A Toda lattice in dimension 2 and Nevanlinna theory

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Abstract

It is shown how to study the 2-D Toda system for SU(n+1) using Nevanlinna theory of meromorphic functions and holomorphic curves. The results generalize recent results of Jost – Wang and Chen – Li.

We consider the 2-dimensional open Toda system for $SU(n+1)$:

$$-\frac{1}{2}\Delta u_j = -e^{u_{j-1}} + 2e^{u_j} - e^{u_{j+1}}, \quad 1 \leq j \leq n, \quad u_0 = u_{n+1} = 0, \qquad (1)$$

where u_j are smooth functions in the complex plane $\mathbf{C}$, and n is a positive integer.

This system was recently studied by several authors (see the reference list in [8]). When $n = 1$ we obtain the Liouville equation

$$-\Delta u_1 = 4e^{u_1}. \qquad (2)$$

Jost and Wang [8] classified all solutions of (1) satisfying the condition

$$\int_{\mathbf{C}} e^{u_j} < \infty, \quad 1 \leq j \leq n. \qquad (3)$$

In this paper, such classification will be given for a larger class of solutions, namely those that satisfy

$$B(r) = \int_{|z| \leq r} e^{u_1} = O(r^K) \quad \text{for some} \quad K > 0. \qquad (4)$$

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This will also give another proof of the result of Jost and Wang. Following their suggestion in [8, p. 278], we apply the Nevanlinna theory of holomorphic curves.

It is well known, and goes back to Liouville [9], that the general solution of (2) has the form

$$u = u_1 = \log \frac{2|f'|^2}{(1+|f|^2)^2}, \tag{5}$$

where $f : \mathbf{C} \to \mathbf{P}^1$ is a meromorphic function with no critical points (that is $f'(z) \neq 0,\ z \in \mathbf{C}$ and f has no multiple poles).

The general solution of the Toda system can be similarly described in terms of holomorphic curves $\mathbf{C} \to \mathbf{P}^n$. System (1) appears for the first time in the work on the value distribution of holomorphic curves [1, 14, 15]. In [2], Calabi proved that every solution of (1) comes from a holomorphic curve; this result will be stated precisely below.

Strictly speaking, the present paper contains no new results. Its purpose is to translate some old results of value distribution theory to the language of PDE, and thus to bring these results to the attention of a wider audience. We begin with the simpler case of the Liouville equation.

1. Liouville equation. We recall that the order of a meromorphic function f can be defined by the formula

$$\lambda = \limsup_{r \to \infty} \frac{\log A(r,f)}{\log r}. \tag{6}$$

where

$$A(r,f) = \frac{1}{\pi} \int_{|z| \leq r} \frac{|f'|^2}{(1+|f|^2)^2}.$$

If u is related to f by (5) and satisfies (4) then

$$\lambda \leq K. \tag{7}$$

So (5) establishes a bijective correspondence between solutions of the Liouville equation satisfying (4) and meromorphic functions of finite order without critical points. This class of meromorphic functions was completely described by F. Nevanlinna in [10]. It coincides with the set of all solutions of differential equations

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = P, \tag{8}$$

where P is an arbitrary polynomial.

The expression in the left hand side of this equation is called the Schwarzian derivative. It is well known (after Schwarz) that (8) is equivalent to the linear differential equation

$$w'' + \frac{1}{2}Pw = 0. \tag{9}$$

More precisely, solutions f of (8) are ratios w_1/w_2 of two linearly independent solutions of (9), and every such ratio is a solution of (8). This can be verified by a direct computation.

The order λ of the meromorphic function f is the same as the orders of all non-zero solutions of (9), and it is given by the formula

$$\lambda = (\deg P)/2 + 1, \quad \text{if} \quad P \neq 0.$$

If $P = 0$ then all solutions of (8) are fractional-linear functions.

So we have the following recipe for writing all solutions of (2) satisfying (4).

Theorem 1 *Let u be a solution of the Liouville equation* (2) *that satisfies* (4)*. Then*

$$u = \log \frac{2|f'|^2}{(1+|f|^2)^2},$$

where $f = w_1/w_2$, and the w_i are two linearly independent solutions of $w'' + (1/2)Pw = 0$, where P is a polynomial of degree at most $2(K-1)$.

In the opposite direction, if P is a polynomial of degree $d \geq 0$ and u and f are defined as above, then u is a solution of (2) *satisfying* (4) *with $K = d/2 + 1$.*

The well known asymptotic behavior of solutions of the equation (9) permits to derive precise asymptotic formulas for solutions u of (2). We only consider some special cases. First we infer that the order of growth λ of the function B in (4) can only assume a discrete sequence of values: 0, 1, 3/2, 2, 5/2 . . . .

If $\lambda = 0$, then $P = 0$ and f is fractional-linear. This case was studied by Chen and Li [3]. They noticed a curious fact that spherical derivative of any fractional-linear function always has rotational symmetry about some center.

If $\lambda = 1$, then $P = \text{const}$, and f is a fractional-linear function of e^{az}, where $a \neq 0$ is a complex number. So we obtain a family of elementary solutions of (1).

If $\lambda = 3/2$, then $\deg P = 1$, and u can be expressed in terms of the Airy function.

In the general case, we have the following asymptotic behavior. Let $P(z) = az^d + \ldots$, where $d = \deg P$. The Stokes lines of the equation (9) are defined by

$$\mathrm{Im}(\sqrt{a} z^{d/2+1}) = 0.$$

They break the complex plane into $d+2$ sectors of opening $2\pi/(d+2)$. In each of these sectors, $u(z)$ tends to $-\infty$, while on the Stokes lines it grows like $(d/2)\log|z|$.

2. Toda system. By a holomorphic curve we mean in this paper a holomorphic map

$$f : \mathbf{C} \to \mathbf{P}^n,$$

where $\mathbf{P}^n$ is the complex projective space of dimension n. In homogeneous coordinates, a holomorphic curve is described by a vector-function

$$F : \mathbf{C} \to \mathbf{C}^{n+1}, \quad F = (f_0, f_1, \ldots, f_n)$$

such that f_j are entire functions without common zeros. These entire functions are defined up to a common multiple which is an entire function without zeros in the plane. To each holomorphic curve correspond derived curves defined in homogeneous coordinates as

$$F_k : \mathbf{C} \to \mathbf{C}^{n_k}, \quad n_k = \binom{n+1}{k+1}, \quad 1 \leq k \leq n,$$

$$F_k = f \wedge f' \wedge \ldots \wedge f^{(k)}.$$

It will be convenient to set

$$F_0 = F, \quad \text{and} \quad F_{-1} = 1. \tag{10}$$

Notice that $F_n : \mathbf{C} \to \mathbf{C}$ is an entire function; it is equal to the Wronskian determinant of $f_0, \ldots, f_n$.

f is linearly non-degenerate, that is its image is not contained in any hyperplane, if and only if $F_k \neq 0$ for $1 \leq k \leq n$.

The following relations are sometimes called “local Plücker formulas”, see [1, 8, 13, 15].

$$\Delta \log |F_k|^2 = 4\frac{|F_{k-1}|^2|F_{k+1}|^2}{|F_k|^4}, \quad 0 \leq k \leq n-1. \tag{11}$$

Here $|\,|$ is the Euclidean norm. As F_n is an entire function, we have

$$\Delta \log |F_n|^2 = 0. \tag{12}$$

These relations (11) and (12) hold in the classical sense, that is outside the zeros of F_k.

For a given holomorphic curve $f : \mathbf{C} \to \mathbf{P}^n$ we set for $k = 1, \ldots, n$:

$$u_k = \log |F_{k-2}|^2 - 2\log |F_{k-1}|^2 + \log |F_k|^2 + \log 2. \tag{13}$$

Notice that the u_k do not change if all f_k are multiplied by a common factor, so $(u_1, \ldots, u_k)$ depend only on the holomorphic curve f rather than the choice of its homogeneous representation.

It is verified by simple computation using (11), (12) and our conventions (10) that these functions u_k satisfy the Toda system (1).

Calabi [2] proved the converse statement: every solution of the system (1) in the plane arises from a holomorphic curve $f : \mathbf{C} \to \mathbf{P}^n$, satisfying

$$F_k(z) \neq 0,\ z \in \mathbf{C}, \quad 1 \leq k \leq n, \tag{14}$$

via (13).

Conditions (14) are equivalent to

$$F_n(z) \neq 0,\ z \in \mathbf{C}. \tag{15}$$

This can be proved by a computation in local coordinates, see, for example [13]. The following arguments are taken from Petrenko's book [11], see also [5].

Holomorphic curves that satisfy (15) are called unramified.

Proposition 1 *The class of unramified holomorphic curves $f : \mathbf{C} \to \mathbf{P}^n$ coincides with the class of curves that have homogeneous representations of the form $F = (f_0, \ldots, f_n)$, where $f_1, \ldots, f_n$ is a basis of solutions of a differential equation*

$$w^{(n+1)} + P_n w^{(n)} + \ldots + P_0 w = 0, \tag{16}$$

where P_j are arbitrary entire functions.

Proof. If f is unramified we can choose a homogeneous representation where $F_n = W(f_0, \ldots, f_n) \equiv 1$. Then

$$\begin{vmatrix} w^{(n+1)} & w^{(n)} & \ldots & w \\ f_0^{(n+1)} & f_0^{(n)} & \ldots & f_0 \\ f_1^{(n+1)} & f_1^{(n)} & \ldots & f_1 \\ \ldots & \ldots & \ldots & \ldots \\ f_n^{(n+1)} & f_n^{(n)} & \ldots & f_n \end{vmatrix} = 0$$

is the required equation.

In the opposite direction, suppose that $(f_0, \ldots, f_n)$ is a fundamental system of solutions of an equation (16). If the Wronskian $W = W(f_0, \ldots, f_n)$ has a zero, $W(z_0) = 0$, then the columns of the matrix of $W(z_0)$ are linearly dependent, and we obtain a non-trivial linear combination g of $f_0, \ldots, f_n$ such that $g^{(k)}(z_0) = 0$ for $k = 0, \ldots, n$. As g is a solution of the same equation (16), we conclude from the uniqueness theorem that $g \equiv 0$, but then $W \equiv 0$ which is impossible because $f_0, \ldots, f_n$ are linearly independent. This proves the Proposition.

So we obtained a parametrization of all smooth solutions of the Toda system in the plane: every solution has the form (13) where F_k are the derived curves of a curve f whose coordinates form a basis of solutions of a linear differential equation with entire coefficients.

Now we turn our attention to condition (4). We recall that the order of a holomorphic curve can be defined by equation (6) where

$$A(r, f) = \frac{1}{2\pi} \int_{|z| \leq r} \Delta \log |F_0| = \frac{1}{\pi} \int_{|z| \leq r} \frac{|F_1|^2}{|F_0|^4}.$$

The geometric meaning of $A(r, f)$ is the normalized area of the disc $|z| \leq r$ with respect to the pull-back of the Fubini-Study area form via f.

Let f be a holomorphic curve associated with a solution of the Toda system via (13). Then $B(r) = 2\pi A(r, f)$, and condition (4) implies that f is of finite order, $\lambda \leq K$. It is known that a holomorphic curve f of finite order has a homogeneous representation $F = (f_0, \ldots, f_n)$ where the f_j are entire functions of finite order, in fact maximum of their orders equals the order of f (see, for example [4]).

Let us fix such a representation $(f_0, \ldots, f_n)$. By Proposition 1, these entire functions constitute a basis of solutions of a differential equation

$$w^{(n+1)} + P_n w^{(n)} + \ldots + P_0 w = 0, \tag{17}$$

Now we use a theorem of M. Frei [6]:

Proposition 2 *If a linear differential equation of the form* (17) *has* $n+1$ *linearly independent solutions of finite order then all* P_j, $j = 0 \ldots, n$ *are polynomials.*

The converse is also true: all solutions of the equation (17) with polynomial coefficients have finite order.

There is a simple algorithm which permits to find orders of solutions of (17). We follow [16].

Suppose that $P_j \neq 0$ for at least one $j \in [0, n]$. Plot in the plane the points with coordinates $(k, \deg P_k - k)$, for $0 \leq k \leq n+1$, $\deg P_{n+1} = 0$. Let C be the convex hull of these points, and Γ the part of the boundary ∂C, which is visible from above. This polygonal line Γ is called the Newton diagram of the equation (17). Then the orders of solutions are among the negavive slopes of segments of Γ. Let $-\lambda$ be the negative slope of a segment of Γ, which has the largest absolute value among all negative slopes of segments of Γ. According to a result of Pöschl [12], a solution of exact order λ always exists. Then λ is the order of the holomorphic curve defined by (17), and we have

$$\lambda = \max_{0 \leq k \leq n} \frac{n + 1 - k + \deg P_k}{n + 1 - k}. \tag{18}$$

Now we state our final result.

Theorem 2 *Every solution* u *of the Toda system in* $\mathbf{C}$ *that satisfies* (4) *is of the from* (13)*, where* F_k *are the derived curves of a holomorphic curve whose homogeneous coordinates form a basis of solutions of the equation* (17) *with polynomial coefficients. The degrees of these coefficients, when substituted to* (18)*, give* $\lambda \leq K$*, where* K *is the constant from* (4)*.*

Every basis of solutions of any equation (17) *defines a solution* (u_k) *of the Toda system by the above rule. This solution satisfies* (4) *with* $K = \lambda$*, where* λ *is defined by* (18)*.*

Some special cases are:

1. If $P_0 = \ldots = P_n = 0$ in (17), then a basis of solutions is $(1, z, \ldots, z^n)$ which is the rational normal curve. Thus we recover the main result of the paper [8].

2. Suppose that all P_j are constants. Then solutions of (17) are generalized exponential sums, and we obtain a class of explicit solutions of (1).

The asymptotic behavior of solutions of (1) is more complicated in the general case $n > 1$ than for $n = 1$.

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