A Determination of the Number of Real and Imaginary Roots of the Hypergometric Series
Author(s): Edward B. Van Vleck
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A DETERMINATION OF THE NUMBER OF REAL AND IMAGINARY ROOTS OF THE HYPERGEOMETRIC SERIES*

BY

EDWARD B. VAN VLECK

If the axis of x between 1 and ∞ is considered to be a cut, the hypergeometric series

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \cdots,$$

with its analytic continuation, will define a function which is one-valued over the remainder of the plane of x. The number of roots of this function between 0 and 1 was ascertained first by STIELTJES † and HILBERT ‡ for the special case in which $\alpha = -n$, when the series reduces to a polynomial. Later the determination for the general case was effected by KLEIN § in a memoir notable both for its results and for its method. The number of roots between 0 and $-\infty$ can be obtained from KLEIN's results by means of the equation

$$F(\alpha, \beta, \gamma, x) = (1-x)^{-\alpha} F\left(\alpha, \gamma - \beta, \gamma, \frac{x}{x-1}\right).$$

So far as I am aware, the number of imaginary roots has not been known, and is ascertained for the first time in the present paper.

For this purpose KLEIN's geometrical method has been further developed. In the memoir above cited KLEIN made use of the conformal representation which is effected by the quotient of any two solutions of the hypergeometric differential equation. This quotient, as SCHWARZ showed, builds the positive half plane of x upon a triangle, bounded by arcs of circles, the sides of which

* Presented to the Society October 26, 1901. Received for publication December 3, 1901.

† Comptes Rendus, vol. 100 (1885), p. 620.

‡ Crelle, vol. 103 (1887), p. 337.

§ Mathematische Annalen, vol. 37 (1890), p. 573.

Other methods of finding the number have been given since by the following writers: HURWITZ, Mathematische Annalen, vol. 38, p. 452; GEGENBAUR, Wiener Berichte, vol. 100, p. 225, and Monatshefte für Mathematik und Physik, vol. 2, p. 124; PORTER, American Journal of Mathematics, vol. 20, p. 193, and Bulletin of the American Mathematical Society, vol. 6, p. 280. The simplicity of the form in which the results are obtained by HURWITZ is worthy of note.

correspond to the three segments into which the axis of x is divided by the singular points, $0, 1, \infty$, of the differential equation. KLEIN derives a formula for the number of times which any side returns upon or overlaps itself, and shows that either this number, or this number increased by 1, must be equal to the number of roots of any real solution of the differential equation within the corresponding segment of the axis. By taking the side which corresponds to the segment $(0, 1)$, the number of roots of the hypergeometric series between 0 and 1 is determined to within a unit. To decide, however, between the two values thus obtained, KLEIN abandons the triangle and settles the question by considering the sign of $F(\alpha, \beta, \gamma, x)$ when x approaches 1.

This departure from the fundamental principle of many of his investigations,—to wit, the determination of the properties of the integrals of a differential equation from the shape of the corresponding triangle—is, however, unnecessary. For, as will be shown here, the number of roots of certain particular integrals in each segment of the real axis can be ascertained directly from the triangle. These integrals correspond to the exponents of the singular points. Since $F(\alpha, \beta, \gamma, x)$ is such an integral, the number of its roots in each interval of the axis can be determined without any other aid than the triangle.

The completion of KLEIN's method leads immediately to the determination of the number of roots of the hypergeometric series in the imaginary domain. The theory can also be extended to any regular linear differential equation of the second order with real parameters (real singular points, exponents, etc.). If the analytic continuation of its solutions across the real axis from the positive into the negative half plane is forbidden, the fundamental integrals which correspond to the exponents of the singular points will define functions which are one-valued throughout the positive half plane. To find the number of roots of each function within the half plane, or in any of the segments into which the real axis is divided by the singular points, it is necessary only to construct the circular polygon into which the positive half plane is built by the quotient of any two solutions whatsoever of the differential equation.

One other important question is solved by means of the polygon. The differential equation in special cases may possess one or more integrals whose values are altered only by multiplicative constants for circuits around each of two or more singular points. The shape of the polygon reveals the existence or non-existence of such integrals, and, when they exist, it indicates what integrals have this property.

I. The General Theory.

§ 1. *Notation and preliminary explanations.*

Let

$$p_0(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x) = 0 \tag{1}$$

be any regular linear differential equation in which $p_0(x)$, $p_1(x)$, $p_2(x)$ are polynomials with real coefficients. We will suppose also that the roots of $p_0(x)$, which are the finite singular points of the differential equation, and the exponents of these points are real. The singular points will be denoted by $e_i (i = 1, 2, \cdots, n)$, the subscripts being so assigned as to indicate the order in which they succeed each other upon the axis of x. If the point at infinity is itself a singular point, we shall include it as the last of these points, e_n. Lastly, we shall denote the larger of the two exponents of e_i by λ_i', the smaller by λ_i''.

If the exponent difference $\lambda_i = \lambda_i' - \lambda_i''$ is not an integer, there are two integrals of the equation which in the vicinity of e_i have the form

$$P^{\lambda_i'} = (x - e_i)^{\lambda_i'}[1 + (x - e_i) P_1(x - e_i)],$$

$$P^{\lambda_i''} = (x - e_i)^{\lambda_i''}[1 + (x - e_i) P_2(x - e_i)],$$

P_1, P_2 being ordinary series in ascending powers of their arguments with real coefficients. We shall use the symbols $P^{\lambda_i'}$, $P^{\lambda_i''}$ to represent not merely the above expressions but also their analytic continuations over the positive half plane of x, inclusive of its boundary. Their continuation across the boundary is to be excluded. Upon this understanding $P^{\lambda_i'}$, $P^{\lambda_i''}$ are one-valued within the half plane, and each is a definite linear combination of $P^{\lambda_j'}$, $P^{\lambda_j''}$.

As is well known, the *conform* of the positive half plane which is obtained from the quotient of any two solutions of (1),

$$\eta = \frac{aP^{\lambda_i'} + bP^{\lambda_i''}}{cP^{\lambda_i'} + dP^{\lambda_i''}}, \tag{2}$$

is a polygon bounded by arcs of circles. The side which corresponds to the segment $e_i e_{i+1}$ will be denoted by $E_i E_{i+1}$. The angle at E_i is equal to $\lambda_i \pi$.

The term polygon must be interpreted from the point of view of the theory of functions. Not only may the point at infinity be contained within the polygon, but its surface may be composed of several leaves or partial leaves. If, for example, $\lambda_i > 2$, the surface will wind around E_i so as to overlap itself. It is possible also for a side to overlap, including one or more complete circumferences. We shall not find it necessary to enter into any further discussion of the form of the polygon except for the special case in which it is a triangle (§ 5). For any further information desired the reader is referred to KLEIN,* SCHÖNFLIES† and SCHILLING.‡

* *Lineare Differentialgleichungen* and *Hypergeometrische Function.*

† Mathematische Annalen, vols. 42 and 44.

‡ Ibid., vol. 44, p. 162.

§ 2. *On the connection between the roots of the fundamental integrals and the shape of the polygon.*

We shall now place

$$\eta = \frac{P^{\lambda_i'}}{P^{\lambda_i''}}, \tag{3}$$

so that E_i shall coincide with the origin. Since $P^{\lambda_i'}$, $P^{\lambda_i''}$ are real between e_i and e_{i+1}, the side E_iE_{i+1} is rectilinear and falls upon the real axis. The same two integrals will be real between e_{i-1} and e_i if multiplied by $e^{i\pi\lambda_i'}$ and $e^{i\pi\lambda_i''}$. Hence $E_{i-1}E_i$ is also rectilinear, making an angle $\lambda_i\pi$ with the axis. The second intersection of these two sides, produced if necessary, will be denoted by E_i'. In this case it lies at infinity.

Consider now the zeros of $P^{\lambda_i'}$ and $P^{\lambda_i''}$. It is a familiar fact that two independent integrals of (1) can vanish simultaneously only in the singular points. Such zeros need not be considered here. The remaining zeros of $P^{\lambda_i'}$ and $P^{\lambda_i''}$ give rise respectively to the zeros and infinities of η. The number of zeros of $P^{\lambda_i'}$ within the positive half plane of x is therefore equal to the number of times the polygon includes the origin of the η-plane in its interior, and the number of zeros in either of the segments $e_{i-1}e_i$ and e_ie_{i+1} is equal to the number of times the corresponding side passes through the η-origin. In general the remaining sides do not pass through the origin, and the real roots of $P^{\lambda_i'}$ are therefore usually included in the above segments. In special cases, however, some of the sides may pass through the origin, and every such passage of a side indicates the existence of a root in the segment corresponding to the side. The zeros of $P^{\lambda_i''}$ are indicated in like manner by the passage of the sides and interior of the polygon through the point at infinity.

Let now any other two solutions $\alpha P^{\lambda_i'} + \beta P^{\lambda_i''}$ and $\gamma P^{\lambda_i'} + \delta P^{\lambda_i''}$ be substituted for our two integrals. Obviously the polygon undergoes the transformation

$$\eta' = \frac{\alpha\eta + \beta}{\gamma\eta + \delta}. \tag{4}$$

The origin and point at infinity will be converted into the intersections of $E_{i-1}E_i$ and E_iE_{i+1} in the transformed polygon, but the number of times the surface of the polygon or any one of its sides passes over either intersection is in no wise altered by the transformation. We reach therefore the following result:

THEOREM. *If $\lambda_i' - \lambda_i''$ is not an integer, and if λ_i' denotes the larger of the two exponents of e_i, the number of zeros of $P^{\lambda_i'}$ within the positive half plane of x is equal to the number of times that the interior of the polygon corresponding to any two solutions of (1) passes over E_i. The number of its zeros in any segment of the axis between two successive singular points is equal to the number of times the corresponding side passes over E_i. The zeros* *of* $P^{\lambda_i''}$ *are indicated in like manner by the passage of the sides and interior of the polygon over* E_i', *the second intersection of the sides* $E_{i-1}E_i$ *and* E_iE_{i+1}, *produced if necessary.*

When λ_i is a positive integer, the expression for the integral belonging to the larger exponent is the same as before, but, in general, the form of the other integral must be modified by the introduction of a logarithmic term so that it becomes

$$P^{\lambda_i''} = (x - e_i)^{\lambda_i''} P_2 (x - e_i) + CP^{\lambda_i'} \log (x - e_i).$$

For the class of equations which we are considering, C and the coefficients of P_2 are real. The integrals have also the same form when the two exponents are equal. In this case necessarily $C \neq 0$, and the two integrals can therefore be distinguished by the presence or absence of a logarithmic term. We denote the non-logarithmic integral by $P^{\lambda_i'}$.

Suppose now that in all these cases we put η equal to the quotient (3). It is evident that when $x = e_i$, the quotient must vanish, and the vertex E_i will coincide again with the origin. From this it follows that

If λ_i *is a positive integer or* 0, *the roots of* $P^{\lambda_i'}$ *will be indicated in the same manner as above.*

We cannot, however, reach a similar conclusion concerning the other integral. For, though E_iE_{i+1} will coincide again with the real axis, $E_{i-1}E_i$ will be, in general, the arc of a circle tangent to the axis. The point at ∞ is therefore no longer an intersection of the two sides, and it is in no wise apparent what point in E_iE_{i+1} is to take the place of this point, when the polygon is transformed by (4).

§ 3. *On the coincidence of fundamental integrals belonging to two different singular points.*

The values of $P^{\lambda_i'}$, $P^{\lambda_i''}$ are altered only by a multiplicative constant when x describes a circuit around e_i. If, however, λ_i is an integer and $C \neq 0$, $P^{\lambda_i'}$ does not have this property. The case in which $C = 0$ is also an exceptional one, inasmuch as the two integrals are then multiplied by the same constant. Hence every solution of (1) must be modified in like manner. This is the only case in which any other independent solution shares with the two fundamental integrals the property under consideration. The occurrence of this exceptional case is shown at once by the polygon, for the two sides which meet in E_i at the angle $\lambda_i\pi$ are then arcs of a common circle, and only then. We may therefore dismiss from further consideration in this paragraph the singular points which correspond to such vertices, and confine our attention to the remainder.

We proceed to determine when there is a solution which is altered only by a multiplicative constant for a circuit around either of two singular points, e_i and e_j. One of the two fundamental integrals of e_i will then coincide, except for a numerical factor, with a fundamental integral of e_j. Since this coincidence is a special property of the differential equation, it must, of course, manifest itself in some feature of the polygon which is unaltered by linear transformation.

Suppose first that the integrals which thus coincide are the two which belong to the larger exponents of e_i and e_j. We shall take $\eta = P^{\lambda_j'}/P^{\lambda_j''}$ so that E_j will be situated at the origin. Then, in consequence of the hypothesis just made, η must also have the form

$$\frac{aP^{\lambda_i'}}{cP^{\lambda_i'} + dP^{\lambda_i''}}. \tag{5}$$

But the latter expression vanishes for $x = e_i$. In other words, E_i coincides with the origin and hence with E_j.

Conversely, when these two vertices coincide, the two integrals differ only by a numerical factor. For if by a linear transformation of the η-plane the coincident vertices are brought to the origin, η will vanish both for $x = e_i$ and for $x = e_j$. It follows that $b = 0$ in (2), and η has accordingly the form (5). Since it has a similar form at e_j, we conclude that $P^{\lambda'}$ and $P^{\lambda_j'}$ coincide.

We will next suppose that $P^{\lambda_i'}$ and $P^{\lambda_j''}$ coincide. If η is taken as before and λ_j is not an integer, $E_{j-1}E_j$ and E_jE_{j+1} will not only meet at the origin but will be rectilinear. Accordingly E_j' lies at ∞. But in consequence of our hypothesis, η must also have the form

$$\frac{aP^{\lambda_i'} + bP^{\lambda_i''}}{cP^{\lambda_i'}}, \tag{6}$$

from which it follows that E_i lies at ∞ and coincides with E_j'. Conversely when these two points coincide (Fig. 1), the two integrals must coincide. For

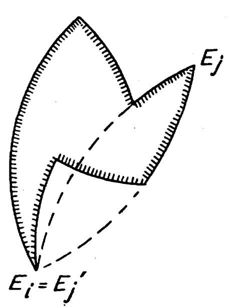

FIG. 1.

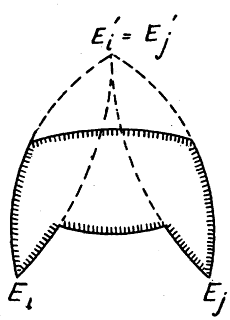

FIG. 2.

let E_j be brought to the origin by a linear transformation of η and at the same time let the two coincident points be removed to ∞. Then on the one hand η must take the form (6), while on the other hand it must be the quotient of the two fundamental integrals of e_j. It follows that $P^{\lambda_i'}$ and $P^{\lambda_j''}$ differ only by a numerical factor.

The coincidence of $P^{\lambda_i''}$ and $P^{\lambda_j''}$ may be discussed in similar fashion provided neither λ_i nor λ_j is an integer. The conclusion thus reached may be recapitulated as follows:

THEOREM.— *When the fundamental integrals $P^{\lambda_i'}$ and $P^{\lambda_j'}$, which belong to the larger exponents of e_i and e_j respectively, coincide except as to a numerical factor, this is revealed in the polygon by the coincidence of the vertices E_i and E_j. If E_i coincides with E_j', the second intersection $E_{j-1}E_j$ and E_jE_{j+1} (produced if necessary), and if λ_j is not an integer, $P^{\lambda_i'}$ and $P^{\lambda_j''}$ differ only by a numerical factor. Lastly, when neither λ_i nor λ_j is an integer, the coincidence of $P^{\lambda_i''}$ and $P^{\lambda_j''}$ is indicated by the coincidence of E_i', and E_j'* (Fig. 2).

An interesting application of this theorem may be made to the case in which three or more consecutive sides, or sides produced, pass through a common point. Let these sides be $E_iE_{i+1}, E_{i+1}E_{i+2}, \cdots, E_{i+r-1}E_{i+r}$. Then there is one integral which is modified only by a constant factor for circuits around any of the singular points $e_{i+1}, e_{i+2}, \cdots, e_{i+r-1}$. When all the sides pass through a common point, the polygon may be made rectilinear by removing the point to ∞. The differential equation then possesses an integral whose value is changed only by a constant factor for any circuit described in the x-plane. This equation and the corresponding polygon have been studied by KLEIN.

II. ON THE DISTRIBUTION OF THE ZEROS OF THE HYPERGEOMETRIC SERIES.

§ 4. *Introductory remarks.*

As is well known, the hypergeometric differential equation

$$x(x-1)\frac{d^2y}{dx^2}-(\gamma-(a+\beta+1)x)\frac{dy}{dx}+a\beta y=0$$

has three singular points, $e_1=0$, $e_1=1$, and $e_3=\infty$, with the exponent differences

$$\lambda_1=|1-\gamma|,\quad \lambda_2=|\gamma-a-\beta|,\quad \lambda_3=|a-\beta|.$$

The two exponents of e_1 are 0 and $1-\gamma$, and the corresponding fundamental integrals are $F(a,\beta,\gamma,x)$ and

$$F_1(x)=x^{1-\gamma}F(a-\gamma+1,\beta-\gamma+1,2-\gamma,x).$$

As we wish to consider here the functions which are obtained by continuing these two series analytically over the positive half plane of x—inclusive of its boundary—*the usual meaning of the symbols F and F_1 will be extended so as to include the two analytical continuations over this half plane.*

In accordance with § 2, we can find the number of roots of either integral in each of the segments into which the axis of x is divided by the singular points, and also the number within the half plane, by constructing the triangle which corresponds to the differential equation. The theory fails only when λ_1, and hence γ, is an integer, and then only for the integral which belongs to the smaller exponent. Now when γ is a negative integer or zero, $F(\alpha, \beta, \gamma, x)$, which is this integral, is devoid of meaning. If $\gamma = 1$, the two integrals coincide, and either is the non-logarithmic integral of e_1. Lastly, if $1 - \gamma$ is a negative integer, $F_1(\alpha, \beta, \gamma, x)$ has no meaning. We conclude therefore that as long as either integral has a meaning, the distribution of its zeros can be obtained by the construction of the triangle.

By properly choosing the two solutions whose quotient is taken for the conformal representation, the vertices of the triangle may be made to take any assigned position. Its essential shape depends therefore only upon the magnitude of the angles. Since these are equal to $\lambda_i \pi (i = 1, 2, 3)$, our problem is to construct the triangle when the exponent differences are given.

§ 5. *Construction of the hypergeometric triangle.*

The construction of the triangle is usually somewhat complicated, but KLEIN * has shown how it can be constructed from a simpler or *reduced* triangle. By a reduced † triangle is to be understood one in which there is no angle greater than 2π and not more than one greater than π.

We shall first explain how the angles of the reduced triangle are to be obtained. For this purpose put

$$\lambda_1 = m_1 + \lambda_1', \quad \lambda_2 = m_2 + \lambda_2', \quad \lambda_3 = m_3 + \lambda_3',$$

in which m_i denotes the integral part of λ_i and λ_i' the fractional remainder. Two cases are to be distinguished. In the first, some one of the integers m_1, m_2, m_3 — call it m_i — is greater than the sum of the other two, m_j and m_k. We then make use of the *reduction :*

$$\begin{aligned} \lambda_i &= m_j + m_k + 2n + \lambda_i' + \epsilon_i, \\ \lambda_j &= m_j + \lambda_j', \\ \lambda_k &= m_k + \lambda_k', \end{aligned} \tag{7}$$

in which n is a non-negative integer and ϵ_i is equal to either 0 or 1. If then we set

$$\lambda_i'' = \lambda_i' + \epsilon_i, \quad \lambda_j'' = \lambda_j', \quad \lambda_k'' = \lambda_k', \tag{8}$$

the angles of the reduced triangle are $\lambda_i''\pi$, $\lambda_j''\pi$, $\lambda_k''\pi$.

* Mathematische Annalen, vol. 37. See also his *Hypergeometrische Function*, p. 404–424, where SCHILLING'S definition of the reduced triangle is used.

† SCHILLING, loc. cit., p. 217.

In the second case each of the three integers m is equal to or less than the sum of the other two. In this case, if

$$M = m_1 + m_2 + m_3$$

is an even integer, place

$$(9) \qquad a_1 = \frac{m_2 + m_3 - m_1}{2}, \quad a_2 = \frac{m_3 + m_1 - m_2}{2}, \quad a_3 = \frac{m_1 + m_2 - m_3}{2}.$$

Then

$$(10) \qquad \lambda_1 = a_2 + a_3 + \lambda_1', \quad \lambda_2 = a_3 + a_1 + \lambda_2', \quad \lambda_3 = a_1 + a_2 + \lambda_3',$$

and to obtain the angles $\lambda_i''\pi$ of the reduced triangle we have merely to take

$$(11) \qquad \lambda_i'' = \lambda_i' \qquad (i = 1, 2, 3).$$

On the other hand, if M is an odd integer, we will set *

$$(12) \quad a_1 = \frac{m_2 + m_3 - m_1 - 1}{2}, \quad a_2 = \frac{m_3 + m_1 - m_2 \pm 1}{2}, \quad a_3 = \frac{m_1 + m_2 - m_3 \mp 1}{2},$$

so that a_1, a_2, a_3 will again be non-negative integers. Where the ambiguities in sign occur, the upper sign is to be selected unless $\lambda_2' \geqq \lambda_1' + \lambda_3'$, when, for a reason which will appear later, the lower sign should be taken. From (12) we get

$$(13) \quad \lambda_1 = a_2 + a_3 + \lambda_1', \quad \lambda_2 = a_3 + a_1 + \lambda_2' + \epsilon_2, \quad \lambda_3 = a_1 + a_2 + \lambda_2' + \epsilon_3,$$

where ϵ_2, ϵ_3 have the values 1 and 0 respectively, unless $\lambda_2' \geqq \lambda_1' + \lambda_3'$, when their values are to be interchanged. The angles of the reduced triangle are then specified by the equations:

$$(14) \qquad \lambda_1'' = \lambda_1', \quad \lambda_2'' = \lambda_2' + \epsilon_2, \quad \lambda_3 = \lambda_3' + \epsilon_3.$$

The various types of reduced triangles will be given later. After the proper one has been picked out, the construction of the triangle may be completed by the attachment of circles to the reduced triangle. The term *circle* is here to be understood in the general sense of the theory of functions. It may, according to circumstances, signify the portion of the plane within or without the bounding circumference, and in special cases the radius of the circle may be infinitely great so that the circle becomes a half plane.

Two modes of making the attachment have been given by KLEIN. By the first mode a circle is added *laterally* along a side, as in Fig. 3, where it is attached to E_jE_k. If two successive lateral attachments are made upon the same side, the one adds the portion of a plane exterior (interior) to the bounding circumference, and the other adds the portion interior (exterior) to the same circle. Hence the two together add an entire plane. Each lateral attachment on a side

* A very slight change is here made in the form of KLEIN's reduction.

$E_j E_k$ increases the angles E_j and E_k by π, and the triangle is bounded alternately by $E_j A E_k$ and the complementary arc $E_j B E_k$.

The second mode of adding a circle is known as *polar attachment.* A circle of the same radius as one of the sides $E_j E_k$ (Fig. 4) is taken and placed above

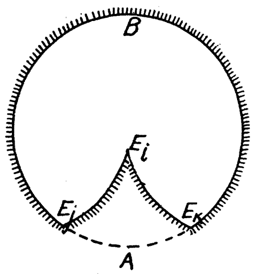

FIG. 3.

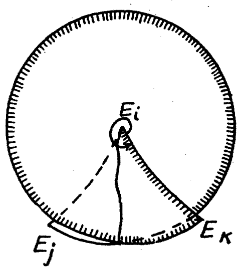

FIG. 4.

or below the triangle so that its circumference shall coincide, in part, with $E_j E_k$. A common cut is then made in the triangle and circle from the side to the opposite vertex, and the triangle and circle are then connected in the manner customary in the construction of a Riemann surface.* Each polar attachment increases a single angle by 2π and adds an entire circumference to the opposite side.

The reduced and completed triangles have, of course, the same vertices. An inspection of (7) and (8) shows that to complete the triangle when the first reduction is used, m_j and m_k lateral attachments must be made upon the sides $E_i E_j$ and $E_i E_k$, while n circles are to be hung to a cut from E_i to the opposite side. If the second reduction is employed, the construction is completed by the lateral attachment of a_1, a_2 and a_3 circles upon $E_2 E_3$, $E_3 E_1$ and $E_1 E_2$ respectively.

§ 6. *On the reduced triangle.*

All the various types of reduced triangles are shown in the accompanying plate.† The triangles are there divided into three sets of five each, which correspond to the three distinct positions which three intersecting circles may take relatively to one another. If the circles pass through a common point, this point may be removed to infinity by a linear transformation of η, and then the triangle becomes rectilinear as in the second section of the plate. *We shall pay*

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* If one part of the triangle is placed above and one part below the circle before the pieces are connected, the completed figure will not intersect itself. See Fig. 4.

† Copied from KLEIN'S *Hypergeometrische Function*, p. 405. See also § 16 of the article by SCHILLING previously cited. Triangles 2 and 8 in the plate should be turned over in order that the interior may lie to the left of $E_i E_j$.

no special attention to the cases in which two of the circles are tangent to each other, since these are merely the limiting cases of those here considered.

For each position of the three circles there is a triangle in which the sum of the angles is equal to or less than the sum in any other triangle bounded by arcs of the same circles. This triangle is called the *minimal* triangle and is placed first in each of the three sections of the plate. If λ_0, μ_0, ν_0 denote the magnitudes of its three angles in terms of π, the triangle is distinguished from the remaining triangles by means of the relations

$$\lambda_0 + \mu_0 \leqq 1, \quad \mu_0 + \nu_0 \leqq 1, \quad \nu_0 + \lambda_0 \leqq 1,$$

and it will belong to section I, II or III according as

$$(15) \qquad \begin{aligned} \lambda_0 + \mu_0 + \nu_0 &> 1 \text{ (triangle 1)}, \\ &= 1 \text{ (triangle 2)}, \\ &< 1 \text{ (triangle 3)}. \end{aligned}$$

The angles of the remaining reduced triangles are expressed in terms of λ_0, μ_0, ν_0.

The expressions for the angles given in the plate will enable us to decide which of the reduced triangles should be selected for given values of λ_1, λ_2, λ_3. It will not, however, be necessary in the subsequent work to distinguish between triangles 1 and 4, nor among 7, 10 and 13.

For convenience of treatment, we shall first divide the triangles into two groups, the first group comprising nos. 1–6, in which all the angles are acute, while the second group contains the remainder. An inspection of the reduction processes will show that if M is an even integer, $\lambda_i'' = \lambda_i'$ $(i = 1, 2, 3)$, and consequently the angles of the reduced triangle will all be acute. On the other hand, if M is odd, one angle will be obtuse.

In the first group we have already distinguished the first three triangles from the others. No. 6 is characterized by a relation of the form

$$\lambda_j' + \lambda_k' - \lambda_i' = (1 - \mu_0) + (1 - \nu_0) - \lambda_0 > 1$$

or

$$(16) \qquad \lambda_j' + \lambda_k' > 1 + \lambda_i',$$

while for no. 5 we have

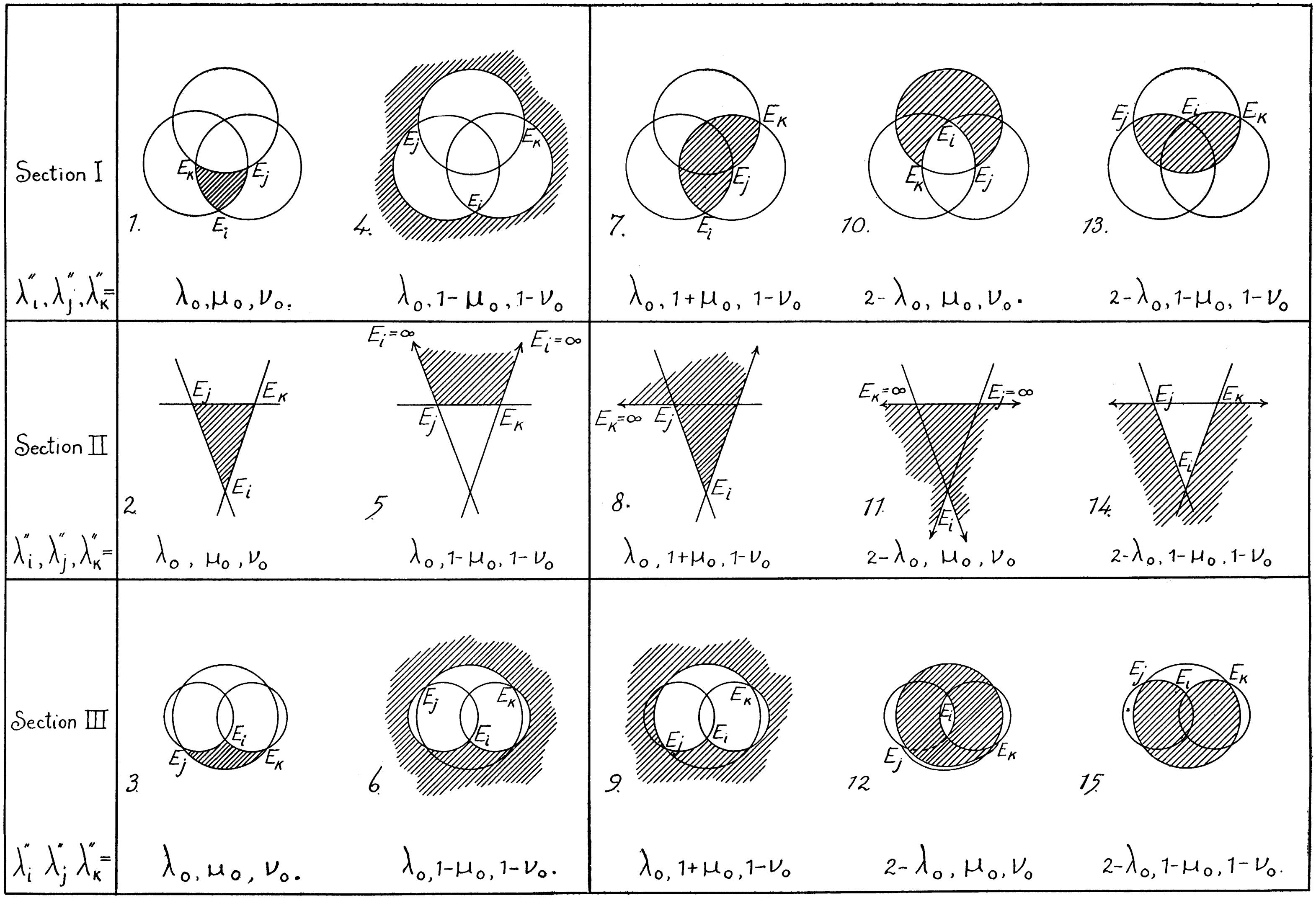
$$(17) \qquad \lambda_j' + \lambda_k' = 1 + \lambda_i'.$$

In the second group of reduced triangles let the obtuse angle be denoted by $\lambda_r''\pi$. Then $\lambda_r' = 1 + \lambda_r'$, while for each of the two remaining angles $\lambda_i'' = \lambda_i'$. The expressions for the angles of no. 9 give

$$(1 + \lambda_j') + \lambda_i' - \lambda_k' = (1 + \mu_0) + \lambda_0 - (1 - \nu_0) < 1,$$

or

$$(18) \qquad \lambda_k' > \lambda_i' + \lambda_j',$$


Section I
λ''i, λ''j, λ''k =
1.
Ek
Ej
Ei
λo, μo, νo.
4.
Ej
Ek
Ei
λo, 1- μo, 1- νo
7.
Ek
Ej
Ei
λo, 1+μo, 1- νo
10.
Ei
Ek
Ej
2- λo, μo, νo.
13.
Ej
Ei
Ek
2- λo, 1- μo, 1- νo
Section II
λ''i, λ''j, λ''k =
2.
Ej
Ek
Ei
λo, μo, νo
5
Ei = ∞
Ei = ∞
Ej
Ek
λo, 1- μo, 1- νo
8.
Ek = ∞
Ej
Ei
λo, 1+μo, 1- νo
11.
Ek = ∞
Ej = ∞
Ei
2- λo, μo, νo
14.
Ej
Ek
Ei
2- λo, 1- μo, 1- νo
Section III
λ''i, λ''j, λ''k =
3.
Ei
Ej
Ek
λo, μo, νo.
6
Ej
Ek
Ei
λo, 1- μo, 1- νo.
9.
Ek
Ej
Ei
λo, 1+μo, 1- νo
12
Ei
Ej
Ek
2- λo, μo, νo
15.
Ej
Ei
Ek
2- λo, 1- μo, 1- νo.

in which *one of the two subscripts on the right hand side refers to the obtuse angle.* Similarly for triangle 12 we obtain

(19) $$\lambda'_i > \lambda'_j + \lambda'_k,$$

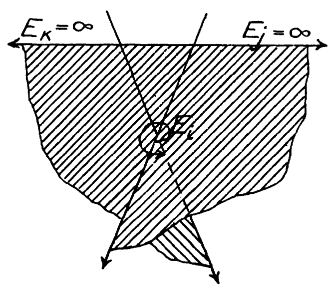
n which the subscript i refers to the obtuse angle. It can be easily verified that these relations hold for no other triangles of the second group. Triangle 15 is distinguished by means of the inequality

(20) $$\lambda'_1 + \lambda'_2 + \lambda'_3 > 2,$$

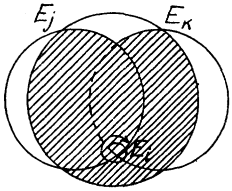
and the characteristic relations for nos. 8, 11, and 14 are obtained by merely replacing the sign $>$ by $=$ in the last three inequalities.

The possibility of making the attachments required by the reductions has yet to be considered. From a glance at the plate it is apparent that a circle can be attached laterally to any side of a reduced triangle with the exception of nos. 11 and 12. In each of these triangles lateral attachment upon $E_j E_k$, the side opposite to the obtuse angle, is impossible owing to the fact that the side returns upon itself. But it can be shown that such attachment is not required by the reductions. For, by (7) and (8), when the first reduction is used, the attachments upon the side opposite to an obtuse angle of the reduced triangle must be polar. On the other hand, when the second reduction is employed, it follows from (14) that E_2 is the vertex of the obtuse angle unless $\lambda'_2 \geqq \lambda'_1 + \lambda'_3$. The reduced triangle would then be of type 11 or 12. But in this exceptional case we so modified the form of the reduction as to make E_3 the vertex of the obtuse angle and thereby avoided the use of the two triangles. The requisite lateral attachments can therefore always be made.

Polar attachment is demanded only by the first reduction and the attachment is then made to a single side. If the reduced triangle has an obtuse angle, the side lies opposite to this angle. An inspection of triangles 7–15 and of 1–4



TRIANGLE 16.



TRIANGLE 17.

shows that in these triangles the attachment is always feasible. In nos. 5 and 6 polar attachment to $E_j E_k$ is impossible, since in no. 5 the boundary of the half plane to be attached would pass through E_i, while in no. 6 the cut would cross the boundary of the circle to be attached. When the form of the reduction leads to these exceptional cases, the construction of the triangle is to be effected as follows. Instead of making the first attachment triangles 16 and 17 are to be substituted. The angle E_i is thereby increased by 2π just as in polar attachment. The remaining $n-1$ polar attachments, as well as the lateral attachments required, may then be made in the usual manner. For convenience of statement we shall hereafter include these two triangles under the term reduced triangle.

§ 7. *On the distribution and number of roots of* $F(\alpha, \beta, \gamma, x)$ *when* $1-\gamma<0$.

All the needful preparation for the determination of the number of roots of $F(\alpha, \beta, \gamma, x)$ in each segment of the real axis and in the imaginary domain has now been made. When $1-\gamma<0$, $F(\alpha, \beta, \gamma, x)$ belongs to the larger exponent of $'e_1$, and its roots are indicated by the passage of the sides and interior across E_1.

The number of real roots in the interval $(0, 1)$ is exactly equal to the number of times E_1E_2 crosses E_1 or overlaps itself. Now the only reduced triangles which contain an overlapping side are nos. 11, 12 and 16. But E_1E_2 will be identical with this side of no. 11 or no. 12 only when $m_3>m_1+m_2$ and then only if M is odd and $\lambda'_3 \geqq \lambda'_2+\lambda'_1$. The corresponding conditions for no. 16 are that M should be even and $1+\lambda'_3=\lambda'_1+\lambda'_2$. In every other case the overlapping of E_1E_2 is due solely to polar attachments upon this side. Each such attachment adds an entire circumference which covers E_1. Now the attachments upon E_1E_2 are polar only if $m_3>m_1+m_2$, and by (7) the number of such attachments (which we before denoted by n) is equal to the integral part of $(m_3-m_2-m_1)/2$. The first attachment, however, is not to be counted when triangle 16 or 17 is employed, that is to say, if $1+\lambda'_3 \geqq \lambda'_1+\lambda'_2$. The form of this condition suggests the introduction of the number*

$$E\left(\frac{\lambda_3-\lambda_1-\lambda_2+1}{2}\right)$$

in place of n, and this will be seen at once to agree exactly with the number of times E_1E_2 overlaps itself.†

Further attention should, perhaps, be called to triangles 11 and 16. In each of these triangles two vertices coincide. If the two are E_1 and E_2 the side E_1E_2 just closes, and there is a root of $F(\alpha, \beta, \gamma, x)$ in $e_2=1$. If this root is included in our enumeration, we reach the following result:

* By $E(q)$ is to be understood a number which is equal to the integral part of q if $q>0$, and which is equal to 0 if $q \leqq 0$.

† We come thus to KLEIN's formula for the number of times any side overlaps itself.

If $1-\gamma<0$, *the number of roots of* $F(\alpha, \beta, \gamma, x)$ *between* 0 *and* 1 *inclusive is* $E\{(\lambda_3-\lambda_1-\lambda_2+1)/2\}$.

From considerations of symmetry it follows immediately that *the number of roots between* 0 *and* $-\infty$ *must be* $E\{(\lambda_2-\lambda_1-\lambda_3+1)/2\}$.

To find the number of imaginary roots we must determine how often the surface of the triangle passes over E_1. Now the interior of a reduced triangle never crosses any of its vertices, and obviously it can only be made to do so by lateral attachment. If, in particular, it crosses E_1, the attachments must be made to the opposite side, E_2E_3.

Before taking up these attachments it will be convenient to dispose first of the case in which $m_1>m_2+m_3$. As the attachments upon E_2E_3 are then polar, we draw at once the following conclusion:

Case 1. *If* $1-\gamma<0$ *and* $m_1>m_2+m_3$, *the number of imaginary roots of* $F(\alpha, \beta, \gamma, x)$ *within the positive half plane is equal to* 0.

We return now to the consideration of lateral attachments upon E_2E_3. When two consecutive lateral attachments are made upon any side of a reduced triangle, an entire plane is added which generally passes over the opposite vertex. The only exceptions are triangles 11 and 16 in which the vertices E_j and E_k coincide. It is therefore impossible to make the surface cross either vertex by lateral attachment. These two triangles can be obtained only if the first reduction is used and then only under the following conditions:

$$m_i>m_j+m_k \quad \begin{cases} M \text{ even}, \ 1+\lambda_i'=\lambda_j'+\lambda_k', \\ M \text{ odd}, \quad \lambda_i' \ =\lambda_j'+\lambda_k'. \end{cases}$$

But these are precisely the conditions which make $(\lambda_i'-\lambda_j'-\lambda_k'+1)/2$ an integer.

The effect of an even number of lateral attachments upon E_2E_3 has thus been ascertained. If the total number is odd, there remains one more attachment to be considered. Suppose first that M is even, and let a single circle be added laterally along a side of the reduced triangle. It will fail to cover the opposite vertex unless attached to E_jE_k in no. 6 or to a side ending in E_i in no. 17. As we are considering only attachments upon E_2E_3, the conditions for the occurrence of these exceptional cases must be

1) $$\lambda_2'+\lambda_3'>1+\lambda_1'$$
and

2) $$m_i>m_1+m_k, \quad \lambda_1'+\lambda_k'>1+\lambda_i'.$$

Suppose next that M is an odd integer. As by hypothesis $m_1\leqq m_2+m_3$, the vertex of the obtuse angle in the reduced triangle must be either E_2 or E_3. Hence we have only to consider the effect of a lateral attachment upon one of the sides passing through the vertex of the obtuse angle. It will be found that the circle added will pass over the vertex opposite the side of attachment unless it is attached to triangles 11 and 12 or to E_iE_j in 8 or 9. The conditions for the occurrence of these exceptional cases are

1) $$m_i > m_j + m_1, \quad \lambda_i' \geqq \lambda_j' + \lambda_1',$$
and
2) $$\lambda_1' \geqq \lambda_2' + \lambda_3'.$$

The effect of the attachments upon E_2E_3 has now been completely determined. It remains only to ascertain their number and to apply our results. Three cases must be distinguished according to the number of attachments made.

Case 2: $m_2 > m_3 + m_1$. The form of reduction (7) shows that their number is

$$m_3 \equiv \frac{m_2 + m_3 - m_1 + 1}{2} - \frac{m_2 - m_3 - m_1 + 1}{2}.$$

Except in the special cases which have just been singled out the number of imaginary roots within the half plane will be $E(m_3/2)$ or $E\{(m_3 + 1)/2\}$ according as M is even or odd. The form of the conditions for the existence of the exceptional cases suggests, however, the introduction of the number

$$U = E\left(\frac{\lambda_2 + \lambda_3 - \lambda_1 + 1}{2}\right) - E\left(\frac{\lambda_2 - \lambda_3 - \lambda_1 + 1}{2}\right),$$

in terms of which our final result can be most simply expressed.

If $1 - \gamma < 0$ *and* $m_2 > m_1 + m_3$, *the number of imaginary roots of* $F(\alpha, \beta, \gamma, x)$ *within the positive half plane of* x *is equal to* $E(U/2)$ *unless* $\lambda_2 - \lambda_1 - \lambda_3$ *is an odd integer when the number is equal to* 0.

Case 3: $m_3 > m_1 + m_2$. The result is the same as in case 2 with the interchange of the subscripts 2 and 3.

Case 4: No one of the integers m_i greater than the sum of the other two. The number of lateral attachments upon E_2E_3 is $E\{(m_2 + m_3 - m_1)/2\}$, and we obtain at once the following result:

If $1 - \gamma < 0$ *and each of the integers* m_1, m_2, m_3 *is equal to or less than the sum of the other two, the number of imaginary roots within the positive half plane is* $E(V/2)$, *where*

$$V = E\left(\frac{\lambda_2 + \lambda_3 - \lambda_1 + 1}{2}\right).$$

It is interesting to note how the changes in the number of imaginary roots take place when $\lambda_1, \lambda_2, \lambda_3$ are continuously varied. Since the roots of $F(\alpha, \beta, \gamma, x)$ are symmetrically situated with respect to the real axis and since also a multiple root of any solution of the differential equation must coincide with a singular point, the change can conceivably take place in just two ways. Either a number of roots of $F(\alpha, \beta, \gamma, x)$ unite for an instant with a singular point and then separate and distribute themselves differently between the real and imaginary domains, or a root of a second branch of the function we are considering must cross the cut $e_2 e_3$ and thus become a root of the branch $F(\alpha, \beta, \gamma, x)$.

Consider the first alternative. When $x = 0$, $F(\alpha, \beta, \gamma, x) = 1$. It is impossible therefore for roots of $F(\alpha, \beta, \gamma, x)$ to unite with e_1 so long as this symbol continues to have a meaning. We shall not consider here the changes which ensue when γ passes through a negative integral value. The union of the roots with e_2 or e_3 is shown by the coincidence of E_1 with E_2 and E_3 respectively. But, as we have seen, the number of imaginary roots is then 0. For an instant they are all absorbed into the singular point. It is possible also for two real roots to unite simultaneously with the same point, one being taken from each of the two segments which terminate in the point. Hence when the roots separate again, the number of imaginary roots in each half plane may be increased by a unit.

When the change takes place in the second manner and a root crosses the cut, $E_2 E_3$ for the moment passes through E_1. The three sides of the reduced triangle then meet in a common point and it accordingly belongs to the second section of the plate. Now the only triangles of this section in which it is possible to make a side pass completely through the opposite vertex by lateral attachment are nos. 5 and 8. This happens when an odd number of attachments is made upon $E_j E_k$ and $E_i E_j$ respectively. If we impose the condition that $E_2 E_3$ shall be this side, we obtain the following results:

If $1 - \gamma < 0$, *the number of real roots of* $F(\alpha, \beta, \gamma, x)$ *included between* 1 *and* ∞ *is equal to* 0 *unless* $(\lambda_2 + \lambda_3 - \lambda_1 + 1)/2$ *is a positive integer. Then if* m_3 *in case* 2, m_2 *in case* 3, *or* $E\{(m_2 + m_3 - m_1)/2\}$ *in case* 4 *is an odd integer, there will be a single root between* 1 *and* ∞, *and in no other case.*

§ 8. *On the number and distribution of the roots when* $1 - \gamma > 0$.

If $1 - \gamma > 0$, the roots of $F(\alpha, \beta, \gamma, x)$ are indicated by the passage of the sides and interior of the triangle across E_1', the second intersection of the sides $E_1 E_2$ and $E_1 E_3$. We will determine first the number of real roots, ascertaining for this purpose the number of times which $E_1 E_2$ passes over E_1'.

Case 1: $m_1 > m_2 + m_3$.* When M is odd, the vertex of the obtuse angle is

* The same four cases are distinguished here as in the article by HURWITZ, but the number of roots is here expressed in terms of the exponent differences, while HURWITZ gives it in terms of α, β, γ. The change to the latter form is easily made.

E_1. The point E_1' is contained within $E_1 E_2$ only if the triangle is of type 6, 9, 12 or 17, and one of the following sets of conditions must then hold:

(1) M is even and either $\lambda_1' + \lambda_3' > 1 + \lambda_2'$ or $\lambda_2' + \lambda_3' > 1 + \lambda_1'$;

(2) M is odd and either $\lambda_1' > \lambda_2' + \lambda_3'$ or $\lambda_2' > \lambda_1' + \lambda_3'$.

The lateral attachments upon $E_1 E_2$ have no effect upon its position if their number be even, but if their number be odd, it must be replaced by the complementary arc. Now when one of the two complementary arcs contains E_1', the other will not. Exceptions arise only from the coincidence of E_1' and E_2, when both arcs terminate in E_1'. Such a coincidence occurs in figures 8 and 5 and then only if $\lambda_1' + \lambda_3' = \lambda_2'$ and $1 + \lambda_2' = \lambda_1' + \lambda_3'$ respectively. The total number of lateral attachments upon $E_1 E_2$ is

$$m_2 \equiv m_1 - \frac{m_1 + m_3 - m_2 + 1}{2} - \frac{m_1 - m_2 - m_3 - 1}{2},$$

and the final result in each triangle depends upon whether this number is even or odd. Taking proper account of the exceptions noted, we obtain the following result:

If $1 - \gamma > 0$ *and* $m_1 > m_2 + m_3$, *the number of roots of* $F(a, \beta, \gamma, x)$ *between* 0 *and* 1 *will be* 0 *or* 1 *according as*

$$(21) \qquad X \equiv E(\lambda_1) - E\left(\frac{\lambda_1 + \lambda_3 - \lambda_2 + 1}{2}\right) - E\left(\frac{\lambda_1 - \lambda_2 - \lambda_3 + 1}{2}\right)$$

is even or odd, unless $(\lambda_1 + \lambda_3 - \lambda_2 + 1)/2$ *is an integer. In this special case there is a single root, in the interval, which coincides with* $x = 1$.

Case 2: $m_2 > m_1 + m_3$. If there is an obtuse angle, its vertex is E_2. Then $E_1 E_2$ can not contain E_1'. As also the number of lateral attachments on this side is m_1, we conclude at once that

If $1 - \gamma > 0$ *and* $m_2 > m_1 + m_3$, *the number of roots between* 0 *and* 1 *will be either* 0 *or* 1 *according as* $E(\lambda_1)$ *is even or odd.*

Case 3: $m_3 > m_1 + m_2$. The attachments upon $E_1 E_2$ are polar, and their number is equal to the integral part of

$$\frac{m_3 - m_2 - m_1}{2} = \frac{m_1 + m_3 - m_2}{2} - m_1.$$

Each adds a circumference containing E_1'. If M is odd, $E_1 E_2$ lies opposite to the obtuse angle and contains E_1' unless $\lambda_2' > \lambda_1' + \lambda_3'$. If M is even, this point is included only if $\lambda_1' + \lambda_3' > 1 + \lambda_2'$ or $\lambda_1' + \lambda_2' \geqq 1 + \lambda_3'$. In the latter case the substitution of figure 17 for figure 6 takes the place of the first attachment. The final conclusion is as follows:

If $1-\gamma>0$ *and* $m_3>m_1+m_2$, *the number of roots of* $F(\alpha,\beta,\gamma,x)$ *in the interval* $(0,1)$ *is*

$$E\left(\frac{\lambda_1+\lambda_3-\lambda_2+1}{2}\right)-E(\lambda_1);$$

one of them coincides with $x=1$ *if* $\lambda_1+\lambda_3-\lambda_2$ *is an odd integer.*

We give without further discussion the result for

Case 4: *If* $1-\gamma>0$ *and if no one of the integers* m_1, m_2, m_3 *is greater than the sum of the other two, the number of roots in the interval* $(0,1)$ *will be either* 0 *or* 1 *according as*

$$E(\lambda_1)-E\left(\frac{\lambda_1+\lambda_3-\lambda_2+1}{2}\right)$$

is even or odd, unless $\lambda_1+\lambda_3-\lambda_2$ *is an odd integer. In the latter case there is a single root which coincides with* $x=1$.

We proceed next to determine the number of imaginary roots, observing for this purpose how often the interior of the triangle passes across E_1'. The only reduced triangle which can contain the point E_1' in its interior is no. 15, and E_1 must then be the vertex of the obtuse angle. This holds in case 1. Furthermore, this case is the only one in which the surface of the triangle can be made to include E_1' by polar attachment. On this account we shall postpone its consideration to the last.

In the remaining cases we have only to trace the effect of the lateral attachments. Each pair of consecutive attachments to a side adds an entire plane which necessarily contains E_1'. If the number of attachments is odd, there remains one more attachment to be taken account of. Suppose first that this is upon E_1E_2. Then if E_1' was originally contained within this side, it becomes an interior point in consequence of the attachment. Now we have already determined, in studying the number of real roots, under what conditions E_1' will be contained in E_1E_2 in the reduced triangle. The result applies with change of subscript to E_1E_3.

The effect of a single attachment upon E_2E_3 remains to be considered. If M is even, the circle attached will cover E_1' unless $\lambda_1'+\lambda_2'+\lambda_3'\leqq 1$ (triangle 3) or $\lambda_1'+\lambda_k'\geqq 1+\lambda_i'$. When M is odd, the side necessarily passes through the vertex of the obtuse angle (E_2 or E_3 being its vertex), and it will be seen that E_1' is made an interior point by the attachment only if $\lambda_1'+\lambda_2'+\lambda_3'>2$ (no. 15) or if, when E_j is the vertex of the obtuse angle, $\lambda_k'>\lambda_1'+\lambda_j'$ (no. 9).

We are now prepared for the consideration of case 2. If the number of attachments upon E_2E_3 is written in the form

$$\frac{m_1+m_2+m_3+1}{2}-\frac{m_1+m_2-m_3+1}{2},$$

we are led to express the final result as follows:

Case 2: *If* $1-\gamma>0$ *and* $m_2>m_1+m_3$, *the number of imaginary roots of* $F(\alpha, \beta, \gamma, x)$ *within the positive half plane of* x *is* $E(\lambda_1/2)+E(q/2)$ *in which*

$$q=E\left(\frac{\lambda_1+\lambda_2+\lambda_3+1}{2}\right)-E\left(\frac{\lambda_1+\lambda_2-\lambda_3+1}{2}\right)$$

unless $\lambda_1+\lambda_2-\lambda_3$ *is an odd integer, when the number is* $E(\lambda_1/2)$.

It is evident also that in

Case 3: $1-\gamma>0$, $m_3>m_1+m_2$. *The same result holds after the interchange of the subscripts* 2 *and* 3.

In case 4 there are three sets of lateral attachments to be taken account of. If M is odd, one of these is upon the side opposite to the obtuse angle. Now this side is E_1E_3 unless $\lambda'_2>\lambda'_1+\lambda'_3$ when the triangle is of type 9 and E_1E_3 coincides with E_iE_j. It follows that if M is odd, E'_1 is contained in E_1E_3 unless $\lambda'_2>\lambda'_1+\lambda'_3$ or $\lambda'_3>\lambda'_1+\lambda'_2$. On the other hand, when M is even, this point is excluded unless $\lambda'_1+\lambda'_2>1+\lambda'_3$. These exceptions suggest that the introduction of

$$S=E(\lambda_1)-E\left(\frac{\lambda_1+\lambda_2-\lambda_3+1}{2}\right) \tag{22}$$

in place of a_2, the number of lateral attachments upon E_1E_3. For a corresponding reason we shall express the number of attachments upon E_2E_3 in the form:

$$a_1=E\left(\frac{m_1+m_2+m_3+1}{2}\right)-E\left(\frac{m_1-m_2+m_3+1}{2}\right)-E\left(\frac{m_1+m_2-m_3+1}{2}\right).$$

Finally, to simplify the result, we shall introduce analogous expressions in terms of the exponent differences. The simplest form for the result which I have been able to find is the following:

If $1-\gamma>0$ *and if no one of the integers* m_1, m_2, m_3 *is greater than the sum of the other two, the number of roots of* $F(\alpha, \beta, \gamma, x)$ *within the positive half plane is equal to*

$$E(W/2)+E(S/2)+E(T/2)+\epsilon',$$

in which S *is defined by equation* (22), T *is a like expression with the subscripts* 2 *and* 3 *interchanged, and*

$$W=E\left(\frac{\lambda_1+\lambda_2+\lambda_3+1}{2}\right)-E\left(\frac{\lambda_1-\lambda_2+\lambda_3+1}{2}\right)-E\left(\frac{\lambda_1+\lambda_2-\lambda_3+1}{2}\right),$$

while $\epsilon=1$ *or* 0, *the former value being taken unless* M *is even and simultaneously neither* $\lambda'_1+\lambda'_2>1+\lambda'_3$ *nor* $\lambda'_1+\lambda'_3>1+\lambda'_2$, *or unless* M *is odd and either* $\lambda'_3>\lambda'_1+\lambda'_2$ *or* $\lambda'_2>\lambda'_1+\lambda'_3$.

We return now to the case in which $m_1 > m_2 + m_3$. The interior of the reduced triangle will contain E'_1 only if $\lambda'_1 + \lambda'_2 + \lambda'_3 > 2$. Each circle added by polar attachment to E_2E_3 will cover E'_1 if this point lies on the same side of E_2E_3 as does E_1, or, in other words, if the reduced triangle belongs to the third section of the plate. The total number of polar attachments is

$$E\{(m_1 - m_2 - m_3)/2\},$$

but it should be remembered that if $\lambda'_2 + \lambda'_3 > 1 + \lambda'_1$ and M is even, triangle 17 is to be substituted for no. 6, and this substitution takes the place of the first polar attachment. The effect of the lateral attachments depends upon the number X which we introduced in considering the number of real roots in case 1, but in certain cases this number should be replaced by $X + 2$. If we introduce a similar number to correspond to the side E_1E_3, we arrive at the following result:

Case 1: *If* $1 - \gamma > 0$ *and* $m_1 > m_2 + m_3$, *the number of imaginary roots of* $F(\alpha, \beta, \gamma, x)$ *within the positive half plane is equal, in general, to*

$$E(X/2) + E(Y/2),$$

in which X *is defined by equation* (22), *and* Y *is a similar expression with the subscripts* 2 *and* 3 *interchanged. If, however, any one of the following sets of conditions is fulfilled*

$$(23)\begin{cases} 1) \quad m_1 + m_2 + m_3 \text{ even and } \lambda'_1 + \lambda'_2 + \lambda'_3 < 1 \text{ or } \lambda'_i + \lambda'_j > 1 + \lambda'_k \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (k = 1, 2, \text{ or } 3), \\ 2) \quad m_1 + m_2 + m_3 \text{ odd and } \lambda'_1 + \lambda'_2 + \lambda'_3 > 2 \text{ or } \lambda'_k > \lambda'_i + \lambda'_j. \end{cases}$$

the number of such imaginary roots must be increased by

$$E\left(\frac{\lambda_1 - \lambda_2 - \lambda_3 + 1}{2}\right) + \epsilon,$$

in which ϵ *is* 0 *unless either* M *is even and* $\lambda - \lambda'_i - \lambda'_j + 1$ $(i = 2, 3)$, *or* M *is odd and either* $\lambda'_1 > \lambda'_2 + \lambda'_3$, *or* $\lambda'_1 + \lambda'_2 + \lambda'_3 > 2$, *when its value is* 1.

One interesting remark may be made concerning the number of real roots between $e_2 = 1$ and $e_3 = \infty$. If in (23) the sign of inequality is replaced by the sign of equality, we shall have the conditions that the triangle shall belong to section II of the plate. The side E_2E_3 passes through E'_1, and the number of real roots between 1 and ∞ is then usually $E\{(\lambda_1 - \lambda_2 - \lambda_3 + 1)/2\}$. This shows that the additional imaginary roots, noted just above, enter the half plane by crossing the cut simultaneously. This is the only case in which the number of roots between 1 and ∞ ever exceeds 1.

We leave to the reader all further consideration of the transitional cases which arise when the triangle belongs to the second section of the plate.

In conclusion, it may be pointed out that the number of roots of $F_1(a, \beta, \gamma, x)$ can be obtained by interchanging the conditions $1 - \gamma > 0$ and $1 - \gamma < 0$. The number of imaginary roots in the entire plane is, of course, double the number in the half plane.

WESLEYAN UNIVERSITY, MIDDLETOWN, CONN.