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FINELY OPEN SETS IN THE LIMIT SET OF A FINITELY GENERATED KLEINIAN GROUP

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Abstract. We prove a weak form of the Ahlfors conjecture on the limit set of a Kleinian group. Let U be an invariant subset of the limit set. We show that U has empty fine interior or the capacity of $\mathbb{C}_\infty \backslash U$ is zero. In particular the limit set has empty fine interior unless it is equal to $\mathbb{C}_\infty$. The method extends to related examples such as the iteration of rational functions and suggests a strong form of Ahlfors conjecture; the proof is strictly two dimensional.

Key words: Julia set, iteration of mappings, fine topology, holomorphic functions, finely holomorphic functions, Ahlfors conjecture, equicontinuity.

1. Introduction

Consider a finitely generated Kleinian group Γ of Möbius transformations of the sphere $\mathbb{C}_\infty$. Then Ahlfors conjectured that the closed limit set Λ has measure zero or is the full sphere. This problem is hard and many important contributions have been made to understanding this and similar problems (for example analysing the Julia set of a rational function).

This paper studies the possibility that invariant fine analytic structure could exist in the limit set of a Kleinian group Γ and more generally asks whether the fine and conventional Fatou/Julia decomposition could differ. The answer is no, at least in this two dimensional case and for nice families of iterating functions (but the arguments are rather special to this dimension).

Recall that a set E is said to be thin at a point e if one can find a superharmonic function u which separates e from E; that is to say so that $u(x) > 1$ for $x \neq e$ providing $x \in E$ and $| x - e |$ is small enough while $u(e) < 1$. In this case the set $\mathbb{C}_\infty \setminus E$ is said to be a fine neighbourhood of e. Wiener gave a necessary and sufficient quantitative condition in terms of logarithmic capacity for this to occur [8]. From Wiener's criterion, and Beurling's projection lemma [2] it follows easily that in 2 dimensions, any fine neighbourhood of e contains circles of arbitrarily small radii centred on e. In three or higher dimensions the Lebesgue thorn provides a counterexample showing that such nested spheres do not always exist. The fine neighbourhoods generate a topology finer than the usual one called the fine topology, this topology

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has been well studied [5, 6] and is a reasonable one. A significant result is that although not first countable, the topology is quasi-Lindelöf.

The main interest in the fine topology is that the notions of harmonicity and holomorphy can be extended to functions defined on a finely open set and that finely harmonic and holomorphic functions share many of the properties of their classical counterparts. For example, finely holomorphic functions can be defined in terms of the existence of only one complex derivative, are finely open mappings, infinitely finely differentiable and uniquely determined on a given fine component by their formal power series at a single point [5], [7]. Finely harmonic functions satisfy an appropriate mean value property, are finely continuous, and satisfy the maximum principle. The composition of a finely harmonic function, with a finely holomorphic function is always finely harmonic. Both classes are preserved under uniform limits. The functions on a compact set K which can be uniformly approximated by harmonic functions defined on neighbourhoods of K are exactly the continuous functions on K which are finely harmonic on the fine interior of K [4]. A bounded finely harmonic function on a classical open set will always be a classical harmonic function. On the other hand, there do exist finely meromorphic functions mapping the Riemann sphere $\mathbb{C}_\infty$ to itself which are not rational functions.

THEOREM 1. *Suppose that Γ is a finitely generated Kleinian group and that Λ is its limit set. Suppose that U is a proper Γ-invariant subset of $\mathbb{C}_\infty$ in the sense that* $\text{Cap}(\mathbb{C}_\infty \setminus U) > 0$ *and* $\text{Cap}(\gamma U \setminus U) = 0$ *for every γ in Γ; then either U meets the Fatou set $\mathbb{C}_\infty \setminus \Lambda$ or it has no fine interior. In particular, $\Lambda \cap U$ has no fine interior.*

Proof: Recall that Möbius transforms map circles to circles; and that Γ is generated by a finite set $\Gamma_0 = \{l_i : i = 1, \dots, j\}$. Any circle having spherical radius less than $\pi/2$ is said to have an interior and an exterior (the smaller and larger components of the complement respectively). In particular, as Γ_0 is a finite collection of such maps, there is an absolute radius such that if a circle has smaller radius then the interior gets mapped to the interior by every member of Γ_0.

Suppose that U is a proper Γ-invariant subset of $\mathbb{C}_\infty$ and that it has fine interior; fix x in the fine interior and a compact fine neighbourhood K of x in U, then the elements of Γ are all continuous and analytic on a neighbourhood of K. Moreover, they all map K into U and hence omit a common set of positive capacity. By Theorem 3 (below) one may find a second compact fine neighbourhood L of x and on which the restrictions of the functions in Γ are a normal family. As the functions are equicontinuous on L, there will be a small circle C_0 centred on x in L such that its image

particular any element λ of Γ_0 will map the interior of any of these small circles C_γ to the interior of another $C_{\lambda\gamma}$. A simple induction extends this to any finite product, and hence to Γ itself.

It follows that Γ is equicontinuous on the interior of C_0 which therefore lies in the complementary set to Λ. ∎

Notice that the argument works just as well for the iterates of a rational function on $\mathbb{C}_\infty$ providing one considers the interior of the rather complicated curve $f^n(C_0)$ to be everything except the largest component of the complement.

2. Equicontinuity of Holomorphic Functions Omitting a Set of Positive Capacity

It was shown in [7] that if K is a compact subset of $\mathbb{R}^d$ then any uniformly bounded family of continuous functions on K which are finely harmonic on the fine interior of K are in fact finely equicontinuous. That is to say for any uniformly bounded family $\mathcal{K}$ of finely harmonic functions continuous on K, and for any point x in the fine interior K' of K, there is a compact subset L of K which is a fine neighbourhood of x and on which the family $\mathcal{F}$ is equicontinuous.

It follows immediately that any family of functions $\mathcal{F}$ on K which extend to be holomorphic on a neigbourhood of K and which omit a common open set from their range will be equicontinuous on the set L.

Suppose that the family of functions $\mathcal{F}$ on K omit a common set E from their range, and that although E is not open, it is of positive logarithmic capacity. Then it is also possible, although the proof is less natural, to show that $\mathcal{F}$ will be equicontinuous on the set L. This result cannot be improved on.

Indeed, the inverse image of a set of capacity zero under a finely holomorphic function is easily seen to have capacity zero and countable unions of sets of capacity zero have capacity zero; it follows that for any sequence of finely holomorphic functions on a finely open set U and any set E of zero capacity it is possible to delete a capacity zero (and hence finely closed and nowhere dense) set E' from U so that the sequence omits all values in E. On the other hand it is easy to construct a sequence of holomorphic functions which is not equicontinuous off any set of zero capacity.

Before we can give the proof of the theorem we must establish a couple of lemmas.

LEMMA 1. *Let E be a closed set of positive capacity in the complex sphere $\mathbb{C}_\infty$, and let $\mathcal{H}(E)$ denote the set of bounded continuous functions on $\mathbb{C}_\infty$*

E. Then the function

$$u(r) = a \log | r - x | + b \log | r - y | - (a+b) \log | r - z |$$

can be uniformly approximated off any neighbourhood of x, y, z by a function in $\mathcal{H}(E)$.

Proof: Without loss of generality, we may assume the points x, y, z are all in the complex plane. For convenience, denote by $B(\epsilon, w)$ the ball of radius ϵ in the usual planar metric centred on w. Now, by assumption, the capacity of $E \cap B(\epsilon, x)$ is strictly positive for any ϵ and so there exists a probability measure μ_x concentrated on $B(\epsilon, x) \cap E$ whose logarithmic potential $p_x(v) = \int \log | w - v | \mu_x(dw)$ is bounded. Moreover, standard potential theory tells us we may choose μ_x so that p_x is a continuous function on the plane. Now, put $\delta = \epsilon^{1/2}$, then off $B(\delta, x)$ one has $| p_x(r) - \log | r - x | | \leq \frac{\delta}{1-\delta}$. It follows that if $h(r) = a p_x(r) + b p_y(r) - (a+b) p_z(r)$ then $h \in \mathcal{H}(E)$ and

$$(1) \qquad | u(r) - h(r) | \leq 2(| a | + | b |) \frac{\delta}{1-\delta}$$

off $B(\delta, x) \cup B(\delta, y) \cup B(\delta, z)$. As δ is arbitrary the proof is complete. ■

LEMMA 2. *Let E be a closed set of positive capacity in the complex sphere $\mathbb{C}_\infty$, and let $\mathcal{H}(E)$ denote the set of bounded continuous functions on $\mathbb{C}_\infty$ that are harmonic off E. Let x and y be two distinct points not in E. Then either*

A. *E is a subset of a circle, x, y are conjugate relative to the circle, and $h(x) = h(y)$ for every function in $\mathcal{H}(E)$, or*

B. *there is a function h in $\mathcal{H}(E)$ such that $h(x) \neq h(y)$.*

Proof: Any function $h \in \mathcal{H}(E)$ is uniquely determined by its values on E. Suppose E is a subset of a circle, then inversion in that circle preserves $\mathcal{H}(E)$ and fixes each function h in the class, hence if x, y are conjugate relative to the circle, $h(x) = h(y)$.

To prove that B holds if A fails we can apply the previous lemma. Fix three regular points x, y, z in E. The functions $u(r)$ defined above separate the points in the disk with boundary circle through x, y, z as one varies a and b. In the case where E is not contained in a circle further choices of x, y, z allow one to separate all points off E. Because the functions $u(r)$ can be approximated locally uniformly by elements of $\mathcal{H}(E)$ the lemma is proved. ■

THEOREM 2. *The following quantity*

$$(2) \qquad \tau = \inf\{\sup\{| h(x) - h(y) | : \| h \| < 1, h \in \mathcal{H}(E)\} : d(x,y) \geq \rho\}$$

is strictly positive for any choice of ρ ($d(.,.)$ is the spherical distance on $\mathbb{C}_\infty$).

Proof: A deep theorem of Ancona [1] shows that any set E of positive capacity contains a set of positive capacity F which is compact and regular at all of its points. It follows that every continuous function on F can be extended to a function in $\mathcal{H}(E)$ and it is enough to prove the corollary with F in place of E. If x, y are both in F then it is clear that $\sup\{| h(x) - h(y) | ; \| h \| < 1, h \in \mathcal{H}(F)\} = 2$, and that if either x or y is not in F then $\sup\{| h(x) - h(y) | ; \| h \| < 1, h \in \mathcal{H}(F)\} > 0$. The subset of $\mathbb{C}_\infty \times \mathbb{C}_\infty$ comprising those points (x,y) for which $d(x,y) \geq \rho$ is compact, and $\sup\{| h(x) - h(y) | ; \| h \| < 1, h \in \mathcal{H}(F)\}$ is strictly positive there. As lowersemicontinuous functions on compact sets attain their lower bounds the infimum must also be strictly positive. This completes the proof of the theorem. ■

THEOREM 3. *Let $\mathcal{F}$ be a family of functions which are continuous on K (compact in $\mathbb{C}_\infty$) and finely holomorphic on the fine interior K' of K, and suppose these functions omit a common set E of positive logarithmic capacity from their range; then for any point x in K', there is a compact subset L of K which is a fine neighbourhood of x and on which the family $\mathcal{F}$ is spherically equicontinuous.*

Proof: Let K be a compact subset of $\mathbb{R}^d$, and fix a point x in the fine interior K' of K; then the set $\mathcal{G}$ of continuous functions on K uniformly bounded by 1 and finely harmonic on the fine interior of K will be finely equicontinous. Let L be a compact subset of K, which is also a fine neighbourhood of x, and on which the family $\mathcal{G}$ is equicontinuous in the usual metric topology. The claim is that the family $\mathcal{F}$ is also equicontinuous on L. Let $x \in L$. Choose ρ, and fix δ so that if g is in $\mathcal{G}$ and y is in $B(\delta, x) \cap L$ then $| g(x) - g(y) | < \tau$. Now if $h \in \mathcal{H}(F)$ and if $\| h \| < 1$ and if $f \in \mathcal{F}$ then $g = h(f)$ is in $\mathcal{G}$; varying h it follows from the previous lemma that the spherical distance $d(f(x), f(y))$ must be at most ρ. Consequently the restrictions of the functions in $\mathcal{F}$ to L are equicontinuous when thought of as spherical functions; moreover the sphere is compact and so the Arzelà–Ascoli Theorem applies to show that they are in fact a normal family. ■

The above proof establishes the equicontinuity of $\mathcal{F}$ in the case where the case we have only shown equicontinuity if we regard the functions in $\mathcal{F}$ as taking their values in the quotient of $\mathbb{C}_\infty$ obtained by identifying conjugate points. We now deal with this annoying technical point.

LEMMA 3. *A sequence of continuous functions $\mathcal{F}$ from the compact set L into the sphere which are equicontinuous when conjugate points (with respect to some fixed circle) are identified contains a subsequence which converges uniformly on the sphere and the given sequence is thus also equicontinuous in the usual metric on the sphere.*

Proof: Suppose a sequence of functions f_i in $\mathcal{F}$ were equicontinuous on L in the quotient sense but not in the usual sense. One may assume the sequence converges uniformly in the quotient metric. Define L_1 to be that subset of L where $f_i(l)$ has its limit point on the circle, and let L_2 be the complementary open set $L \setminus L_1$. For each point l in L_2, the sequence has either one or two conjugate accumulation points, and it eventually remains bounded away from the circle. The equicontinuity and continuity of the functions f_i ensures that on a neighbourhood U_l of l a subsequence of the sequence will remain on one side of the circle only and so converge uniformly in the spherical metric. Choosing a countable covering of L_2 by sets U_l a simple diagonal selection argument thus shows that a subsequence of f_i converges locally uniformly on L_2 and uniformly on L_1 and in particular converges pointwise on the sphere to a function f.

In fact the subsequence which we again denote by f_i converges uniformly to f on L. For simplicity let the circle be regarded as the equator, and let d, d' be the usual spherical metric and quotient pseudo–metrics respectively. Fix ϵ, and choose N so that $d'(f_i, f_j) < \epsilon/5$ uniformly on the sphere if $i, j > N$. The d'–uniform equicontinuity of the f_i permits one to choose δ so that if $d(l,m) < \delta$ then $d'(f_i(l), f_i(m)) < \epsilon/5$. Now $d(f_i(l), \text{Circle}) \leq d(f_i(l), f_i(m)) + d(f_i(m), f(m))$ for any m in L_1 and so for $i > N$ and for $d(l, L_1) < \delta$ one has $d(f_i(l), \text{Circle}) \leq 2\epsilon/5$ and under this hypothesis the spherical distance between $f_i(l)$ and its conjugate is at most $4\epsilon/5$. We conclude that the spherical distance between $f_i(l)$ and $f_j(l)$ is at most ϵ uniformly for $d(l, L_1) < \delta$ and $i, j > N$.

On the other hand the set $d(l, L_1) \geq \delta$ is compact in L_2 and so the subsequence f_i converges uniformly there. Choose M so that the spherical distance between $f_i(l)$ and $f_j(l)$ is at most ϵ on this set if $i, j > M$.

It follows that the spherical distance between $f_i(l)$ and $f_j(l)$ is at most ϵ on all of L if $i, j > \max(N, M)$. The arbitrary choice of ϵ shows that the convergence of the subsequence was indeed uniform in the spherical metric and that the original family was normal. ■

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