On the Second Main Theorem of Cartan

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November 6, 2014

Abstract

The possibility of reversion of the inequality in the Second Main Theorem of Cartan in the theory of holomorphic curves in projective space is discussed. A new version of this theorem is proved that becomes an asymptotic equality for a class of holomorphic curves defined by solutions of linear differential equations.

2010 MSC: 30D35, 32A22. Keywords: Nevanlinna theory, Second Main theorem, linear differential equations.

1 Introduction

We consider holomorphic curves $f:\mathbf{C}\to\mathbf{P}^n$. In homogeneous coordinates such curves are represented as $(n+1)$-tuples of entire functions

$$f=(f_0:\ldots:f_n),$$

where not all f_j are equal to 0. A homogeneous representation is called reduced if the f_j do not have zeros common to all of them. A reduced representation is defined up to a common entire factor which is zero-free.

In the following definitions we use a reduced homogeneous representation, however one can easily check that the definitions of $N(r,a,f), T(r,f), N_1(r,f)$, $m(r,a,f)$ and $m_k(r,f)$ are independent of the choice of a reduced homogeneous representation.

Let a be a hyperplane in $\mathbf{P}^n$. It can be described by an equation

$$\alpha_0w_0+\ldots+\alpha_nw_n=0,\quad\text{where}\quad \alpha=(\alpha_0,\ldots,\alpha_n)\neq(0,\ldots,0). \qquad (1)$$

*Supported by the NSF grant DMS-1361836.

The intersection points of the curve $f(z)$ with the hyperplane a are zeros of the entire function $g_a = (\alpha, f) = \alpha_0 f_0 + \ldots + \alpha_n f_n$. Let $n(r, a, f)$ be the number of these zeros in the disc $|z| \leq r$, counting multiplicity, then the Nevanlinna counting function is defined as

$$N(r, a, f) = \int_0^r (n(t, a, f) - n(0, a, f)) \frac{dt}{t} + n(0, a, f) \log r. \tag{2}$$

The Cartan–Nevanlinna characteristic $T(r, f)$ can be defined as follows:

$$\begin{aligned} T(r, f) &= \frac{1}{2\pi} \int_0^r \left(\int_{|z| \leq t} \Delta \log \|f(z)\| dm_z \right) \frac{dt}{t} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \|f(re^{i\theta})\| d\theta - \log \|f(0)\|, \end{aligned}$$

where $\|f\| = \sqrt{|f_0|^2 + \ldots + |f_n|^2}$, and dm is the element of the area. Here $\Delta \log \|f\|$ is the density of the pull-back of the Fubini–Study metric, and equality holds by Jensen's formula. The order ρ of f is defined by the formula

$$\rho = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

The proximity functions are

$$m(r, a, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{\|\alpha\| \|f(re^{it}\|}{|g_a(e^{it})|} dt.$$

Here the integrand is

$$\log \frac{1}{\operatorname{dist}(f(z), a)},$$

where dist is the "chordal distance" from the point $f(z)$ to the hyperplane a. Now we consider the Wronskian determinant $W_f = W(f_0, \ldots, f_n)$ which is an entire function; it is identically equal to zero if and only if f is *linearly degenerate*, that is if $f_0, \ldots, f_n$ are linearly dependent. We denote by $n_1(r, f)$ the number of zeros of W_f in the disc $\{z : |z| \leq r\}$ and define the function $N_1(r, f)$ by a formula similar to (2).

A set A of hyperplanes is usually called admissible if any $n+1$ hyperplanes of the set have empty intersection. If the set A contains at least $n + 1$ hyperplanes, admissibility is equivalent to

$$\operatorname{codim}(a_1 \cap \ldots \cap a_k) = k \tag{3}$$

for every $k \in [1, n+1]$ and every k hyperplanes of the set A. We use the convention that $\operatorname{codim} x = n+1$ iff $x = \emptyset$. We use (3) to extend the definition of admissibility to systems of arbitrary cardinality. So a system of hyperplanes will be called *admissible* if any $k \leq n+1$ vectors α defining these hyperplanes as in (1) are linearly independent.

With these definitions, the Second Main Theorem (SMT) of Cartan says:

For every linearly non-degenerate holomorphic curve and for every finite admissible set A,

$$\sum_{a \in A} m(r, a, f) + N_1(r, f) \leq (n+1)T(r, f) + S(r, f), \tag{4}$$

where S is an "error term" with the property that $S(r, f) = o(T(r, f))$ for $r \to \infty$, $r \notin E$, where E is an exceptional set of finite length.

Better estimates of the error term are available, but they do not concern us here. When $n = 1$, Cartan's SMT coincides with the Second Main Theorem of Nevanlinna for the meromorphic function $f = f_1/f_0$. When $n = 1$, the assumption that the set A is admissible is vacuous.

Nevanlinna's SMT was considered from the very beginning as a partial generalization of the Riemann–Hurwitz formula [1]. However, the Riemann-Hurwitz formula is an equality, while the SMT is only an inequality. This inspired the research on the reversion of the SMT: roughly speaking, the question is whether one can replace the $\leq$ sign with the $=$ sign in (4) for $n = 1$. A survey of the early results on this topic is contained in the book by Wittich [17, Ch. IV]. The general conclusion one can make from these results is that for all simple, "naturally arising" meromorphic functions an asymptotic equality indeed holds. But of course, (4) cannot be literally true for all meromorphic functions in the form of equality, because there are meromorphic functions f with $m(r, a, f) \neq o(T(r, f))$ for an uncountable set of a, and an exceptional set E of r does not help.

Recently, K. Yamanoi [18] found a way to overcome this difficulty for $n = 1$. He defined the modified proximity function

$$\overline{m}_q(r, f) = \sup_{(a_1, \ldots, a_q) \in \overline{\mathbf{C}}^q} \frac{1}{2\pi} \int_{-\pi}^{\pi} \max_{1 \leq j \leq q} \log \frac{1}{\operatorname{dist}(f(re^{it}), a_j)} dt.$$

With this definition, he proved the following theorem.

Let $f : \mathbf{C} \to \overline{\mathbf{C}}$ be a transcendental meromorphic function. Let $q : \mathbf{R}_{>0} \to \mathbf{N}$ be a function satisfying

$$q(r) \sim \left(\log^+ \frac{T(r,f)}{\log r}\right)^{20}.$$

Then

$$\overline{m}_{q(r)}(r,f) + N_1(r,f) = 2T(r,f) + o(T(r,f)), \quad r \notin E,$$

where E is a set of zero logarithmic density.

For functions of finite order, this result was improved in [19]: it holds with any function $q(r)$ that satisfies $\log q(r) = o(T(r,f))$.

In this paper, we discuss the possibility of an asymptotic equality in Cartan's SMT for arbitrary $n > 1$. First we show by an example that the admissibility condition creates a new difficulty which is not present for $n = 1$: even for very simple curves there can be no admissible system for which (4) holds with equality. Then we propose a modified form of Cartan's SMT which does not involve the admissibility condition, and show that in this modified form asymptotic equality holds for a class of holomorphic curves.

2 Example

The simplest non-trivial examples in value distribution theory for $n = 1$ are meromorphic functions $f = w_1/w_0$, where $w_0,\ w_1$ are two linearly independent solutions of a differential equation of the form

$$w'' + Pw = 0, \tag{5}$$

where P is a polynomial. These functions f, which were studied in detail by F. Nevanlinna [9] and R. Nevanlinna [10], can be characterized by the properties: f is of finite order, and $N_1(r,f) \equiv 0$.

For each such f, there is an integer p and a finite set of points $\{a_1, \ldots, a_q\}$ in $\overline{\mathbf{C}}$ such that

$$m(r, a_j, f) = (2m_j/p)T(r,f) + O(\log r), \quad r \to \infty, \tag{6}$$

where m_j are positive integers, and

$$\sum_{j=1}^{q} m_j = p.$$

So we have an asymptotic equality in (4).

This result is related to two other results:

1. If f is a meromorphic solution of arbitrary linear differential equation with polynomial coefficients, then we have an asymptotic equality in the SMT for f, with $A = \{0, \infty\}$, [17, Ch. IV].

2. If f has finitely many critical and asymptotic values, then an asymptotic equality holds in the SMT for f, if A is the set of critical and asymptotic values [13, 17]. Functions $f = w_1/w_0$, where w_0, w_1 are linearly independent solutions of (5) have no critical values and their asymptotic values are exactly those a_j in (6).

These results suggest that in searching for improvements of (4) one has to look first at the holomorphic curves whose homogeneous coordinates are linearly independent solutions of a differential equation

$$w^{(n+1)} + P_n w^{(n)} + \ldots + P_0 w = 0, \tag{7}$$

with polynomial coefficients P_j. This class of curves can be characterized by the properties that the order is finite and $N_1(r, f) \equiv 0$, [11, 5].

The following example was mentioned in [4]:

$$w''' - zw' - w = 0. \tag{8}$$

This is equivalent to

$$w'' - zw = c, \quad c \in \mathbf{C}. \tag{9}$$

This is a non-homogeneous Airy equation, and we can describe the asymptotic behavior of all solutions using the well-known asymptotic formulas [14, 15]. All non-trivial solutions are entire functions of order $\rho = 3/2$, and for description of their behavior we use the Phragmén–Lindelöf indicator:

$$h_w(t) = \lim_{r \to \infty} r^{-3/2} \log |w(re^{it})|.$$

First of all, we have three solutions w_0, w_1, w_2 (Airy's functions) for $c = 0$. These satisfy

$$w_0 + w_1 + w_2 = 0, \tag{10}$$

and have the indicators

$$H_0(t) = -\cos\left(\frac{3}{2}t\right), \quad |t| \le \pi, \quad H_j(t) = H_0(t \pm 2\pi/3), \quad j = 1, 2. \tag{11}$$

The rest of solutions of (8), which correspond to non-zero values of c in (9) can be expressed in terms of Airy functions by the method of variation of constants. These explicit asymptotic expressions show that the list of possible indicators for $c \neq 0$ is this:

$$G_0(t) = \left(-\cos\left(\frac{3}{2}t\right)\right)^+, \quad |t| \leq \pi, \quad G_j(t) = G_0(t \pm 2\pi/3), \quad j = 1, 2. \tag{12}$$

Another way to obtain these indicators is to notice that (8) has a formal solution

$$w^*(z) = \sum_{n=0}^{\infty} \frac{(3n)!}{3^n n!} z^{-3n-1}.$$

According to the general theory [15], there exists a solution w_3 such that $w_3(z)$ has w^* as the asymptotic expansion in the sector

$$S_0 = \{z : |\arg z| < \pi/3\}.$$

For this solution, $h_{w_3}(t) = 0$, $|t| \leq \pi/3$. As the equation (8) is invariant under the substitution $z \mapsto e^{2\pi i/3}z$, in each of the three sectors S_0, $S_{\pm 1} = e^{\pm 2\pi i/3} S_0$ there exists a solution with zero indicator.

Notice that for every $t \notin \{\pi, \pm\pi/3\}$, the set of solutions with $h_w(t) \leq 0$ is at most two dimensional. Indeed, if there were three linearly independent solutions with $h_w(t) \leq 0$, then every solution would satisfy $h(t) \leq 0$, but this is not so because $\max\{H_0, H_1, H_2\}$ is positive at every point $t \notin \{\pi, \pm\pi/3\}$. As for every t there exists a solution w with $h_w(t) = 0$, we obtain that for every t, the set of solutions w with $h_w(t) < 0$ is at most one-dimensional. This shows that our list (11), (12) of possible indicators of solutions is complete.

Now let f be the holomorphic curve whose homogeneous coordinates are three linearly independent solutions of (8). Then the entire functions $g_a = (a, f)$ are exactly the non-trivial solutions of (8). Let $A = \{a_1, \ldots, a_q\}$ be an admissible system of hyperplanes. Let h_j be the indicators of entire functions g_{a_j}, and let h be their pointwise maximum. Then

$$h(t) = |\cos((3/2)t)|,$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h(t)dt = \frac{3}{2\pi}\int_{-\pi/3}^{\pi/3} \cos\left(\frac{3}{2}t\right) dt = \frac{2}{\pi},$$

therefore

$$T(r, f) = \left(\frac{2}{\pi} + o(1)\right) r^{3/2}.$$

We claim that

$$\sum_{j=1}^{q}\int_{-\pi}^{\pi}(h(t)-h_j(t)dt \leq 8\int_{-\pi/3}^{\pi/3}\cos\left(\frac{3}{2}t\right)=\frac{32}{3}.$$

This follows from the fact that on each of the three components of the set $\{t\in(-\pi,\pi):h(t)>0\}$ at most one of the h_j can be negative, and at most two of the h_j can be non-positive, and in addition, we cannot have negative indicators in all three components, because the three solutions w_0, w_1, w_2 satisfying (10) cannot be all present in an admissible set. So we have

$$\sum_{j=1}^{q} m(r,a_j,f) \leq \left(\frac{16}{3\pi}+o(1)\right)r^{3/2} \leq \left(\frac{8}{3}+o(1)\right)T(r,f).$$

The Wronski determinant of three linearly independent solutions of (8) is zero-free, $N_1(r,f)\equiv 0$, and we cannot have asymptotic equality in (4).

This example shows that if one desires (4) with asymptotic equality then non-admissible sets of hyperplanes A should be permitted. In the next section we state and prove a version of (4) which applies to an arbitrary finite system of hyperplanes.

3 Modified Second Main Theorem

Let us consider the projective space $\mathbf{P}^n$ equipped with the chordal metric dist. The distance between two subsets of $\mathbf{P}^n$ is defined in the usual way, as the $\inf \operatorname{dist}(x,y)$, where x is in one set and y is in another set.

Let us fix an *arbitrary* finite set A of hyperplanes. Intersections of various subsets of hyperplanes in A are projective subspaces of various codimension. We call these subspaces "*flats* generated by A", and denote the set of all these flats by $F(A)$. We also denote by $\operatorname{codim}(x)$ the codimension of a flat $x\in F(A)$. If $\operatorname{codim}(x)=k$, then there exists an admissible set $\{a_1,\ldots,a_k\}\subset A$ such that $x=a_1\cap\ldots\cap a_k$. If $\emptyset\in F(A)$, then flats of all codimensions $1,\ldots,n+1$ exist in $F(A)$. Such systems A will be called *complete*. A system of hyperplanes is complete if the vectors α corresponding to this system as in (1) span $\mathbf{C}^{n+1}$.

We frequently use the following fact, without special mentioning: if $a_1,\ldots,a_k$ is an admissible set of hyperplanes, and $X=a_1\cap\ldots\cap a_k$ then

$$C_1\max_{1\leq j\leq k}\operatorname{dist}(w,a_j)\leq \operatorname{dist}(w,X)\leq C_2\max_{1\leq j\leq k}\operatorname{dist}(w,a_j),\quad w\in\mathbf{P}^n,$$

with positive constants C_1, C_2 depending only on the set of hyperplanes.

For $w \in \mathbf{P}^n$ and $k \in \{1, \ldots, n\}$, we define $d_k(w)$ as the shortest distance from w to a flat of codimension k in $F(A)$. It is also convenient to set $d_{n+1}(w) = 1$.

For a holomorphic curve $f : \mathbf{C} \to \mathbf{P}^n$ and $k \in \{1, \ldots, n+1\}$, we define the *k-proximity functions*

$$m_k(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{1}{d_k(f(re^{it}))} dt.$$

So $m_1 \geq m_2 \geq \ldots \geq m_n \geq m_{n+1} = 0$. Functions m_k depend on A which is not reflected in the notation. Proximity functions for flats of arbitrary codimension were considered for the first time by H. and J. Weyl's [16]. With this definition we have

Theorem 1. *Let $f : \mathbf{C} \to \mathbf{P}^n$ be a linearly non-degenerate holomorphic curve. Let A be an arbitrary finite complete set of hyperplanes. Then*

$$\sum_{k=1}^{n} m_k(r, f) + N_1(r, f) \leq (n+1)T(r, f) + S(r, f), \tag{13}$$

where $S(r, f)$ is the same error term as in Cartan's theorem.

When $n = 1$, we have

$$m_1(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \max_{a \in A} \log \frac{1}{\operatorname{dist}(f(re^{it}), a)} dt + O(1),$$

so in the case when $m(r, a, f) = o(T(r, f))$ for all but finitely many a, Theorem gives essentially the same as Yamanoi's result.

Let $f_0, \ldots, f_n$ be linearly independent polynomials whose maximal degree is k. Then there exist linear combinations $g_0, \ldots, g_n$ of these polynomials whose degrees satisfy $k_0 < k_1 < \ldots < k_n = k$. Then we have

$$m_j(r, f) = (k - k_{j-1}) \log r + O(1), \quad T(r, f) = k \log r + O(1).$$

Computing the degree of the Wronskian $W(g_0, \ldots, g_n)$, we obtain

$$N_1(r, f) = \left(\sum_{j=0}^{n} k_j - n(n+1)/2 \right) \log r + O(1). \tag{14}$$

Thus

$$\sum_{j=1}^{n} m_j(r,f) + N_1(r,f) = (n+1)T(r,f) - \frac{n(n+1)}{2}\log r + O(1).$$

When k is large, $T(r,f)$ is large in comparison with $\log r$, and we obtain a relation close to (13). So (13) can be considered as an extension of the formula for the degree of the Wronskian to the transcendental case, compare [15, Introduction, (II″)].

The proof of Theorem 1 is a combination of Cartan's argument with the following elementary

Lemma 1. *Let A be a finite complete set of hyperplanes in $\mathbf{P}^n$. Then there exists a constant $C > 0$ depending only on A, such that for every $w \in \mathbf{P}^n$ we have*

$$\prod_{k=1}^{n} d_k(w) \geq C \min_B \prod_{a \in B} \mathrm{dist}(w,a),$$

where the infimum is taken over all admissible systems $B = \{a_1, \ldots, a_{n+1}\}$ of hyperplanes in A.

Proof. First we notice that if $x \in F(A)$ and $\mathrm{codim}\, x = k$, then there exists an admissible subset $\{a_1, \ldots, a_{n+1}\} \in A$ such that $x = a_1 \cap \ldots \cap a_k$. Indeed, by passing from hyperplanes to their defining vectors, this is equivalent to the familiar statement from linear algebra: if a finite set A of vectors spans the space, then every linearly independent subset of A can be completed to a basis consisting of vectors of A.

Now we prove the statement by contradiction. For w not in the union of hyperplanes of A, we set

$$\phi(w) = \frac{\prod_{k=1}^{n} d_k(w)}{\min_B \prod_{a \in B} \mathrm{dist}(w,a)}.$$

Suppose that there is a sequence w_j for which $\phi(w_j) \to 0$. By choosing a subsequence, we may assume that $w_j \to w_\infty \in \mathbf{P}^n$. If w_∞ does not belong to any hyperplane $a \in A$, then $\phi(w_\infty) > 0$, and we obtain a contradiction because ϕ is continuous in the complement of hyperplanes.

If w_∞ belongs to some flat of $F(A)$, let $x \in F(A)$ be the flat of maximal codimension to which w_∞ belongs. Then $d_j(w)$ are bounded away from zero for w in a neighborhood V of w_∞ and $j > k = \mathrm{codim}\, x$. Suppose that $x = a_1 \cap \ldots \cap a_k$. Then, by the remark in the beginning, there is an admissible system $B = \{a_1, \dots, a_{n+1}\} \subset A$ beginning with $a_1, \dots, a_k$, and $w_\infty \notin a_j$ for $j > k$ by definition of x. Then for $w \in V$, we have

$$\prod_{j=1}^{n} d_j(w) \geq C_1 \prod_{j=1}^{k} d_j(w) \geq C_2 \prod_{j=1}^{k} \mathrm{dist}(w, a_j) \geq C_3 \prod_{j=1}^{n+1} \mathrm{dist}(w, a_j).$$

This contradicts our assumption that $\phi(w_j) \to 0$ and proves the lemma.

Proof of Theorem 1. Fix a reduced representation of f. Normalize all hyperplane coordinates so that $\|\alpha\| = 1$ in (1). Let

$$u = \log \|f\|, \quad u_a = \log |g_a|, \quad a \in A. \tag{15}$$

Then

$$-\log \mathrm{dist}(f(z), a) = u(z) - u_a(z).$$

According to Lemma 1, for every $z \in \mathbf{C}$, we can find an admissible system $B(z)$, $|B(z)| = n + 1$, in A such that

$$\begin{aligned} -\sum_{k=1}^{n} \log |d_k(f(z))| &\leq -\sum_{a \in B(z)} \log \mathrm{dist}(f(z), a) + O(1) \\ &= (n+1)u(z) - \sum_{a \in B(z)} u_a(z) + O(1). \end{aligned} \tag{16}$$

Let $W = W(f_0, \dots, f_n)$ be the Wronskian determinant. If $B = \{a_1, \dots, a_{n+1}\}$ is an admissible system, then

$$|W_B| = |W(g_{a_1}, \dots, g_{a_{n+1}})| = C(B)|W|. \tag{17}$$

Let

$$L_B(z) = \log^+ \left| \frac{W_B(z)}{\prod_{a \in B(z)} |g_a(z)|} \right|.$$

Then

$$-\sum_{a \in B(z)} u_a(z) \leq -\log |W_B(z)| + |L_B(z)| + O(1) \leq -\log |W(z)| + R(z), \tag{18}$$

where $R(z)$ is the sum of non-negative quantities $L_B(z)$ over all admissible systems of cardinality $n + 1$. The Lemma on the Logarithmic derivative implies that

$$\int_{-\pi}^{\pi} R(re^{it}) dt = S(r, f),$$

see [3], [7, p. 222]. Jensen's formula gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|W(re^{it})|dt=N_1(r,f)+O(1),$$

and the definition of $T(r,f)$ can be rewritten as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}u(re^{it})dt=T(r,f)+O(1).$$

Combining (16) and (18), integrating over circles $|z|=r$, and using the last three equations we obtain

$$\sum_{k=1}^{n}m_k(r,f)+N_1(r,f)\leq(n+1)T(r,f)+S(r,f).$$

This completes the proof of Theorem 1.

Now we compare Cartan's formulation of the SMT with Theorem 1.

Proposition. *Let A be a finite admissible system of hyperplanes, $|A|\geq n+1$, and f a non-constant holomorphic curve whose image in not contained in any hyperplane of A. Then*

$$\sum_{a\in A}m(r,a,f)\leq\sum_{k=1}^{n}m_k(r,f)+O(1).$$

Proof. Let $A=\{a_1,\ldots,a_q\}$. Define u and $u_j=\log|g_{a_j}|$ by formulas (15). Fix $z\in\mathbf{C}$ and order the functions u_j by magnitude of $u_j(z)$,

$$u_{j_1}(z)\leq u_{j_2}(z)\leq\ldots\leq u_{j_q}(z),$$

where the j_k depend on z. Then for $k\leq n$ we have

$$u(z)-u_{j_k}(z)=-\log\mathrm{dist}(f(z),x_k)+O(1)\leq-\log d_k(f(z))+O(1),$$

where $x_k=a_{j_1}\cap\ldots\cap\ a_{j_k}$, and the $O(1)$ depends only on A. For $k\geq n+1$ we obtain $u(z)-u_{j_k}(z)=O(1)$. Adding these inequalities we obtain

$$\sum_{k=1}^{q}u(z)-u_{j_k}(z)\leq-\sum_{k=1}^{n}d_k(f(z))+O(1).$$

Integrating this inequality, over circles $|z| = r$ we obtain the statement of the proposition.

Remark. Unlike the usual proximity functions $m(r, a, f)$, the $m_k(r, f)$ can be substantially greater than $T(r, f)$. For example, if f is the curve considered in the previous section, then $m_1(r, f) = 2T(r, f) + O(1)$. It is a challenging problem to obtain the exact upper estimates of the quantities

$$\delta_k = \liminf_{r\to\infty} \frac{m_k(r, f)}{T(r, f)}$$

for every $k \in [1, n]$. These are analogs of Nevanlinna defects. There is a conjecture that $\delta_2 \leq 1$ for $n = 2$.

4 Curves defined by solutions of linear ODE

Let $\mathfrak{F}$ be the set of all entire functions y which satisfy differential equations of the form

$$y^{(N)} + P_{N-1}y^{(N-1)} + \ldots + P_0 y = 0 \tag{19}$$

with polynomial coefficients P_j. This class contains exponential polynomials. For the curves of the form $f(z) = (e^{\lambda_0 z} : \ldots : e^{\lambda_n z})$ asymptotic equality holds in Cartan's SMT [2].

Theorem 2. *Let $f : \mathbf{C} \to \mathbf{P}^n$ be a transcendental linearly non-degenerate holomorphic curve, whose homogeneous coordinates belong to $\mathfrak{F}$. Then there exists a finite complete system A of hyperplanes such that*

$$\sum_{k=1}^{n} m_k(r, f) + N_1(r, f) = (n + 1 + o(1))T(r, f), \quad r \to \infty.$$

These curves are of finite order, so there is no exceptional set of r. The result seems to be new even for $n = 1$.

To prove Theorem 2, we use the following two facts about the class $\mathfrak{F}$:

1. $\mathfrak{F}$ is a differential ring [6]. This means that $\mathfrak{F}$ is closed under addition, multiplication and differentiation.

2. For every differential equation (19) and every θ, there exists $\epsilon > 0$, and N linearly independent solutions $y_1, \ldots, y_N$ of (19) such that

$$y_k(z) \sim e^{Q_k(z^{1/p})} z^{s_k} \log^{m_k} z, \quad z = re^{it}, \quad r \to \infty, \tag{20}$$

uniformly with respect to t when $|t-\theta| \leq \epsilon$. Here Q_k are polynomials, $Q_k(0)=0$, p is a positive integer, $s_k \in \mathbf{C}$ and m_k are integers. All triples (Q_k, n_k, m_k), $1 \leq k \leq N$, in (20) are distinct. For a proof we refer to [15].

3. All solutions y of (19) are entire functions of completely regular growth in the sense of Levin–Pflüger [8], the notion which we recall now.

Let f be a holomorphic function in an angular sector $S=\{re^{i\theta} : |\theta-\theta_0| < \epsilon,\ r>0\}$. We say that f has *completely regular growth* with respect to order $\rho > 0$ if the following finite limit exists

$$\lim_{r\to\infty,\, re^{i\theta}\notin E} \frac{\log|f(re^{i\theta})|}{|r|^\rho} =: h_f(\rho,\theta), \tag{21}$$

uniformly with respect to θ, for $|\theta-\theta_0| < \epsilon$. Here $E \subset S$ is an exceptional set which can be covered by discs centered at z_j of radii r_j, such that

$$\sum_{j:|z_j|<r} r_j = o(r), \quad r \to \infty.$$

Such sets E are called C_0-sets in [8].

The limit $h_f(\rho,\theta)$ is called the indicator. It is always continuous as a function of $\theta \in (-\epsilon, \epsilon)$. Notice that if f has completely regular growth with respect to order ρ, then it has completely regular growth with respect to any larger order, and the indicator with respect to the larger order is zero.

An entire function f is said to be of completely regular growth, if it has completely regular growth in any sector with respect to its order $\rho = \rho(f)$.

If f_1 and f_2 are two functions of completely regular growth with respect to the same order ρ then evidently

$$h_{f_1+f_2}(\rho,\theta) \leq \max\{h_{f_1}(\rho,\theta), h_{f_2}(\rho,\theta)\},$$

and equality holds if $h_{f_1}(\rho,\theta) \neq h_{f_2}(\rho,\theta)$.

Petrenko [11, Sect. 4.3] proved that all entire functions satisfying differential equations of the form (19) have completely regular growth.

Let $V \subset \mathfrak{F}$ be a vector space of finite dimension $n+1$. Let ρ be the maximal order of elements of V. From now on, all indicators will be considered with respect to this order ρ, and we suppress it from notation.

Choose a ray $\{z : \arg z = \theta_0\}$. Each function $f \in V$ is a linear combination of some finite set of entire functions w_k which have asymptotics of the form (20) in an angular sector containing our ray. It is clear that functions w_k have trigonometric indicators of the form $c_k \sin \rho(\theta - \theta_k)$. Two distinct trigonometric functions of this form can be equal only at a finite set of points.

We conclude that for each V there exist finitely many rays such that for any sector S complementary to these rays the possible indicators of elements of V are strictly ordered:

$$h_1(\theta) < h_2(\theta) < \ldots < h_m(\theta), \quad e^{i\theta} \in S. \tag{22}$$

Here $m \geq 1$ is the number of distinct indicators in S. Such sectors will be called *admissible* for a vector space V.

We fix an admissible sector S of our vector space V, and construct a *special basis* in V. Let h_j be the indicator of some element of V. Then we define $V_j \subset V$ be the subspace consisting of functions whose indicator at most h_j. If all possible indicators are ordered as in (22), then

$$V_1 \subset V_2 \subset \ldots \subset V_m = V.$$

We choose $\dim V_1$ linearly independent functions in V_1, then $\dim V_2 - \dim V_1$ functions in V_2 which represent linearly independent elements of the factor space V_2/V_1, and so on. So that the basis elements chosen from $V_j \backslash V_{j-1}$ are linearly independent as elements of V_j/V_{j-1}.

Let $w_0, w_1, \ldots, w_n$ be this basis, ordered in such a way that the indicators increase,

$$h_{w_0}(\theta) \leq h_{w_1}(\theta) \leq \ldots \leq h_{w_n}(\theta), \quad e^{i\theta} \in S. \tag{23}$$

Notice that, the indicator of any linear combination of the form

$$c_0 w_0 + \ldots + c_{n-1} w_{n-1} + w_n \tag{24}$$

is the same as the indicator of w_n. This sequence (w_j) is called a special basis of V corresponding to the sector S.

Lemma 2. *Outside of a C_0 exceptional set E as in (21), the special basis satisfies*

$$\log |W(w_0, \ldots, w_n)| = \sum_{j=0}^{n} \log |w_j| + o(r^\rho)$$

in the sector S.

Proof. If f_1 and f_2 are two functions of completely regular growth in S, then the limit in (21) also exists for their ratio $f = f_1/f_2$ and this limit is equal to $h_{f_1}(\theta) - h_{f_2}(\theta)$. Let

$$\mathcal{L}(w_0, \ldots, w_n) = \frac{W(w_0, \ldots, w_n)}{w_0, \ldots, w_n}.$$

The statement of the Lemma is equivalent to $h_{\mathcal{L}}(\theta) \equiv 0$.

As $\mathcal{L}$ is a determinant consisting of the logarithmic derivatives of functions of class $\mathfrak{F}$, we have $h_{\mathcal{L}}(\theta) \leq 0$ by the Lemma on the logarithmic derivative [7]. It remains to prove that $h_{\mathcal{L}}(\theta) \geq 0$.

We prove this by induction in n. The statement is evident when $n = 0$. When $n = 1$ we set $f = w_1/w_0$. Then $\mathcal{L} = f'/f$. If $h_{\mathcal{L}}(\theta_0) < 0$, we integrate f'/f along the ray $\arg z = \theta_0$. If the exceptional set E intersects which ray, we bypass it by a curve close to the ray consisting of arcs of circles. The result is that

$$f = c + O(e^{-\delta r^\rho}).$$

This implies that

$$h_{w_1 - cw_0}(\theta_0) < h_{w_1}(\theta_0),$$

which contradicts the definition of the special basis.

Suppose now that the statement of the Lemma holds for spaces V of dimension at most $m + 1$, with some $m \geq 1$. We have to prove it for $n = m + 1$. Assume by contradiction that $h_{\mathcal{L}(w_0,\ldots,w_n)}(\theta_0) < 0$ for some θ_0. Define functions B_j as solutions of the following system of linear equations

$$\sum_{j=0}^{n-1} B_j w_j^{(k)} = w_n^{(k)}, \quad k = 0, \ldots, n-1.$$

By Cramer's rule,

$$B_j = \pm \frac{W_j}{W_n},$$

where W_j is the Wronskian of size n made of functions w_i with $i \neq j$. We use the formula for differentiation of the logarithm of the quotient of Wronskians [12, Part VII, Probl. 59], [7, p. 251]

$$\frac{d}{dz} \log \left(\frac{W_j}{W_n} \right) = \frac{W_{j,n} W}{W_j W_n} = \frac{\mathcal{L}_{j,n} \mathcal{L}}{\mathcal{L}_j \mathcal{L}_n}, \tag{25}$$

where $W_{j,n}$ is the Wronskian of size $n-1$ with w_j and w_n deleted, and W is our Wronskian of size $n+1$. Notation $\mathcal{L}, \mathcal{L}_j, \mathcal{L}_{j,n}$ has similar meaning. Using the induction assumption, we conclude that the right hand side of (25) has negative indicator. Integrating with respect to z along an appropriate curve near the ray $\arg z = \theta_0$, that avoids the exceptional set E, we obtain $B_j = c_j + O(e^{-\delta r^\rho})$, $0 \leq j \leq n-1$, where $c_j \neq 0$ and $\delta > 0$ are constants. So we conclude that the indicator of

$$w_n - \sum_{j=0}^{n-1} c_j w_j$$

at the point θ_0 is strictly less than $h_{w_n}(\theta_0)$. This contradicts the property (24) of the special basis. The contradiction completes the proof of Lemma 2.

Proof of Theorem 2. Let $f : \mathbf{C} \to \mathbf{P}^n$ be a linearly non-degenerate holomorphic curve whose homogeneous coordinates are functions of $\mathfrak{F}$.

Let ρ be the order of our curve; it is equal to the maximal order of components f_j.

Let $V \subset \mathfrak{F}$ be the subspace spanned by the homogeneous coordinates. To such a space V we associated finitely many exceptional rays, whose complement consists of admissible sectors. Let us fix any admissible sector S, and a special basis $w_0, \ldots, w_n$ in S.

Let $w_j = (f, \alpha_j), 0 \leq j \leq n$, then the vectors $\{\alpha_0, \ldots, \alpha_n\}$ are linearly independent. We define subspaces

$$X_k = \{w \in \mathbf{C}^{n+1} : (w, \alpha_0) = \ldots = (w, \alpha_{k-1}) = 0\}, \quad 1 \leq k \leq n,$$

so that $\operatorname{codim} X_k = k$. We use the notation $u = \log \|f\|$, $u_j = \log |w_j|$. If z is outside of an exceptional set E, we have

$$u_j(z) \leq u_{j+1}(z) + o(|z|^\rho), \quad 0 \leq j \leq n-1,$$

view of (23). So

$$\begin{aligned} \log d_k(z) &\leq \log \operatorname{dist}(f(z), X_k) \\ &= \max_{0 \leq j \leq k-1} \log |(f(z), \alpha_j)| - \log \|f\| = u_{k-1}(z) - u(z) + o(r^\rho). \end{aligned}$$

Then, using Lemma 2 and $u = u_n + o(r^\rho)$, we obtain

$$\sum_{j=1}^{n} \log \frac{1}{d_k(z)} \quad \geq \quad -\sum_{j=0}^{n-1} u_j(z) + nu + o(r^\rho)$$

$$
\begin{aligned}
&= -\sum_{j=0}^{n} u_j(z) + (n+1)u(z) + o(r^\rho) \\
&= -\log|W(w_0,\ldots,w_n)| + (n+1)u(z) + o(r^\rho).
\end{aligned}
$$

Integrating this with respect to θ on the sector S, and then adding over all admissible sectors, we obtain

$$\sum_{j=1}^{n} m_k(r,f) + N_1(r,f) \geq (n+1)T(r,f) + o(r^\rho).$$

Integrals over the exceptional set E contribute $o(r^\rho)$ [8]. For curves f with components in $\mathfrak{F}$ we always have $T(r,f) = cr^\rho$, so the error term is $o(T(r,f)$.

The opposite inequality follows from Theorem 1, where exceptional set is absent because we deal with functions of finite order.

Remark. A special case of Theorem 2 is that the homogeneous coordinates of f are linearly independent solutions of (7) with $N = n+1$. In this case we have $N_1(r,f) = 0$. For such curves Theorem 2 gives

$$\sum_{k=1}^{n} m_k(r,f) = (n+1+o(1))T(r,f).$$

These curves are analogous to meromorphic functions considered in [9, 10].

The author thanks Jim Langley for finding a mistake in the previous version of this paper.

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