Theorems of Stephenson and Baker

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September 17, 2021

This is an appendix to my paper [2] with the proofs of two theorems used in that paper.

1. **Theorem of Stephenson.** *Let* f_1 *and* f_2 *be meromorphic functions in the plane. If they are both real on a non-degenerate curve* γ *then there exist meromorphic functions* F_1, F_2, ϕ *such that* F_j *map the real line* $\mathbf{R}$ *to* $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$, $\phi(\gamma) \subset \overline{\mathbf{R}}$, *and*

$$f_j = F_j \circ \phi.$$

Let us prove this first for rational functions f_j, to demonstrate the idea. The image of $(f_1, f_2) : \mathbf{C} \to \overline{\mathbf{C}} \times \overline{\mathbf{C}}$ is contained in some algebraic curve. Let

$$H(f, g) = 0, \quad H \not\equiv 0$$

be the equation of this curve. We may assume that H is irreducible. It is clear that the curve γ contains an analytic curve, and without loss of generality we may assume that it is simple and analytic. Let ψ be a biholomorphic map of some neighborhood of 0 onto a neighborhood of a point on γ sending a real interval around 0 into γ. Then functions $g_j = f_j \circ \psi$ are real on an interval of the real line so they commute with the reflection in the real line. These functions satisfy $H(g_1, g_2) = 0$, and they also satisfy $H^*(g_1, g_2) = 0$ where

$$H^*(u, v) = \overline{H(\overline{u}, \overline{v})}.$$

Thus the curves $H(u, v) = 0$ and $H^*(u, v) = 0$ have infinite intersection, namely $\{(f_1(z), f_2(z)) : z \in \gamma\}$, and these curves are irreducible, so they must coincide. We conclude that H is a real polynomial, up to a constant factor. Now the curve $H(u,v) = 0$ receives a non-constant map from the plane, so this curve considered as a Riemann surface X:

a) Must be $\overline{\mathbf{C}}$, or to have $\mathbf{C}$ as the universal cover.

b) Admits an anti-conformal involution with infinite set of fixed points.

The involution is induced by the map $(u,v) \mapsto (\overline{u}, \overline{v})$ of $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$.

All anti-conformal involutions with fixed points of $\mathbf{C}$ or $\overline{\mathbf{C}}$ are conjugate to $z \mapsto \overline{z}$ by conformal automorphisms.

Therefore there is a conformal map $\theta : \mathbf{C} \to X$ which splits the complex conjugation with the involution on X. Let $f : \mathbf{C} \to X$ be the map induced by (f_1, f_2), and P_i projections of $X \subset \mathbf{C}^2$ onto coordinate axes in $\mathbf{C}^2$. Then f lifts to a map ϕ to the universal cover of X, we set $F_i = P_i \circ \theta$, and obtain $f_i = F_i \circ \phi$.

The problem with extending this proof to the transcendental case is that the image of f in $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$ does not have to be a curve in the usual sense. For example, this image can be dense in $\mathbf{C}^2$. This is remedied essentially by proper definitions.

The definition of "Analytische Gebilde" (analytic entity, as translated by M. Heins) is essentially due to Weierstrass.

Let A' be the set of pairs (f_1, f_2) where f_i are meromorphic germs at 0 in $\mathbf{C}$, which separate points in a neighborhood of 0, that is f_1 and f_2 have no common right compositional factor of the form z^n, where $n \geq 1$. We define the equivalence relation on the set of these pairs by saying that $(f_1, f_2) \sim (g_1, g_2)$ if there exists a biholomorphic germ $\theta : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ such that $f_j = g_j \circ \theta, j = 1, 2$. The set of equivalence classes is denoted by $A = A'/\sim$.

Topology and complex analytic structure are defined on A in the usual way. To define a neighborhood U of (f_1, f_2) we consider some representatives of f_j which are holomorphic in a region D containing zero, and the neighborhood is represented by pairs of germs $(f_1(z-c), f_2(z-c))$ where $c \in D$. We obtain the map $U \to D$ assigning to the pair of germs this number c used to define them, and this map is a coordinate chart defining the complex structure. With this complex structure, A is a disjoint union of Riemann surfaces. A connected component of A is called an analytic entity. There is a natural map $A \to \overline{\mathbf{C}} \times \overline{\mathbf{C}}$ given by $(f_1, f_2) \mapsto (f_1(0), f_2(0))$.

There is also an anti-conformal involution $I : A \to A$ defined by

$$(f_1, f_2) \mapsto \left(\overline{f_1(\overline{z})}, \overline{f_2(\overline{z})} \right).$$

An analytic entity is called symmetric if this involution maps it into itself. The simple but crucial fact is that if one pair of germs in an analytic entity is fixed by the involution then the whole analytic entity which contains this pair is symmetric.

This is proved by an "analytic continuation". Let (f_1^0, f_2^0) be a symmetric pair. Every other pair of the same analytic entity is obtained by a finite sequence of pairs (f_1^k, f_2^k) such that (f_1^{k+1}, f_2^{k+1}) is in a neighborhood of (f_1^k, f_2^k) as defined above. To specify (f_1^{k+1}, f_2^{k+1}) in this neighborhood one chooses a representative of (f_1^k, f_2^k) and a number c_k as above. Choosing the conjugate numbers c_k and conjugate representatives leads to an analytic continuation to the conjugate germ.

Now to prove Stephenson's theorem, one considers the analytic entity defined by two entire functions f_1 and f_2 as in the theorem. Taking the germs at a point on γ gives a symmetric pair. So our analytic entity is a Riemann surface with an anti-conformal involution. The proof is completed exactly as in the algebraic case.

2. Theorem of Baker. *Let f be an entire function of order less than 1 whose zeros lie in the sector $|\arg z - \pi| < \pi/2 - \delta$, for some $\delta > 0$. If f admits a factorization*

$$f = F \circ \phi$$

with transcendental F then ϕ is a polynomial of degree 1.

Proof. Since the order of f is less than 1, it has infinitely many zeros. By a theorem of Pólya, F must be of zero order, unless ϕ is a polynomial, so F has infinitely many zeros $w_k \to \infty$. All solutions of all equations

$$\phi(z) = w_k, \quad k \geq 0,$$

must lie in the sector $|\arg z - \pi| < \pi/2 - \delta$. We may assume without loss of generality that $w_0 = 0$ by replacing ϕ by $\phi - w_0$ and $F(w)$ by $F(w + w_0)$. This immediately leads to a contradiction if ϕ is a polynomial of degree at least 2. So in the rest of the proof we assume that ϕ is transcendental.

The order of ϕ is less than 1, so ϕ has infinitely many zeros, and without loss of generality we assume that $f(0) \neq 0$. Then

$$\phi(z) = c \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right), \quad \text{where} \quad \sum |z_n|^{-1} < \infty.$$

iSince each factor has absolute value at least 1 when $|\arg z| < \delta$, and tends to ∞ as $z \to \infty$ in this sector, we conclude that $\phi(z) \to \infty$ as $z \to \infty, \ |\arg z| < \delta$. Moreover,

$$\frac{\phi'(z)}{\phi(z)} = \sum_{n=1}^{\infty} \frac{1}{z - z_n}.$$

By the same argument that is used in the proof of the Gauss-Lucas theorem we conclude that all zeros of ϕ' belong to the left half-plane.

Let

$$D(z) := \frac{d \log \phi(z)}{d \log z} = z\frac{\phi'(z)}{\phi(z)} = \sum_{n=1}^{\infty} \frac{z}{z - z_k}.$$

We claim that for some $K > 0$ we have

$$|D(z)| > \frac{4\pi}{\delta}, \quad \text{for} \quad |\arg z| < \delta/2, \quad |z| > K. \tag{1}$$

Indeed, as $|\arg z_n - \pi| < \pi/2 - \delta$ and $|\arg z| < \delta/2$, we conclude that $|\arg(z/(z - z_0))| < \pi/2 - \delta/2$. So

$$|D(z)| \geq \sum_{n=1}^{\infty} \mathrm{Re}\left(\frac{z}{z - z_0}\right) \geq \cos(\delta/2) \sum_{n=1}^{\infty} \left|\frac{z}{z - z_n}\right| \to \infty,$$

as $z \to \infty, \ |\arg z| < \delta/2$, so there is $K > 0$ such that our claim (1) holds.

As $\phi(z) \to \infty$ as $z \to +\infty$ on the positive ray, we can find a sequence $x_n > 0$ such that $|\phi(x_n)| = |w_n|$. Let γ_n be the component of the set

$$\{z : |\phi(z)| = |w_n|\} \cap \{z : |\arg z| \leq \delta/2\}$$

which contains x_n. Then γ_n is a simple analytic curve with endpoints z_n' and z_n'', where $\arg z_n' = -\delta/2$ and $\arg z_n'' = \delta/2$. This follows because the intersection of the level set of ϕ with the right half-plane is a smooth analytic curve (ϕ has no critical points in the right half-plane), and this curve cannot escape to infinity in the sector $|\arg z| < \delta$ since $\phi(re^{i\alpha})$ tends to ∞ in this sector as $r \to \infty$, uniformly with respect to α.

Denoting by Δ_γ an increment along γ, we have

$$i\Delta_{\gamma_n} \arg \phi = \Delta_{\gamma_n} \log \phi = \int_{z_n'}^{z_n''} \frac{d \log \phi(\zeta)}{d \log \zeta} d \log \zeta.$$

Now we notice that $\arg \phi(z)$ is a strictly monotone function of the natural parameter on γ_n, so, using (1) we obtain

$$|\Delta_{\gamma_n} \arg \phi(z)| \geq \frac{4\pi}{\delta} |\Delta_{\gamma_n} \log z| \geq \frac{4\pi}{\delta}\delta > 2\pi.$$

This means equations $\phi(z) = w_n$ have some solutions on γ_n which contradicts to the fact that all w_n-points of ϕ lie in the left half-plane. This proves Baker's theorem.

References

[1] I. N. Baker, The value distribution of composite entire functions, Acta Sci. Math. 32 (1971), 87—90.

[2] A. Eremenko, Metrics of constant positive curvature with four conic singularities on the sphere, Proc. AMS 148, 9 (2020) 3957–3965.

[3] K. Stephenson, Analytic functions sharing level curves and tracts, Ann. Math., 123 (1986) 107—144.