $u_k(z) = u(z)$, when $z \in E_k$ and $u(z) < t$. Furthermore $u_k(z) > 0$, $z \in E_k$, so the support of u_k is connected.

Denote by D the component of the set $\{z \in \mathbf{C} : u(z) < t\}$ containing 0. This set is open in the usual topology because the subharmonic function u is upper semi-continuous; it contains zero because $u(0) = 0$ in view of (27). By the maximum principle, D is simply connected.

Now the u_k have disjoint supports (which means that at each point at most one of them is different from zero). This follows from the definition of u_k. Also $u_k(z) = u(z)$, $z \in E_k \cap D$. So

$$\sum_{k=1}^{q} u_k(z) = u(z), \qquad z \in D. \tag{41}$$

It follows from (41) that u_k are subharmonic in D: $\mu_k|_{E_k\cap D} = 2\mu|_{E_k\cap D} \geq 0$ by Lemma 1 and $\mu_k|_{D\setminus E_k} \geq 0$ by Lemma 5 and the fact that $u_k \geq 0$.

Applying consequently (41), (40) and (37), we obtain for the restrictions of measures to D:

$$\sum_{k=1}^{q} \mu_k = 2\mu \geq 2 \bigvee_{k=1}^{q} \mu_k. \tag{42}$$

An immediate consequence from (42) is

$$q \geq 2. \tag{43}$$

In particular, D is unbounded. The restrictions of u_k to D have connected supports by Lemma 4.

Lemma 6 *Suppose that non-negative subharmonic functions $u_1, \ldots, u_q$ are defined in a simply connected domain D and have disjoint connected supports. Assume that their Riesz measures satisfy* (42). *Then there exists a Riemann surface F with a two-sheeted ramified covering $p : F \longrightarrow D$ and a function h harmonic on F such that $u \circ p = |h|$, where $u = \sum u_k$. Furthermore, the covering p is ramified over at most $q - 2$ points in D and each ramification point of p is a zero of h of order at least* 3.

This is a modified form of the main lemma from [10]. For completeness, a proof will be given in Section 5.

Applying this lemma to our functions u and u_k we obtain a function h harmonic on the Riemann surface F such that $u \circ p = |h|$. Consider a disk $D_0 = D(0, r)$ so small that p is unramified over $D_0\setminus\{0\}$. Then $u(z^2) = |h_1(z)|$, $z \in D_0$ for some harmonic function h_1 on D_0. It follows from (27) that $h_1(0) = 0$ and if we denote by $2q_*$ the number of components of the set $\{z \in D_0 : h_1(z) \neq 0\}$, then (again by (27))

$$q_* \geq 2(\lambda - \delta). \tag{44}$$

It is easy to see that the set $K = \{z \in D : u(z) = 0\}$ contains q_* disjoint simple piecewise analytic curves ℓ_n starting at the origin and tending to ∂D. None of them can have a finite cluster point $z_0 \in \partial D$. Otherwise we would have $u(z_0) \geq t > 0$ and $u(z) = 0$, $z \in \ell_n$ and this would contradict the existence of arbitrary small circles centered at z_0 on which u is positive. So each ℓ_n, $1 \leq n \leq q_*$ tends to infinity.

Thus

$$q \geq q_*. \tag{45}$$

But q is the number of fine components of the set $\{z \in \mathbf{C} : u(z) > 0\}$ for a subharmonic function u subject to the growth restriction (28), so by Lemma 3 and (43) we have $q \leq 2(\lambda+\delta)$. Together with (44) and (45) this gives $|q-2\lambda| \leq 2\delta$. But $\delta > 0$ was chosen arbitrarily small in the beginning of this section, so $q = 2\lambda$.

Recall now that λ was chosen as an arbitrary number between λ_* and λ^*. So

$$\rho = \rho_* = \lambda^* = \lambda_* = n/2 \tag{46}$$

for some natural $n \geq 2$.

The next conclusion is that u has maximal possible number of components of the set $\{z \in \mathbf{C} : u(z) > 0\}$. Together with Lemma 3, this implies that all sets $\{z \in E_k : u(z) > t\}$ are connected for every $t > 0$.

Using this observation, we are going to prove that u is harmonic in $\bigcup E_k$. To do this we reconsider the definition of the set D (see paragraph following Lemma 5). Fix k between 1 and q. Each fine component of the set $\{z \in E_k : u_k(z) < u(z)\}$ is a fine component $D_{k,m,n}$ of some set $\{z \in E_k : u(z) > t_m\}$. But now we know that there is *only one* such component $D_{k,m,n}$ for each k. Denote it by D_k and the corresponding value t by t_k. We have $u_k(z) = u(z) \leq t_k$ if $z \in E_k \backslash D_k$ and $u_k(z) = t_k$ if $z \in D_k$. So $u_k(z) \leq t_k$, $z \in E_k$ and (39) implies $t_k \geq B(R, u)$. So our domain D which was defined as the component of $\{z \in \mathbf{C} : u(z) < \min_k\{t_k\}\}$, $0 \in D$, contains the disc $D(0, R)$. We apply Lemma 6 to $D = D(0, R)$ for arbitrary $R > 0$ and conclude that there is a Riemann surface F, a two-sheeted ramified covering $p : F \longrightarrow \mathbf{C}$ and a harmonic function h on F which satisfies $u \circ p = |h|$. As the surface F has only a finite number of ramification points over $\mathbf{C}$, we may compactify it by adding one or two points over infinity. Let us study the harmonic function h on the Riemann surface F, which satisfies

$$|h(z)| \leq c|p(z)|^{n/2} \tag{47}$$

in view of (27), (28) and (46). Consider a multi-valued analytic function H on the Riemann surface F such that $h = \Re H$. The derivative $s = dH/dp$ is a single valued meromorphic function on F. (The singularities at infinity are at most poles in view of (47).) Again by (47) we conclude that the sum of multiplicities of poles of the function s over infinity is at most $n - 2$. The only zeros of dp are ramification points of F and all these zeros are simple. On the other hand at the ramification points of F, the function h has zeros of at least third order so dH has multiple zeros and s has zeros. Thus the only poles of s may occur in infinite points of F and their total multiplicity is at most $n-2$. Now by (47) the total multiplicity of zeros of s over 0 is at least $n - 2$. So s has no other zeros and the covering $p : F \longrightarrow \bar{\mathbf{C}}$ may be ramified only over 0 and ∞. Thus

$$u(z) = \Re\,(cz^{n/2})$$

with some $c \in \mathbf{C}$. To determine $|c|$ note that $m(r_j, 1/f') = \big(2 + o(1)\big)T(r_j, f)$, so by definition of u we have

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta}) = 2.$$

Thus $|c| = \pi$ and

$$u(re^{i\theta}) = \pi r^{n/2}\left|\cos\frac{n}{2}(\theta-\theta_0)\right| \tag{48}$$

for some $\theta_0 \in [0, 2\pi]$.

We summarize what has been proved in this section:

Proposition 1 *Let f be a meromorphic function which satisfies* (1), (14) *and* (13). *If we choose λ and r_j with the properties* (16) *and* (17) *and define U_j by* (20) *then there exists a subsequence of the sequence r_j such that $U_j \to u$ and u has the form* (48).

4. Conclusion of the proof. By (18), (19) and (46) we have

$$T(t) \leq \left(\frac{t}{r}\right)^{\rho-\delta} T(r), \qquad t \leq \varepsilon_0 r;$$

$$T(t) \leq \left(\frac{t}{r}\right)^{\rho+\delta} T(r), \qquad t \geq {\varepsilon_0}^{-1} r,$$

for every $\delta > 0$ and ε_0 depending on δ.

We conclude that the relations (16) and (17) are satisfied *for every sequence* $r_j \to \infty$. Thus we may apply Proposition 1 to every sequence $r_j \to \infty$. It follows that $T(tr)/T(r) \to t^\rho$, $r \to \infty$, uniformly with respect to $t \in [1, 2]$. Setting $T(r) = r^\rho \ell_1(r)$, we conclude that $\ell_1(tr) \sim \ell_1(r)$, $r \to \infty$ uniformly with respect to $t \in [1, 2]$. This proves (8).

Let us prove (7). Denote by $X = \{u(\,.\,;\theta_0) : \theta_0 \in [0, 2\pi]\}$ the set of all subharmonic functions of the form

$$u(re^{i\theta};\theta_0) = \pi r^\rho |\cos\rho(\theta-\theta_0)|.$$

It is clear that X is a compact subset of ${L^1}_{\text{loc}}$. Remark that ${L^1}_{\text{loc}}$ is a metric space and denote its metric by dist. Consider the family of δ-subharmonic functions

$$U_t(z) = \frac{1}{t^\rho \ell_1(t)}\log\frac{1}{|f'(tz)|}, \qquad t > 0.$$

By (8) (which has already been proved) and Proposition 1, we have

$$\text{dist}\,(U_t, X) \to 0. \tag{49}$$

Let $u_t \in X$ be the closest element to U_t. We claim that

$$\text{dist}\,(u_t, u_{ct}) \to 0, \qquad t \to \infty, \tag{50}$$

uniformly with respect to $c \in [1, 2]$. If this is not the case, we have

$$\text{dist}\,(u_{t_m}, u_{c_m t_m}) \geq \varepsilon > 0 \tag{51}$$

for some sequences $c_m \in [1, 2]$ and $t_m \to \infty$. As $c^{-\rho}u(cz) = u(z)$ for all $u \in X$ and $c > 0$, we have

$$\begin{aligned} u_{c_m t_m}(z) &= U_{c_m t_m}(z) + o(1) \\ &= {c_m}^{-\rho} U_{t_m}(c_m z) + o(1) \\ &= {c_m}^{-\rho} u_{t_m}(c_m z) + o(1) \\ &= u_{t_m}(z) + o(1), \end{aligned}$$

where $o(1)$ stands for a function which tends to zero in ${L^1}_{\text{loc}}$ as $m \to \infty$. This contradicts (51) and proves (50).

If we set $u_t = u\big(\,.\,, \theta_0(t)\big)$, then (50) is equivalent to

$$\theta_0(t) - \theta_0(ct) \to 0 \bmod 2\pi, \qquad t \to \infty,$$

uniformly with respect to $c \in [1, 2]$.

From (49) follows

$$U_t(z) = u(z; \theta_0(t)) + o(1), \qquad t \to \infty$$

in ${L^1}_{\text{loc}}$. This is equivalent to (7), because normal convergence of δ-subharmonic functions implies the convergence by 1-measure.

Once we have (7), all other statements of the Theorem 1 follow easily. Consider the family of curves $\Gamma_\phi = \{re^{i\theta} : \theta - \ell_2(r) = \phi\}$, $0 \leq \phi < 2\pi$. If a curve Γ_ϕ meets an exceptional circle, we replace the part of the curve inside this circle by a part of circumference of the circle. After performing this procedure, we get new curves ${\Gamma_\phi}'$ that do not intersect the exceptional set. We also have ${\Gamma_\phi}' = \{re^{i\theta} : \theta - \ell_2(r) = \phi + o(1)\}$. Integrating f' along the curves ${\Gamma_\phi}'$ and using the asymptotic formula (7), we conclude that for some $a_k \in \mathbf{C}$ there holds

$$\log \frac{1}{|f(re^{i\theta}) - a_k|} = \pi r^\rho \ell_1(r) |\cos \rho(\theta - \ell_2(r))| + o(r^\rho \ell_1(r)),$$

$$\frac{\pi}{2\rho}(2k - 1) \leq \theta - \ell_2(r) \leq \frac{\pi}{2\rho}(2k + 1),$$

as $re^{i\theta} \notin C_0$, $r \to \infty$, uniformly with respect to θ. This implies immediately that $\delta(a) = p(a)/\rho$, where $p(a)$ is the number of a_k's equal to a; the sum of deficiencies is two and all deficient values are asymptotic.

5. Proof of Lemma 6. We may suppose without loss of generality that $D = D(0, R_0)$, $R_0 > 0$.

Denote by D_k the union of the interiors of all Jordan domains whose boundaries are in the set $\{z \in D : u(z) > 0\}$. Then D_k are domains containing the supports of u_k. Each D_k is not relatively compact in D. It follows from the definition that D_k's are simply-connected.

A point $z_0 \in \partial D_k$ is called accessible if there exists a curve $\varphi : [0,1] \longrightarrow \mathbf{C}$ such that $\varphi(t) \in D_k$ if $t \in [0,1)$ and $\varphi(1) = z_0$. Denote by $\partial_0 D_k$ the set of all accessible boundary points. This is a Borel set [17], so it is μ_k-measurable.

Proposition 2 $\mu_k(\partial D_k \backslash \partial_0 D_k) = 0$.

Proof. Fix a number R, $0 < R < R_0$. Consider the domain $G = D_k \cup \{z \in \bar{\mathbf{C}} : |z| > R\}$. The domain G is regular for the Dirichlet problem because each boundary point is contained in a continuum in the boundary [15]. Let g be the Green function with the pole at infinity, extended to be zero on $\mathbf{C}\backslash G$. Then g is a subharmonic function in $\mathbf{C}$ and its Riesz measure is the harmonic measure for G. Choose a constant c so large that $u_k(z) < cg(z)$, $|z| = (R_0 + R)/2$, and apply Lemma 5. We obtain that the restriction of μ_k to $\partial D_k \cap D(0, R)$ is majorized by the harmonic measure of G. It is known (see, for example, [21]) that the harmonic measure is supported by the set of accessible points. But a point z_0, $|z_0| < R$ is accessible from G iff it is accessible from D_k. The Proposition is proved. □

Proposition 3 *The set E of points which are accessible simultaneously from three or more domains D_k is finite and $\mu_k(E) = 0$ for all k.*

Proof. Assume that three points z_1, z_2, z_3 are accessible simultaneously from three disjoint domains D_1, D_2, D_3. Fix arbitrary three points $w_n \in D_n$ and draw in each domain D_n, $1 \le n \le 3$, three curves $\gamma_{n,m}$, connecting w_n to z_m. We may choose the curves to be disjoint except their ends. We obtain a graph with 6 vertices and 9 edges such that each vertex of the group $\{z_m\}$ is connected to each vertex of the group $\{w_n\}$. It is well known that such a graph cannot be embedded in the plane. So each triple of our domains can have at most two common accessible points, thus the set E is finite. Furthermore our subharmonic functions u_k are non-negative, so their Riesz measures cannot charge a finite set.

Denote by $\mu_{k,n}$ the restriction of the measure μ_k to the set $\partial_0 D_k \cap \partial_0 D_n$, $1 \le n \le q$, $n \ne k$, and let $\mu_{k,k}$ be the restriction of μ_k to $D_k \cup \partial_0 D_k \setminus \bigcup_{n \ne k} \partial_0 D_n$. By Propositions 2 and 3 we have

$$\mu_k = \sum_{n=1}^{q} \mu_{k,n}, \tag{52}$$

and there exist Borel supports $B_{k,n}$ of measures $\mu_{k,n}$ such that $B_{k,n} \cap B_{\ell,m} = \emptyset$ unless the non-ordered pairs $\{k,n\}$ and $\{\ell,m\}$ coincide. Using (42) we obtain

$$\sum_{k,n=1}^{q} \mu_{k,n} = \sum_{k=1}^{q} \mu_k \geq 2 \bigvee_{k=1}^{q} \mu_k = 2\Big(\sum_{k=1}^{q} \mu_{k,k} + \sum_{1 \leq k < n \leq q} (\mu_{k,n} \vee \mu_{n,k}) \Big)$$

or

$$\sum_{1 \leq k < n \leq q} (\mu_{k,n} + \mu_{n,k}) \geq \sum_{k=1}^{q} \mu_{k,k} + 2 \sum_{1 \leq k < n \leq q} (\mu_{k,n} \vee \mu_{n,k}). \tag{53}$$

We always have $\mu_{k,n} + \mu_{n,k} \leq 2(\mu_{k,n} \vee \mu_{n,k})$ with the only possible case of equality when $\mu_{k,n} = \mu_{n,k}$. Thus (53) implies

$$\mu_{k,k} = 0, \qquad 1 \leq k \leq q \tag{54}$$

and

$$\mu_{k,n} = \mu_{n,k}, \qquad 1 \leq k < n \leq q. \tag{55}$$

It follows from (54) that the functions u_k are harmonic in D_k.

Proposition 4 *Harmonic measure on $\partial_0 D_k$ is absolutely continuous with respect to μ_k.*

Proof. Fix $z_0 \in D_k$. Then choose $r > 0$ such that $D(z_0, r) \subset D_k$ and set $K = \partial D(z_0, r)$. Let g be the Green function for D_k with pole at z_0 extended to be zero in $D \backslash D_k$. Choose a constant $c > 0$ such that $g(z) \leq c u_k(z)$, $z \in K$, then apply Lemma 5.

Note that the harmonic measure of $\partial D \cap \partial D_k$ with respect to D_k is positive, so there are some points $z_k \in \partial D$, accessible from D_k. Take q copies of the unit disc U_k, $1 \leq k \leq q$ and fix the conformal maps $\varphi_k : U_k \to D_k$ such that $\varphi_k((-1, 0])$ is a curve tending to z_k. The radial limits of φ_k are exactly the accessible boundary points of D_k. We denote by $\varphi_k(x)$, $x \in T_k = \partial U_k$, the radial limit of φ_k at the point x, whenever it exists.

We show that if x, $y \in T_k$, $x \neq y$, and $\varphi_k(x)$, $\varphi_k(y) \in D$, then $\varphi_k(x) \neq \varphi_k(y)$. Otherwise consider a curve γ, consisting of two radii, ending at the points x, y. The closure Γ of the image $\varphi_k(\gamma)$ is a Jordan curve in D. The domain G, bounded by Γ cannot intersect D_n, $n \neq k$ because all D_n are not relatively compact in D. On the other hand, G contains a part of ∂D_k of positive harmonic measure. It follows from Proposition 4, that $\mu_{k,k}(G) > 0$, which contradicts (54).

We say that the domains D_k and D_n are *contiguous* if there exist at least two common boundary points in D accessible from both domains. Obviously, if

$\mu_{k,n} \neq 0$, then D_k and D_n are contiguous. Fix a number k, $1 \leq k \leq q$. Assume that the domain D_n is contiguous to D_k. Consider the set $X_{k,n} \subset T_k$, where φ_k has radial limits, which are accessible from D_n. Set

$$\begin{aligned} b_{k,n} &= \inf\{\theta \in (-\pi, \pi) \; : \; e^{i\theta} \in X_{k,n}\}; \\ a_{k,n} &= \sup\{\theta \in (-\pi, \pi) \; : \; e^{i\theta} \in X_{k,n}\}; \\ T_{k,n} &= (b_{k,n}, a_{k,n}) \subset T_k. \end{aligned}$$

The arc $T_{k,n}$ is called the *contiguity arc.*

We show that none of the contiguity arcs $T_{k,n}$ contain points x in which $\varphi_k(x) \in \partial D$. Otherwise, we would have points x, y, t, $-\pi < x, y, t < \pi$, with radial limits $\varphi_k(e^{ix}) = a$, $\varphi_k(e^{iy}) = b$, $\varphi_k(e^{it}) = c$, where $a \in D$ and $c \in D$ are accessible from D_n and $b \in \partial D$. Join a and c by simple arcs $\gamma_1 \in D_k$ and $\gamma_2 \in D_n$. The Jordan curve $\gamma_1 \cup \gamma_2 \cup \{a, c\}$ bounds a domain G, relatively compact in D. The curve ${\varphi_k}^{-1}(\gamma_1)$ divides U_k into two parts, one of which has on the boundary the point -1, while the other the point e^{iy}. So the images of both these parts are not relatively compact, which is not possible since one of these images lies in G.

We show that the contiguity arcs $T_{k,n}$ and $T_{k,m}$ do not intersect if $m \neq n$. If these two arcs intersect, then we can find three points $-\pi < x < y < t < \pi$, such that there exist pairwise distinct radial limits $\varphi_k(e^{ix}) = a$, $\varphi_k(e^{iy}) = b$, $\varphi_k(e^{it}) = c$; a, b, $c \in D$ such that a and c are accessible from a domain D_m and b is accessible from D_n (k, m and n are pairwise distinct). Join the points a and c by simple arcs in D_k and D_m. We obtain a Jordan curve, bounding a domain G, relatively compact in D. It is easy to see that $b \in G$. So D_n intersects G, but this is a contradiction because D_n is not relatively compact and does not meet the boundary of G.

Observe that on each T_k there is an open set that does not intersect the contiguity arcs, because the number of contiguity arcs is finite and the harmonic measure of $\partial D \cap \partial D_k$ with respect to D_k is positive. Suppose that an arc $\Delta \subset T_k$ intersects no contiguity arcs. Then the radial limits of φ_k belong to ∂D almost everywhere on Δ. Otherwise the limits of the function φ_k on Δ that are in D form a set of positive harmonic measure with respect to D_k, and none of these limits is accessible from D_n, $n \neq k$. Taking into account Proposition 4, we obtain a contradiction with (54).

A similar argument shows that the points on $T_{k,n}$ in which there exist radial limits of φ_k, being accessible boundary points from D_n, are dense in the arc $T_{k,n}$.

We supply the circumference T_k with the positive orientation (anti–clockwise). There is a natural monotone (orientation reversing) mapping $\psi_{k,n}$ of a dense subset of the arc $T_{k,n}$ to a dense subset of the arc $T_{n,k}$: $\psi_{k,n}(x) = y$ if $\varphi_k(x) = \varphi_n(y)$. We extend $\psi_{k,n}$ to a homeomorphism. This can be done since the functions $\psi_{k,n}$ and $\psi_{n,k} = {\psi_{k,n}}^{-1}$ are strictly monotone. We paste together the closures of the circles U_k and U_n, identifying the closed arcs $T_{k,n}$ and

$T_{n,k}$ with the aid of homeomorphism $\psi_{k,n}$. We perform this procedure for each pair $\{k, n\}$ for which D_k and D_n are contiguous and obtain a bordered surface (polyhedron) S'. Its border arises from the arcs on T_k, which are complementary to the contiguity arcs. Denote the interior of S' by S. On the surface S we have a net, consisting of the edges $T_{k,n} = T_{n.k}$ and vertices, which are the points where the edges meet. The map $\varphi(x) = \varphi_k(x)$, $x \in U_k$, is defined on a dense open subset of S as well as on a dense subset of the net.

A closed curve $\Gamma \subset S$ is said to be admissible if it does not meet vertices, intersects transversally and non-tangentially the edges in a finite set of points, and at each point of intersection $x \in \Gamma \cap T_{k,n}$ there exist radial limits $\varphi_k(x) = \varphi_n(x)$. Admissible curves are dense in the set of all closed curves on S. An admissible curve $\Gamma \subset S$ has the image $\varphi(\Gamma)$, which is defined to be the closure of $\bigcup_k \varphi_k(\Gamma \cap U_k)$.

We show that the surface S is homeomorphic to a plane domain. If this is not so, then there exist two admissible curves Γ_1 and Γ_2 intersecting transversally in a unique point, not lying on an edge. Then the closed curves $\varphi(\Gamma_1)$ and $\varphi(\Gamma_2)$ in the disc D intersect transversally in a unique point, which is impossible.

Next we show that S is homeomorphic to a disc. For this purpose consider again the bordered surface S'. If there is more than one boundary component of S', then there exists a Jordan admissible curve Γ that separates some boundary components B_1 and B_2 of S'. Choose two curves Γ_1 and Γ_2 that do not intersect Γ, that tend to some points $b_i \in B_i$, $i = 1, 2$, and such that there exist limits

$$\lim_{x \to b_i,\, x \in \Gamma_i} \varphi(x) = a_i \in \partial D, \qquad i = 1, 2.$$

The image $\varphi(\Gamma)$ is a Jordan curve which separates $\varphi(\Gamma_1)$ and $\varphi(\Gamma_2)$. So one of these two curves is separated from ∂D and we get a contradiction.

Let Γ be an admissible curve. Denote by $n(\Gamma)$ the number of its intersections with edges. If z is a point in the plane or in S, and Γ is arbitrary closed curve not passing through z, denote by $\mathrm{ind}_z \Gamma$ the rotation number of Γ with respect to z. The set of odd vertices (i.e., such vertices in which an odd number of edges meet) will be called Q.

Proposition 5 $n(\Gamma) \equiv \sum_{x \in Q} \mathrm{ind}_x(\Gamma) \pmod 2$.

Proof. By a small deformation we achieve that the curve Γ will have a finite number of self-intersections. We continuously deform the curve Γ, contracting it to a point $x_0 \in U_1$. The deformation is carried out in such a manner that the intermediate curves have a finite number of self-intersections and these self-intersections do not occur on edges and vertices. When during the process of deformation the curve passes through a vertex x so that the number $\mathrm{ind}_z \Gamma$ changes by 1, the number $n(\Gamma)$ obtains an even increment if the vertex x is even, and an odd increment if the vertex x is odd. This proves the proposition.

Consider an arbitrary odd vertex x. Let Γ_m be a sequence of admissible Jordan curves converging to x and such that Γ_{n+1} separates Γ_n from x. Denote by K_m the closure of the domain bounded by $\varphi(\Gamma_m)$. Then $K_{m+1} \subset K_m$ and therefore $\bigcap_{m=1}^{\infty} K_m$ is a non-empty set which we denote by $K(x)$. The curves $\varphi(\Gamma)$, where $\Gamma \subset S$ is an admissible curve, do not intersect the sets $K(x)$.

For each odd vertex x we select a point $\varphi(x) \in K(x)$. For any admissible curve Γ we have

$$\mathrm{ind}_x(\Gamma) = \mathrm{ind}_{\varphi(x)}\varphi(\Gamma). \tag{56}$$

Consider a two-sheeted ramified covering $p : F \to D$ of the disk D by some Riemann surface F, ramified precisely over the points of $\varphi(x), x \in Q$. Let $\tilde{D}_1, \ldots, \tilde{D}_{2q}$ be the p-preimages of the domains $D_1, \ldots, D_q$, where $p^{-1}(D_k) = \tilde{D}_k \cup \tilde{D}_{q+k}$. The definition of contiguity of domains on the Riemann surface F is exactly the same as in D.

Proposition 6 *Let $\tilde{D}_{k_1}, \ldots, \tilde{D}_{k_n}$ be a finite sequence such that each $\tilde{D}_{k_i}$ is contiguous to $\tilde{D}_{k_{i+1}}$, $1 \leq i \leq n$, $k_{n+1} = k_1$. Then n is even.*

This proposition explains the role of the Riemann surface F.

Proof. There exists a closed curve $\gamma \subset F$ such that the intersection of γ with each $\tilde{D}_{k_n}$ is a simple open arc and the ends of this arc are accessible boundary points of $\tilde{D}_{k_n}$. The curve $\gamma_1 = p(\gamma)$ is the φ-image of some admissible curve $\Gamma \in S$,

$$\gamma_1 = \varphi(\Gamma),$$

and we have

$$\sum_{x \in Q} \mathrm{ind}_{\varphi(x)}\gamma_1 \equiv 0 \pmod 2,$$

because $\gamma_1 = p(\gamma)$ for a closed curve γ in F. So, by (56)

$$\sum_{x \in Q} \mathrm{ind}_x\Gamma \equiv 0 \pmod 2.$$

Thus the number $n = n(\Gamma)$ is even by Proposition 5.

Now we may assign to each domain $\tilde{D}_k$, $1 \leq k \leq 2q$, a sign $s(k) = \pm 1$ such that contiguous domains have opposite signs. This is possible by Proposition 6. Define the functions $\tilde{u}_k$ and h in the following way:

$$\begin{aligned} \tilde{u}_k &= s(k)u_k \circ p, & 1 \leq k \leq q; \\ \tilde{u}_{q+k} &= s(q+k)u_k \circ p, & 1 \leq k \leq q; \end{aligned}$$

$$h = \sum_{n=1}^{2q} \tilde{u}_n.$$

It is evident that h is δ-subharmonic on F. Its Riesz charge is

$$\sum_{n=1}^{2q} s(n)\tilde{\mu}_n = \sum_{m,n=1}^{2q} s(n)\tilde{\mu}_{n,m}, \tag{57}$$

where $\tilde{\mu}_n$ is the Riesz charge of $\tilde{u}_n$, while $\tilde{\mu}_{m,n}$ is the restriction of $\tilde{\mu}_n$ to $\partial_0\tilde{D}_n \cap \partial_0\tilde{D}_m$. If $\tilde{\mu}_{m,n} \neq 0$, then $\tilde{D}_n$ and $\tilde{D}_m$ are contiguous, so they have opposite signs. It follows from (55) that $\tilde{\mu}_{m,n} = \tilde{\mu}_{n,m}$, so the expression (57) is equal to zero and the function h is harmonic.

By definition of h we have $u \circ p = |h|$. The number of ramification points of the covering $p : F \to D$ is equal to the number of odd vertices of the net on S.

We estimate the number of vertices in the net. For this purpose consider S' and replace the unique border component of S' by one vertex x_0. The resulting polyhedron S'' is homeomorphic to a sphere. All faces have x_0 on the boundary, so at least q edges meet at x_0. In each vertex meet at least 3 edges. If we denote by v, e, and q, respectively, the number of vertices, edges and faces of S'', then $v - e + q = 2$ by the Euler formula. Furthermore, $e \geq 1/2(q + 3(v-1))$ and we get $v \leq q - 1$ and the number of vertices on S does not exceed $q - 2$ as required.

Finally in a neighborhood of each ramification point on the Riemann surface F the harmonic function h changes sign at least 6 times. So it has a zero at this point of order at least 3. The lemma is proved. □

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