On Certain Differential Equations of the Second Order Allied to Hermite's Equation.

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Hermite's differential equation

$$\frac{d^2y}{du^2} = [n(n+1)p(u) + h]\,y$$

can be thrown by the substitution

$$x = p(u) \text{ or } u = \int \frac{dx}{2\sqrt{f(x) = (x - e_1)(x - e_2)(x - e_3)}}$$

into the form

$$f(x)\frac{d^2y}{dx^2} + \frac{f'(x)}{2}\,\frac{dy}{dx} - \frac{n(n+1)x + h}{4}\,y = 0.$$

As is well known, it admits of two solutions whose product is a polynomial in x. Other differential equations of the second order which have the same or an analogous property have been given by Fuchs,* Brioschi,† Markoff,‡ Lindemann,§ and G. W. Hill.‖ Markoff confines his attention to the hypergeometric equation, Fuchs and Brioschi to differential equations in which the coefficient of $\frac{dy}{dx}$ is one-half the derivative of the coefficient of $\frac{d^2y}{dx^2}$. Lindemann, in his discussion of the "differential equation of the functions of the elliptic cylinder," a limiting form of Hermite's equation, proves that it admits of two solutions whose product is a holomorphic function. Hill's equation is an extension of this equation, and possesses the same property.

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† Annali di Matematica, Ser. II, t. IX, p. 11.
‡ Math. Ann., Bd. 28.
§ Math. Ann., Bd. 22.
‖ Acta Mathematica, Bd. 8.

The object of the first section of this paper is to determine in general what regular differential equations of the second order admit of two solutions whose product is a polynomial. It will be found that there are several distinct classes of such equations under which those hitherto considered are comprised as special cases. Incidentally we shall obtain a class of irregular equations with three singular points, which includes the equations of Lindemann and of Hill.

The properties of the two solutions and of their quotient η will be developed in the second section. In particular, it will be shown that the monodromic group of substitutions of η can be thrown into the form

$$\bar{\eta} = \frac{\alpha}{\eta}, \ \bar{\eta} = \beta\eta,$$

and that, conversely, if the group of any regular differential equation can be thus expressed, there will be two solutions whose product is a polynomial multiplied by certain factors which correspond to the singular points and can be removed by an elementary substitution. So far as I am aware, the identity of these two classes of equations has not been hitherto noted. The other properties developed are for the most part extensions of properties given by Hermite and Klein for Hermite's equation, but to effect the generalization a new method is employed which is independent of elliptic integrals. The third section of the paper is devoted chiefly to an investigation of the position of the real roots of the polynomial product with reference to the singular points, when these points are real and their number is limited to four. Klein's investigation* for Hermite's equation here also paves the way, but the "Oscillation theorem" upon which it is based is inadequate to the more general discussion, and recourse is had to the method of conformal representation.

I.

§1. Any regular linear differential equation of the second order with a singular point at ∞ may be written in the form

$$\frac{d^2y}{dx^2} + \sum_{i=1}^{i=r} \left(\frac{1-\lambda_i'-\lambda_i''}{x-e_i}\right) \frac{dy}{dx} + \left(\frac{\lambda_\infty'\lambda_\infty''-\Sigma\lambda_i'\lambda_i''x^{r-2}+a_1x^{r-3}+\ldots+a_{r-2}}{\Pi(x-e_i)} + \sum \frac{\lambda_i'\lambda_i''}{(x-e_i)^2}\right) y = 0, \quad [1]$$

* Math. Ann., Bd. 40.

where $$\Sigma(\lambda_i' + \lambda_i'') + \lambda_\infty' + \lambda_\infty'' = r - 1. \qquad [2]$$

The singular points e_i will here be supposed to be given, but the "*accessory parameters*" $a_1, \ldots, a_{r-2}$ and the exponents λ_i', λ_i'' are to be so determined that the product of two particular integrals shall be a polynomial P_n of the n^{th} degree. The two fundamental integrals for e_i have in general the form

$$\left.\begin{aligned} P^{\lambda_i} &= [\pm (x - e_i)^{\lambda_i'}][1 + B(x - e_i) + C(x - e_i)^2 + \ldots] \\ P^{\lambda_i''} &= [\pm (x - e_i)^{\lambda_i''}][1 + B'(x - e_i) + C'(x - e_i)^2 + \ldots] \end{aligned}\right\}, \qquad [3]$$

the leading coefficient in each series for convenience being taken equal to unity. When, however, the difference of the two exponents is an integer, one of these integrals must in general be modified by the introduction of a logarithmic term. In the first factor of each expansion a definite sign is to be attached to the binomial, but for the present it is immaterial which sign is selected. The corresponding expansions for the singular point ∞ are

$$P^{\lambda_\infty'} = \left[\pm \left(\frac{1}{x}\right)^{\lambda_\infty'}\right]\left[1 + \frac{B}{x} + \frac{C}{x^2} + \ldots\right]$$

and a similar series for P^{λ} .

The foregoing expansions hold only over a limited portion of the x-plane. When, however, the product of two solutions is a polynomial, the integration of the equation can be effected by familiar methods, and its general integral will be expressed in terms of two particular integrals which hold over the entire plane. Two cases are possible, according as the two solutions forming the polynomial product are identical or distinct. In either case the polynomial itself satisfies the differential equation

$$\frac{d^3y}{dx^3} + 3p\frac{d^2y}{dx^2} + \left(\frac{dp}{dx} + 2p^2 + 4q\right)\frac{dy}{dx} + \left(4pq + 2\frac{dq}{dx}\right)y = 0,$$

where p and q denote the coefficients of $\frac{dy}{dx}$ and of y in [1], and it is obtained by substituting for y in this equation a polynomial of the n^{th} degree with unknown coefficients. When the two solutions are identical, their common value y_1 is the square root of the polynomial. A second integral can be obtained by means of the well-known relation

$$y_1 y_2' - y_1' y_2 = C \prod_{i=1}^{i=r} (x - e_i)^{\lambda_i' + \lambda_i'' - 1}, \qquad [4]$$

which exists between any two independent integrals of the equation. This gives for the quotient of the two integrals

$$\eta = \frac{y_2}{y_1} = \int \frac{C\,dx}{y_1^2 \Pi (x - e_i)^{1 - \lambda_i' - \lambda_i''}}. \qquad [5]$$

In the second case, if y_1, y_2 represent the distinct solutions, differentiating the equation $y_1 y_2 = P_n$ and combining with [3], we find

$$\left. \begin{aligned} y_1 &= C' \sqrt{P_n}\, e^{\,C\int \frac{dx}{P_n . \Pi (x - e_i)^{1 - \lambda_i' - \lambda_i''}}} \\ y_2 &= C'' \sqrt{P_n}\, e^{-C\int \frac{dx}{P_n . \Pi (x - e_i)^{1 - \lambda_i' - \lambda_i''}}} \end{aligned} \right\}, \qquad [6]$$

where C, C' and C'' are constants. These formulæ hold equally well when for P_n a holomorphic function can be substituted.

§2. We proceed now to determine the conditions under which the square of a single solution y_1 can be a polynomial of the n^{th} degree. Let y_1 at any singular point in the finite plane be expressed as $aP^{\lambda_i'} + bP^{\lambda_i''}$. Since the expansion of its square into a series is to begin either with a constant or with a positive integral power of $x - e_i$, the exponents λ_i' and λ_i'' must be restricted in value. If neither a nor b is zero, both exponents must be positive integers (including zero) or each must be the half of an odd positive integer. If, on the other hand, either a or b is zero, y_1 is one of the fundamental integrals for e_i, and only the single exponent which belongs to this integral is thus restricted. It is necessary, therefore, that at least one exponent of each singular point in the finite plane, say λ_i'', shall be equal to the half of a non-negative integer. Also, since the square of y_1 is a polynomial of the n^{th} degree, one of the two exponents for infinity, say λ_∞'', must be equal to $-\frac{n}{2}$. The proposed solution can therefore now be expressed in the form $\Pi (x - e_i)^{\lambda_i''} Y$, where Y is a polynomial whose degree is $n' = \frac{n}{2} - \Sigma\lambda_i'' = -(\lambda_\infty'' + \Sigma\lambda_i'')$. The substitution of this in [1] gives as the differential equation for Y

$$\frac{d^2 Y}{dx^2} + \Sigma \frac{1 - \lambda_i}{x - e_i} \frac{dY}{dx} + \frac{(\lambda_\infty' + \Sigma\lambda_i'')(\lambda_\infty'' + \Sigma\lambda_i'')\, x^{r-2} + A'x^{r-3} + B'x^{r-3} + \ldots}{\Pi (x - e_i)} Y = 0, \qquad [7]$$

where λ_i is the exponent-difference $\lambda_i' - \lambda_i''$. This equation has been shown by Heine* to admit of a polynomial solution of degree n', provided the parameters A', B', ... are properly determined, and the number of such determinations for any given set of exponent-differences λ_i is

$$(n', r-1) = \frac{(n'+1)(n'+2)\ldots\ldots(n'+r-2)}{1. \quad 2. \quad \ldots\ldots \quad (r-2)} .\dagger \qquad [8]$$

We conclude therefore that *the differential equation* [1] *will admit of a particular solution whose square is a polynomial of the* n^{th} *degree only when the exponents satisfy the following conditions:*

(1). *One exponent* λ_i'' *of each singular point in the finite plane must be half of a non-negative integer.*

(2). $\frac{n}{2} - \Sigma\lambda_i''$ *must be a non-negative integer* n'.

(3). *One exponent of the singular point at infinity must be equal to* $-\frac{n}{2}$.

When any set of exponents is given which conform to these conditions, the number of such equations will be $(n', r-1)$.

It will be noticed that when neither a nor b is zero, the exponent-difference λ_i must be an integer. The logarithmic term, which ordinarily appears in the expansion of $P^{\lambda_i'}$ or $P^{\lambda_i''}$ when this is the case, must necessarily be eliminated by the conditions imposed upon the accessory parameters; that is, e_i is an *apparent singular point*. Furthermore, since neither exponent is negative, it follows that e_i cannot be an infinity of any solution of [1]. Hence the product of any two solutions will be holomorphic in the vicinity of the point.

§3. The simplest application of this result is to the differential equation for the hypergeometric series $F(\alpha, \beta, \gamma, x)$. The exponents for this equation are $\begin{pmatrix} 0 & \infty & 1 \\ 1-\gamma & \alpha & \gamma-\alpha-\beta \\ 0 & \beta & 0 \end{pmatrix}$. If, therefore, n is even, the sufficient condition is that α or β shall be equal to $-\frac{n}{2}$; if n is odd, not only must α or β be equal to $-\frac{n}{2}$, but either $1-\gamma$ or $\gamma-\alpha-\beta$ must be the half of an odd positive

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* Berliner Monatsberichte, 1864, or Handbuch der Kugelfunctionen, Bd. I, s. 473.

† If $r=2$, this number is unity.

integer not greater than $\frac{n}{2}$. These results comprise four of the six cases given by Markoff in which the product of two solutions of the equation is a polynomial of the n^{th} degree. In two of these four cases he fails, however, to notice that the polynomial is the square of a single solution.

§4. We have now to consider the conditions under which the product of two distinct solutions will be a polynomial. Let the requirement be first made that it shall be finite and one-valued. In the vicinity of e_i it will have the form

$$y_1 y_2 = a\,(P^{\lambda_i'})^2 + b\,(P^{\lambda_i''})^2 + cP^{\lambda_i'}P^{\lambda_i''}.$$

If neither a nor b nor c is zero, it can be argued in the same manner as before, that the exponents are both non-negative integers or are each the half of an odd positive integer, and that e_i is again an apparent singular point, in the vicinity of which every product of two integrals is holomorphic. The same conclusion holds if either a or b singly is zero. If c is zero, the only condition is that the two exponents are each the half of a non-negative integer. Hence unless e_i is again an apparent singular point, one exponent must be half of an odd positive integer and the other a non-negative integer. Finally, if a and b are both zero, $\lambda_i' + \lambda_i''$ shall be a non-negative integer. *Setting aside the apparent singular points, we have then some such scheme as*

$$\left(\begin{matrix} e_1 & e_2 & e_3 & \ldots\ldots e_r \\ \frac{1}{2} + m_1' & \frac{1}{2} + m_2' & \lambda_3' + \lambda_3'' = m_3 & \ldots\ldots \\ m_1'' & m_2'' & & \ldots\ldots \end{matrix}\right)$$

for the exponents of the singular points in the finite plane, the m being zero or positive integers.

Such a scheme suffices to ensure at each of the points separately the existence of a one-valued finite product which has either the form $a\,(P^{\lambda_i'})^2 + b\,(P^{\lambda_i''})^2$ or $cP^{\lambda_i'}P^{\lambda_i''}$. We have next to learn under what conditions the product of two integrals will be one-valued when x makes a circuit around two singular points. Let e_1 and e_2 be two singular points whose circles of convergence overlap, and suppose also their exponents to have the values written down in the above scheme. Place

$$\left.\begin{aligned} P^{\lambda_1'} &= \alpha P^{\lambda_2'} + \beta P^{\lambda_2''}, \\ P^{\lambda_1''} &= \gamma P^{\lambda_2'} + \delta P^{\lambda_2''}. \end{aligned}\right\} \qquad [9]$$

In the vicinity of e_1 the product can be expressed as

$$a_1(P^{\lambda_1'})^2 + b_1(P^{\lambda_1''})^2 + c_1 P^{\lambda_1'} P^{\lambda_1''},$$

in the vicinity of e_2 as

$$(a_1\alpha^2 + b_1\gamma^2 + c_1\alpha\gamma)(P^{\lambda_2'})^2 + (a_1\beta^2 + b_1\delta^2 + c_1\beta\delta)(P^{\lambda_2''})^2 + (2a_1\alpha\beta + 2b_1\gamma\delta + c_1\alpha\delta + c_1\beta\gamma)\, P^{\lambda_2'} P^{\lambda_2''}. \quad [10]$$

By a circuit about e_1 the sign of c_1 is changed; by one about e_2, the sign of the coefficient of $P^{\lambda_2'} P^{\lambda_2''}$. Comparing [10] with its value after both changes have been made, we obtain as the conditions that the product shall remain unaltered by a circuit around the two points,

$$c_1 = 0, \quad a_1\alpha\beta + b_1\gamma\delta = 0. \quad [11]$$

*There is, therefore, save for a numerical factor, one product of two integrals, and only one, which remains unaltered for a circuit about e_1 and e_2.** In the region common to the two circles of convergence this product can be written in either of the forms

$$a_1(P^{\lambda_1'})^2 + b_1(P^{\lambda_1''})^2, \quad (a_1\alpha^2 + b_1\gamma^2)(P^{\lambda_2'})^2 + (a_1\beta^2 + b_1\delta^2)(P^{\lambda_2''})^2,$$

which shows that *the product is also unaltered for a circuit around e_1 and e_2 separately.*

There remain yet two other possible exponent-schemes for e_1 and e_2 to be examined, namely, $\begin{pmatrix} \frac{1}{2} + m_1' & \lambda_2' + \lambda_2'' = m_2 \\ m_1'' & \end{pmatrix}$ and $(\lambda_1' + \lambda_1'' = m_1,\ \lambda_2' + \lambda_2'' = m_2)$, but in neither case can a one-valued product be obtained without a specialization of the accessory parameters of the differential equation. For, assuming the first case, $a_1(P^{\lambda_1'})^2 + b_1(P^{\lambda_1''})^2$ must in the vicinity of e_2 become equal to $c_2 P^{\lambda_2'} P^{\lambda_2''}$. But if $c_1 = 0$, the coefficients of $(P^{\lambda_2'})^2$ and $(P^{\lambda_2''})^2$ in [10] can vanish only when $\begin{vmatrix} \alpha^2 & \beta^2 \\ \gamma^2 & \delta^2 \end{vmatrix} = 0$, and this imposes a condition upon the parameters of the differential equation. On the second assumption $P^{\lambda_1'}$, $P^{\lambda_1''}$ in the vicinity of e_2 can differ from $P^{\lambda_2'}$, $P^{\lambda_2''}$ only by constant factors, and this involves a two-fold specialization of the parameters.

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* In case the two circles of convergence do not overlap, the reasoning still holds good. The right-hand members of [9] must then be taken to represent what the left-hand members become, when continued analytically along some definite path to the vicinity of e_2.

The conclusions which have been reached for the singular points in the finite plane apply with only slight modifications to the point ∞. When the product of two integrals is here one-valued, either (1) the point is an apparent singular point, and λ'_∞, λ''_∞ are congruent both to $\frac{1}{2}$ or both to 0, mod. 1; or (2) they are congruent to $\frac{1}{2}$ and 0 respectively; or (3) $\lambda'_\infty + \lambda''_\infty$ is an integer. The exponents must be still further restricted if the product is a polynomial of the n^{th} degree. When expanded in series for $x = \infty$, it begins with $\left(\frac{1}{x}\right)^{-n}$. Hence in the first two of the three cases just specified, the exponent which is the smaller algebraically must be $-\frac{n}{2}$, and in the third case the sum of the two exponents must be $-n$.

§5. These considerations suffice for the solution of our problem, when there are three singular points e_1, e_2, ∞. The differential equation then contains no accessory parameter. To obtain a one-valued product we are therefore limited to taking two pairs of exponents which differ by the half of an odd integer. To make this product a polynomial, the exponents must also be so chosen that the product shall be finite in e_1 and e_2 and have at ∞ a pole of the n^{th} order. Accordingly we can take for the exponents either of the two following sets of values, but no others:

$$\text{I}\quad \begin{pmatrix} e_1 & e_2 & \infty \\ \frac{1}{2} + m'_1 & \frac{1}{2} + m'_2 & \lambda'_\infty + \lambda''_\infty = -n \\ m''_1 & m''_2 & \end{pmatrix},$$

$$\text{II}\quad \begin{pmatrix} e_1 & e_2 & \infty \\ \frac{1}{2} + m'_1 & \lambda'_2 + \lambda''_2 = m_2 & \frac{n_\infty}{2} \\ m''_1 & & -\frac{n}{2} \end{pmatrix}, \quad n_\infty > -n$$

the m being positive integers and n_∞ an integer, positive or negative, so chosen as to make in agreement with [2] the sum of the six exponents equal to unity.

The first of these exponent schemes comprises those equations which can be reduced by elementary transformations to the hypergeometric form without destroying the polynomial form of the product. For if $m''_1 = m''_2 = 0$ and $e_1 = 1$, $e_2 = 0$, we have at once the hypergeometric equation. When these constants have other values, an entire linear transformation of the independent variable will reduce e_1, e_2 to 0, 1, and the substitution

$$y = (x - e_1)^{\bar{\lambda}_1} (x - e_2)^{\bar{\lambda}_2} \bar{y}, \qquad [12]$$

in which $\bar{\lambda}_1$, $\bar{\lambda}_2$ denote respectively the smaller of the two exponents at e_1, e_2 will reduce one of the exponents at each of these points to zero. Applying, in particular, the exponent scheme to the differential equation for $F(\alpha, \beta, \gamma, x)$, we see that the product of two distinct solutions of that equation will be a polynomial, when α, β, γ have values in accordance with the following scheme:

$$\begin{pmatrix} 0 & \infty & 1 \\ 1 - \gamma = \frac{1}{2} + m_1 & \alpha + \beta = -n & \gamma - \alpha - \beta = \frac{1}{2} + m_2 \\ 0 & & 0 \end{pmatrix}, \quad m_1 + m_2 = n.$$

This scheme embraces the two cases distinguished in Markoff's investigation, which were not included under §3.

§6. The same line of reasoning may be applied to a differential equation

$$\frac{d^2y}{dx^2} + \left(\frac{1 - \lambda_1' - \lambda_1''}{x - e_1} + \frac{1 - \lambda_2' - \lambda_2''}{x - e_2}\right) \frac{dy}{dx} + \left(\frac{\lambda_1' \lambda_1''}{(x - e_1)^2} + \frac{\lambda_2' \lambda_2''}{(x - e_2)^2} + \frac{A + Bx + Cx^2 + \ldots\ldots}{(x - e_1)(x - e_2)}\right) y = 0$$

with two singular points in the finite plane and an essential singularity at ∞. The product of two solutions will be holomorphic when

$$\lambda_1' = \tfrac{1}{2} + m_1', \quad \lambda_2' = \tfrac{1}{2} + m_2',$$
$$\lambda_1'' = m_1'', \qquad \lambda_2'' = m_2''.$$

To this form both "the differential equation of the functions of the elliptic cylinder"

$$\frac{d^2y}{d\phi^2} = (A \cos^2 \phi + B) y$$

and also the equation

$$\frac{d^2y}{d\phi^2} = (A + B \cos 2\phi + C \cos 4\phi + \ldots\ldots) y$$

which Hill uses in his calculation of the motion of the lunar perigee, "so far as it depends on the mean motions of the sun and moon," can be reduced by the substitution $x = \cos 2\phi$. The resulting finite singular points and exponents are

$$\begin{pmatrix} +1 & -1 \\ \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}.$$

§7. The case in which there are four singular points can be discharged with almost equal rapidity. The sum of the eight exponents is 2 and the differential equation contains one arbitrary parameter. Consider first the following scheme of exponents:

$$\text{III}\begin{pmatrix} e_1 & e_1 & e_3 & \infty \\ m_1' + \frac{1}{2} & m_2' + \frac{1}{2} & m_3' + \frac{1}{2} & \frac{n_\infty}{2} \\ m_1'' & m_2'' & m_3'' & -\frac{n}{2} \end{pmatrix},$$

in which the m and n_∞ have the same significance as before. It has been previously demonstrated that, except for a multiplicative constant, there is one, and only one, product whose value is independent of a circuit about two singular points, and that the same product is independent of a circuit about either separately. Since a circuit about two points is at the same time a circuit around the other two, it follows that there is one, and only one, product which is one-valued over the entire plane. The exponents show that it is everywhere finite except at ∞, where it has a pole of order n. It is therefore a polynomial of the n^{th} degree. Special interest attaches to this case, since no restriction whatever has been placed upon the arbitrary parameters. We shall subsequently see that this is impossible when the number of singular points is greater than four.

The general differential equation given by the foregoing scheme includes Hermite's equation as a special case. To obtain the latter we have only to place $m_1' = m_1'' = \ldots. = m_3'' = 0$ and $n_\infty = n + 1$. As already noticed, the substitution $x = p(u)$ will remove the first derivative from this equation and reduce it to the form

$$\frac{d^2y}{du^2} = [n(n+1)p(u) + h]\, y.$$

A corresponding reduction can be made in the more general equation. First, by a substitution similar to [12], we may reduce the differential equation to one which has an exponent-scheme of the form III but in which one exponent m_i''

of each finite singular point is equal to zero. When this is done, the substitution of the new independent variable

$$u = \tfrac{1}{2}\int \frac{dx}{(x-e_1)^{-m_1'} \ldots (x-e_3)^{-m_3'}\sqrt{(x-e_1)\ldots(x-e_3)}}, \qquad [13]$$

which makes x an elliptic function of u, say $p_1(u)$, will remove the second derivative and reduce the equation to the form

$$\frac{d^2y}{du^2} = \frac{n_\infty n p_1 + B}{(p_1-e_1)^{2m_1'} \ldots (p_1-e_3)^{2m_3'}}.$$

§8. A second group of equations with four singular points can be obtained by combining with two such singular points as occur in III an apparent singular point. Since the sum of the eight exponents is 2, two exponents for the fourth singular point must be chosen whose sum is an integer. According as the apparent singular point is at ∞ or in the finite plane, the exponents will therefore be

$$\text{IV} \quad \begin{pmatrix} m_1'+\frac{1}{2} & m_2'+\frac{1}{2} & & \frac{n_\infty}{2} \\ & & \lambda_3'+\lambda_3''=m_3 & \\ m_1'' & m_2'' & & -\frac{n}{2} \end{pmatrix},$$

$$\text{V} \quad \begin{pmatrix} m_1'+\frac{1}{2} & m_2'+\frac{1}{2} & \frac{m_3'}{2} & \\ & & & \lambda_\infty'+\lambda_\infty''=-n \\ m_1'' & m_2'' & \frac{m_3''}{2} & \end{pmatrix}.$$

The accessory parameters in the differential equation will in either case be determined by the condition that the logarithmic term in the expansions for the apparent singular point must be made to vanish.

With the first of these two exponent schemes a differential equation first given by Brioschi* and later applied by Haentzschel† to the theory of potential is closely connected. Haentzschel's form of the equation is

$$\frac{d^2y}{du^2} = [(m^2-\tfrac{1}{4})p(u) - h]\,y,$$

in which m is an integer equal to Brioschi's $\frac{n-1}{2}$. If we free the equation from doubly periodic coefficients by the substitution $x = p(u)$, it becomes

$$(4x^3 - g_2x - g_3)\frac{d^2y}{dx^2} + (6x^2 - \tfrac{1}{2}g_2)\frac{dy}{dx} - [(m^2-\tfrac{1}{4})x - h]\,y = 0.$$

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* Annali di Matematica, Serie 2, t. 9.

† "Studien über die Reduction der Potentialgleichung," p. 54.

Both writers prove that this equation admits of two integrals whose product is a polynomial multiplied into $\sqrt{x-e_i}$. Brioschi, however, appears to leave h arbitrary, an oversight which is corrected by Haentzschel. The exponent scheme for the equation is $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}(m+\frac{1}{2}) \\ 0 & 0 & 0 & -\frac{1}{2}(m-\frac{1}{2}) \end{pmatrix}$, but by setting $y=\sqrt[4]{x-e_3}\,\bar{y}$ it may be reduced to $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{m+1}{2} \\ 0 & 0 & -\frac{1}{4} & -\frac{m-1}{2} \end{pmatrix}$ and thus brought under IV. Brioschi gives the equation as an instance in which the square of the product of two solutions is a polynomial, but the modification just made shows that the equation does not differ essentially from those which we are here considering. Indeed, more generally, whenever the product of two solutions of a regular differential equation containing any number of singular points is equal to a polynomial multiplied by a product of powers of the binomial $x-e_i$, these factors may be removed and the equation reduced to the form treated in this paper by an appropriate substitution of the form

$$y=\Pi\,(x-e_i)^{a_i}\,\bar{y}. \qquad [14]$$

§9. In the general case, where r, the number of singular points in the finite plane, is greater than 3, the differential equation contains $r-2$ accessory parameters. On these we are at liberty to impose an equal number of conditions in order to secure, if possible, a polynomial product. The consistency of the conditions thus imposed will have its verification in the existence of the polygons hereafter to be introduced in connection with the conformal representation of η, the quotient of two solutions. Consider first the case in which the exponents are

$$\text{I} \quad \begin{pmatrix} \frac{1}{2}+m_1' & \frac{1}{2}+m_2' \ldots\ldots \frac{1}{2}+m_r' & \frac{n_\infty}{2} \\ m_1'' & m_2'' \ \ldots\ldots \frac{1}{2}+m_r'' & -\frac{n}{2} \end{pmatrix}.$$

We have seen that, irrespective of the values of the accessory parameters, there is one product of two integrals which is one-valued for circuits around e_1 and e_2. Let it be required that this product shall be one-valued for circuits around the remaining $r-1$ singular points. If r is even, the exponent difference for the point ∞ is an integer. One condition must consequently be imposed to remove from $P^{\lambda'_\infty}$ or $P^{\lambda''_\infty}$ the logarithmic term which would naturally appear. This leaves $r-2$ singular points, all of the same character, and $r-3$ independent parameters. If r is odd, one exponent for ∞ is an integer and the other is half of an odd integer. Whether then r is even or odd, the singular points which remain for consideration are all of the same character, and their number exceeds by a unit the number of remaining parameters. Of these singular points, two may be disregarded, for it has been shown that when the sum of the exponents of each of the two points is the half of an odd integer, the product of two integrals will be one-valued for circuits around these points, provided it is one-valued for circuits around every other point. The number of singular points left is therefore now one less than the number of parameters. At each of these points let $y_1 y_2$ be expressed in the form $a_i (P^{\lambda'_i})^2 + b_i (P^{\lambda''_i})^2 + c P^{\lambda'_i} P^{\lambda''_i}$. The values of the coefficients here obviously depend upon the accessory parameters of the differential equation. The condition that the product shall be one-valued over the entire plane requires that each coefficient c_i shall vanish. Since this imposes a single condition upon the parameters for each remaining singular point, a one-valued product can be obtained by imposing a total number of conditions which is one less than the number of parameters. When this is effected, the values of the exponents ensure that the product will be a polynomial. *To each set of exponents* I *there belongs therefore a differential equation containing a single arbitrary parameter, for which the product of two particular solutions will be a polynomial.*

This result may be regarded as an extension of one obtained by Brioschi for differential equations in which the coefficient of $\frac{dy}{dx}$ is one-half of the derivative of the coefficient of $\frac{d^2y}{dx^2}$. Such an equation is evidently obtained by placing all the m of scheme I equal to 0.

Similar considerations apply to such exponent schemes as

$$\text{II}\ \begin{pmatrix} \frac{1}{2}+m'_1 \ldots\ldots \frac{1}{2}+m'_r & \\ m''_1 \qquad\qquad m''_r & \lambda'_\infty + \lambda''_\infty = -n \end{pmatrix},$$

$$\text{III}\ \begin{pmatrix} \frac{1}{2}+m'_1 \ldots\ldots \frac{1}{2}+m'_{r-1} & & \frac{n_\infty}{2} \\ & \lambda'_r + \lambda''_r = m & \\ m''_1 \ \ldots\ldots\ \ m''_{r-1} & & -\frac{n}{2} \end{pmatrix}.$$

Since, however, the introduction of a singular point whose exponent-sum is an integer imposes two conditions upon the parameters, the total number of conditions will be equal to the number of accessory parameters. They will, therefore, be completely determined. It follows also that a scheme with more than one pair of such exponents will be in general impossible. It is, however, conceivable that in exceptional cases the conditions imposed at the several singular points might not all be independent. Cases may therefore arise where more than one such pair of exponents is present, as will indeed be obvious later when the conformal representation is considered.

This exhausts the possibilities of our problem except in so far as apparent singular points are introduced instead of those whose exponent-sums are the halves of odd integers. This can be done, since an apparent singular point, like the point it replaces, imposes but a single condition upon the accessory parameters. The number of points whose exponent-sums are the halves of odd integers must not, however, be made less than 2.

II.

§10. To distinguish briefly between the singular points whose exponent-sums are the halves of odd integers and those whose exponent-sums are integers, we will hereafter refer to them respectively as *singular points of the first and second kinds*. When the two solutions are distinct, we can, by a suitable substitution of the form [14], reduce the exponents for a singular point of the first kind to $\frac{1}{2} + m_i$, 0 and those for a singular point of the second kind to $\pm \frac{\lambda_i}{2}$ without destroying the property that the product of the two solutions is a polynomial. In the same manner the exponents for an apparent singular point can be reduced to zero and a positive integer. It becomes then what has been termed a semi-singular point, in the vicinity of which all solutions can be expanded in an ordinary power series. For convenience we will henceforth assume that these reductions have been made for all the singular points. The only effect of the reductions upon the polynomial is to remove from it all the factors $x - e_i$.

§11. At any singular point of the first kind the two solutions can be expressed as follows:

$$y_1 = C(\sqrt{a_i}\, P_i^0 + \sqrt{-b_i}\, P^{m_i+\frac{1}{2}}), \quad y_2 = \frac{1}{C}(\sqrt{a_i}\, P_i^0 - \sqrt{-b_i}\, P^{m_i+\frac{1}{2}}).$$

When x describes a circuit around the point, these will be changed into

$$\bar{y}_1 = C^2 y_2, \quad \bar{y}_2 = \frac{1}{C^2} y_1.$$

The result of a circuit around two such points is therefore to multiply the one solution by a constant ρ, *the other by its reciprocal* $\frac{1}{\rho}$. If, now, only singular points of the first kind are present, a hyperelliptic integral similar to [13] may be introduced as the independent variable in place of x. Since the periods of u are due to circuits of x around pairs of singular points, the proposition last enunciated shows that *there are two solutions of the differential equation, each of which is multiplied only by a constant whenever a period is added to* u. This theorem is well known in the case of Hermite's equation, the two solutions being then ordinary doubly periodic functions of the second class. In Hill's equation* the multiplication results upon the addition of the period 2π to the argument ϕ.

When the circles of convergence of the two singular points overlap, a formula can be given for the computation of ρ. Suppose the two points to be e_1, e_2. By a circuit around these points $\sqrt{a_1}\, P_1^0 \pm \sqrt{-b_1}\, P^{m_1+\frac{1}{2}}$ will be replaced by $(\sqrt{a_1}\alpha \mp \sqrt{-b_1}\gamma)\, P_2^0 - (\sqrt{a_1}\beta \mp \sqrt{-b_1}\delta)\, P^{m_2+\frac{1}{2}}$, or, expressed in terms of P_1^0, $P^{m_1+\frac{1}{2}}$ with the help of equations [9] and [11], by

$$(\sqrt{a_1}\, P_1^0 \pm \sqrt{-b_1}\, P^{m_1+\frac{1}{2}})\left(\frac{\alpha\delta + \beta\gamma \mp 2\sqrt{\alpha\beta\gamma\delta}}{\alpha\delta - \beta\gamma}\right).$$

We have therefore the formula

$$\rho = \frac{\alpha\delta + \beta\gamma \mp 2\sqrt{\alpha\beta\gamma\delta}}{\alpha\delta - \beta\gamma}. \qquad [15]$$

A circuit around an apparent singular point is obviously without effect upon the two solutions. On the other hand, near a singular point of the second kind, each solution is, except for a constant factor, identical with one of the two fundamental integrals, and they will therefore be multiplied, the one by $e^{+2i\pi\lambda_i}$ and the other by $e^{-2i\pi\lambda_i}$, where x describes a circuit around the point. Combining these results with the preceding we obtain the following noteworthy proposition:

*See either Hill's article in the 8th volume of the Acta Mathematica or one by Callandreau, Astronomische Nachrichten, No. 2547.

If the two solutions, whose product is the polynomial, are selected as the bases of the monodromic group of substitutions of the equation, this group will take the form

$$\text{I.}\quad \bar{y}_1 = \sigma y_2, \quad \bar{y}_2 = \frac{y_1}{\sigma},$$

or

$$\text{II.}\quad \bar{y}_1 = \rho y_1, \quad \bar{y}_2 = \frac{y_2}{\rho}.$$

§12. The essential character of the group of a linear equation of the second order is more commonly exhibited by means of the quotient $\eta = \frac{y_1}{y_2}$. For the equation under discussion *the substitutions of* η *have the form*

$$\text{I.}\quad \bar{\eta} = \frac{\sigma^2}{\eta}, \text{ or II.}\quad \bar{\eta} = \rho^2\eta. \qquad [16]$$

In the Autographie of Klein's lectures upon "Linear Differential Equations," 1894, p. 148, a list of 11 cases is given in which the substitutions are simpler than the general substitution $\bar{\eta} = \frac{\alpha\eta + \beta}{\gamma\eta + \delta}$. Most of the differential equations which correspond to these cases are well known, as for instance the equations belonging to the groups of the regular solids. The chief case which has not received a general investigation is that in which the group has the form to which we have just been led by the consideration of the polynomial product. Conversely, if for any regular differential equation *the group of* $\eta = \frac{y_1}{y_2}$ *can be expressed in the form* [16], *the product of the two solutions* y_1, y_2 *must either be a polynomial or a polynomial multiplied by powers of the binomials* $x - e_i$, *and the latter case can evidently be reduced to the former by such a transformation as* [14]. The form of the substitutions of the group shows, in fact, that the product is multiplied by a constant when x describes a loop enclosing one or more singular points. Such a product is expressible as a holomorphic function multiplied into powers of the $x - e_i$ which correspond to the multiplicative constants and to the infinities of the product. Moreover, since the differential equation is supposed regular, the holomorphic factor must have a pole for $x = \infty$, and hence it is a polynomial.

§13. We shall hereafter confine our attention to *real* differential equations, i. e. to those in which all parameters, whether singular points, exponents, or accessory parameters, are real. Subscripts will be assigned to singular points of the first and second kinds according to the order in which they occur on the x-axis, the apparent singular points being, for convenience, omitted. We will now consider some properties of the solutions which relate to the segments into which the axis is thereby divided.

Consider first the four fundamental integrals which belong to the two extremities of any segment. Each integral has been defined by a power-series [3] which holds throughout a portion or the whole of this segment. Since the differential equation is real, the coefficients of each series must be real, and the signs in [3], which till now have been left arbitrary, can be so chosen that the integrals shall be real as long as the series converge. But any solution of the differential equation which is real along a finite portion of the axis, will, if continued analytically, remain real, until the first singular point is reached where an ordinary power-series fails to hold. The four fundamental integrals, when thus continued, will therefore be real throughout the entire segment irrespective of the apparent singular points which it contains. If, now, in [9], whether the circles of convergence of e_1 and e_2 overlap or not, the right-hand members of the equation are taken to represent what the left-hand members become when continued analytically from the vicinity of e_1 to that of e_2, the constants α, β, γ and δ must be real. It follows that ρ in formula [15] is either a real quantity or a complex imaginary with unit modulus according as $\alpha\beta\gamma\delta$ is positive or negative. *The substitutions which result from a circuit around two consecutive singular points* e_{i-1} *and* e_i *of the first kind must therefore be either both hyperbolic or both elliptic.*

Following a precedent set by Klein, we shall apply the terms hyperbolic and elliptic not only to the substitution but to the segment $e_{i-1}e_i$ around which the corresponding circuit is made. Equation [11] shows that the sign of $\alpha\beta\gamma\delta$ will be opposite to that of $\frac{a_1}{b_1}$. Hence in a hyperbolic segment the product can be expressed as $A_i^2(P_i^0)^2 - B_i^2(P^{m_i+\frac{1}{2}})^2$, and in an elliptic segment as $A_i^2(P_i^0)^2 + B_i^2(P^{m_i+\frac{1}{2}})^2$, in both of which A_i and B_i denote real constants. The two component solutions may therefore be so taken as to be real throughout a hyperbolic segment; on the other hand, in an elliptic segment, they will be conjugate imaginaries. A segment, one or both of whose extremities are singular points of the second kind, will here be classed with the hyperbolic segments, since in this segment both solutions can be taken as real. This follows from the fact that in the vicinity of such a point the two solutions differ only by constant factors from the two real fundamental integrals. On the other hand, the segments which terminate in a singular point of the first kind are the one elliptic and the other hyperbolic, because the sign of the second term of $A_i^2(P_i^0)^2 \pm B_i^2(P^{m_i+\frac{1}{2}})^2$ will be changed when x describes a circuit around e_i. The order of succession of the segments *between any singular point of the second kind and the next point of the same kind* is therefore a definite one. *The hyperbolic and elliptic segments alternate with each other, beginning and ending with a hyperbolic segment.* In agreement with this, the number of singular points of the first kind included between two consecutive points of the second kind must be even, as must also be the total number of points of the first kind.

A difference between the two varieties of segments again appears, when the roots of the polynomial are considered. In an elliptic segment the polynomial consists of the sum of two positive terms. Both of these cannot simultaneously vanish at any point of the segment, for if this were possible, two independent solutions of the differential equation would have at this point a common real root, which contradicts a well-known theorem concerning the alternation of the real roots. It follows therefore that *the real roots of the polynomial are situated only in the hyperbolic segments.*

§14. The foregoing theory can be advantageously set forth, and might, indeed, be independently developed, with the aid of the theory of conformal representation. As is well-known, the quotient η of two independent solutions of [1] builds the positive half of the x-plane conformally upon a polygon $E_1 E_2 \dots E_r E_\infty$ whose sides are arcs of circles. The angles at the vertices which correspond to the singular points e are successively equal to $\lambda_1 \pi, \dots, \lambda_\infty \pi$. The conformity of the representation ceases not only at the vertices but also at the points T of the boundary which correspond to the apparent singular points. The latter points will, however, not be here classed with the vertices of the polygon. The angle between the two arcs which meet in such a point is a multiple of π, and, because there is no logarithmic term in the expansion of η at an apparent singular point, the two arcs must be arcs of a common circle. Hence the point is to be regarded as a sort of turning-point (see Fig. 1) where the direction of a side is reversed one or more times.*

* For a further discussion of such points, see my article in the 16th volume of the American Journal.

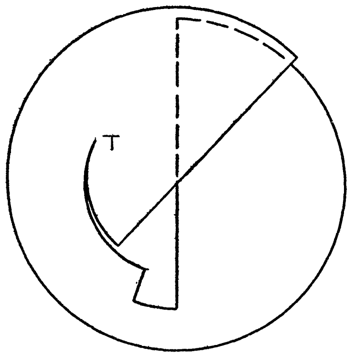

FIG 1

The general shape of the polygon can be determined from the following considerations connected with the substitution-group of η. If the polygon be reflected on any one of its sides, we shall have a new polygon which is the image of the negative half-plane. A reflection of the second polygon upon one of its sides gives a second image of the positive half-plane which is connected with the first by a substitution of the group of η. If we suppose that the first reflection is on the side $E_{i-1}E_i$ and the second upon $E_iE'_{i+1}$ (Fig. 2), the substitution will be due to a circuit around e_i. The invariant points of this substitution will be the intersections of these two sides, produced if necessary, and hence also of the sides $E_{i-1}E_i$ and E_iE_{i+1} of the first polygon. If e_i is a singular point of the second kind, the substitution is of the form (16, II), whose invariant points are $\eta = 0$ and $\eta = \infty$. The two sides $E_{i-1}E_i$ and E_iE_{i+1} are therefore parts of straight lines which meet at the origin [Fig. 2 (a)]. If the singular point is of

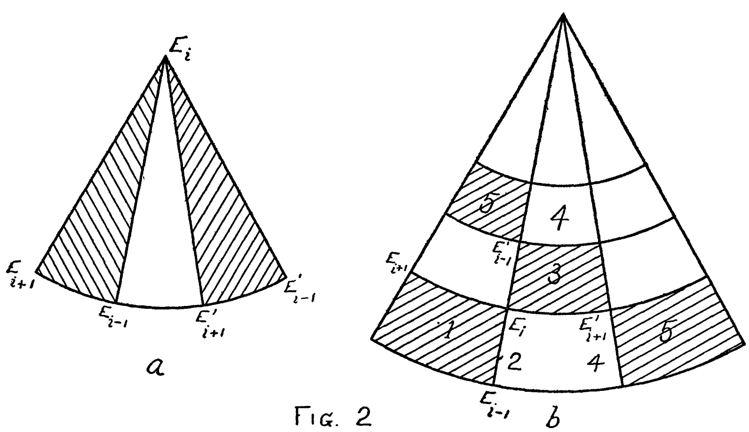

FIG. 2

the first kind, the substitution is of the form (16, I), whose invariant points, $\pm\sigma$, are symmetrically situated with respect to the origin. Since the angle at E_i is $(m_i+\frac{1}{2})\pi$, the two circles of which $E_{i-1}E_i$ and E_iE_{i+1} are arcs which cut each other in these points at right angles. But one of the singular points adjacent to e_i, say e_{i+1}, must likewise be a singular point of the first kind. It follows also that E_iE_{i+1} (see (b) of Fig. 2) must cut a second circle at right angles and in two points which are symmetrically situated with respect to the origin. Evidently therefore E_iE_{i+1} is the arc of a circle whose center is at the origin and $E_{i-1}E_i$ the segment of a straight line which passes through the origin, or *vice versa*. These conclusions concerning the shape of the polygon can be summed up in the following statement:

When the two solutions y_1, y_2 are distinct, the sides of the polygon are arcs of concentric circles and segment of straight lines which cut the circles at right angles.

§15. The methods by which polygons of this character are constructed will be discussed in a later paragraph. In the meantime some of the conclusions already obtained may be easily verified by means of the conformal representation. To a circuit around two consecutive singular points of the first kind corresponds a series of four reflections, as indicated in Fig. 2 (b). These result either in a simple revolution of the initial polygon through an angle ϕ or in increasing the distance of all its points from the origin in the ratio $\rho^2:1$. In other words, the resulting substitution is either elliptic or hyperbolic. Clearly also the straight sides correspond to the hyperbolic and the circular sides to the elliptic segments. The theorem which has been already given concerning the alternation of these two kinds of segments is now immediately evident from an inspection of the figures. Furthermore, the roots of y_1 and y_2 are respectively the zeros and the infinities of their quotient η. Hence we conclude *that if a side E_iE_{i+1} of the polygon passes p times in all through the zero and infinity points of the η-plane, the polynomial has p real roots situated between e_i and e_{i+1}; if also the interior of the polygon includes the zero and infinity points q times in all, q pairs of roots of the polynomial are imaginary.* Since also only the straight sides of the polygon can pass through the origin or infinity, the real roots must lie exclusively in the hyperbolic segments.

§16. The conformal representation also makes apparent the significance of the singular points of the second kind. Should one of the circular sides of a polygon be contracted to a point situated either at the origin or at ∞ (Fig. 3),

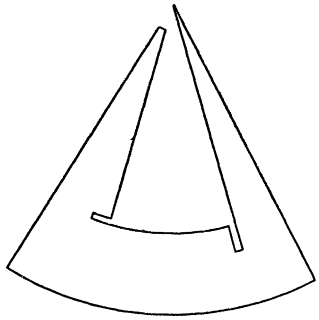

FIG. 3

the union of its extremities would evidently produce a vertex which would correspond to a singular point of the second kind. It is also obvious, conversely, that any such vertex can be regarded as having been formed in this manner. Hence *any differential equation with singular points of the second kind which satisfies the conditions of our problem can be regarded as the limit of an equation containing only singular points of the first kind, each singular point of the second kind being created by the union of two points which terminate an elliptic segment.* Thus, for example, when $m_1 = m_2 = 0$, the hypergeometric equations discussed at the close of §5 are limiting cases of Hermite's equation. It is sometimes possible, also without changing the angles of the polygon, to contract a circular side to a point which does not coincide either with the origin or infinity (see again Fig. 3). In such instances the contraction of an elliptic segment gives rise to an apparent singular point. The result is also the same when it is possible to shrink a hyperbolic segment to a point. From these instances it is clear that the various limiting forms of a given differential equation can be immediately inferred, when the shape of the corresponding polygon is known. In this respect, as in many others, the method of conformal representation has a decided superiority to analytical methods.

III.

§17. Our attention will now be restricted exclusively to such of our differential equations as contain only singular points of the first kind. If the number of these points is greater than 3, the differential equation will contain an arbitrary parameter which can be continuously varied. The polygon undergoes in consequence a continuous deformation, and the properties of the polynomial product also change. The present section will be devoted to a study of some of the changes in its properties which can be discovered by means of the conformal representation. Special attention will be paid to the changes in the distribution of the real roots of the polynomial among the segments of the axis of x.

§18. The general theory of these equations is similar to the well-known theory of Hermite's equation. When the parameter of the latter is continuously varied from $-\infty$ to $+\infty$, for certain critical values the two solutions forming the polynomial product become identical. The equation then becomes a Lamè's equation, and the two identical solutions, when divested of all factors $(x-e_1)^{\lambda_1}$, $(x-e_2)^{\lambda_2}$, $(x-e_3)^{\lambda_3}$, are simply Lamè polynomials. At the same time a change takes place in the distribution of the roots of the polynomial product among the segments of the axis. We will now show that for our more general differential equations the changes in the distribution of the roots occur only when the two solutions become identical. Since the coefficients of the polynomial are real, a change can be supposed to take place in only two ways: either (1) by the passage of one or more roots through a singular point from one segment into the next, or (2) by the conversion of pairs of real roots into conjugate imaginary roots. In the latter case a multiple real root must first be formed. But it is well-known that no solution can have a multiple root at a non-singular point of the plane, neither can two independent solutions have a common real root at such a point. It remains therefore only to examine when the polynomial has a root which coincides with a singular point. This again is impossible when the two solutions are distinct, because then in the vicinity of the point the polynomial may be written in the form $A^2(P_i^0)^2 \pm B^2(P_i^{m_i+\frac{1}{2}})^2$, only the second term of which vanishes for $x=e_i$. *The changes in the distribution of the roots of the polynomial can therefore take place only when the two solutions become identical.*

§19. When this is the case, a change simultaneously occurs in the character of the conformal representation. To determine the shape of the polygon we must take as before the quotient of two independent solutions. One of these, y_1, may be assumed to be, as in section I, the square root of the polynomial, and can accordingly be written in the form $(x-e_1)^{\varepsilon_1\lambda_1}\cdots(x-e_r)^{\varepsilon_r\lambda_r}P$, where each ε is either zero or unity and P denotes a polynomial which does not vanish at any singular point. Formula [5] then shows that the generating substitutions of the group of η will have the form

$$\bar{\eta}=\varepsilon^{2i\pi\lambda_i}\eta+\beta=-\eta+\beta.$$

One of the two invariant points of every such substitution is ∞, and it follows that every side of the polygon, produced if necessary, must pass through this point. *When, therefore, the two solutions forming the polynomial are coincident, the polygon is rectilinear.*

The position of the roots of the polynomial product can be directly deduced from the polygon. For it is clear from [5] that the roots of y_1 are the only infinities of η. *Hence if a side* $E_i E_{i+1}$ *of the polygon passes p-times through* ∞, p *roots are situated between* e_i *and* e_{i+1}; *if the interior of the polygon includes the point* ∞ q *times,* q *pairs of roots are imaginary*; *and lastly, if a vertex of the polygon is situated at* ∞, *the corresponding singular point is a root.* It will be noticed that each of these roots is a double root of the polynomial product unless it coincides with a singular point. In this case the order of its multiplicity is $2\lambda_i$.

§20. We have shown that in every instance the distribution of the roots among the segments is determined by the form of the polygon. To ascertain the changes in their distribution which result from a variation of the parameter, we have need therefore only to determine the changes in the shape of the polygon, and since a change can occur only when the two solutions become identical, it will suffice to follow the successive transitions through a rectilinear form. This will presently be done in detail for the case in which there are only four singular points.

§21. Before doing so, however, it is necessary to say a few words concerning the methods by which the polygons are constructed. The term polygon is to be understood in the broad sense in which it is employed in the Theory of Functions. As has been already said, the polygon may include either in its interior or on its boundary the point ∞. It will be necessary, therefore, in our diagrams to indicate upon which side of its boundary the polygon lies. This will be done by shading the diagrams. The polygon may also contain overlapping portions or leaves somewhat after the manner of a Riemann's surface. To facilitate the construction of the more complex polygons of this character, we shall have recourse to Klein's processes of attachment of circles or planes to polygons of simpler type. A polygon is said to be "*reduced*" when it cannot be constructed by such attachment from any simpler polygon. The different modes of attachment may be most easily illustrated by reference to Fig. 4, which represents the

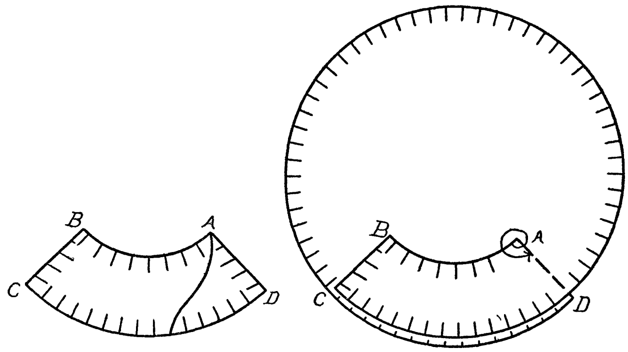

FIG 4

simplest type of a reduced polygon of four sides. To increase A by 2π a circle is taken with the same radius as one of the opposite sides, say CD, and is placed above (or beneath) the polygon so that its boundary shall fall upon this side. The circle and polygon are then cut along a common line from A to CD, and the two are united across the cut like the two leaves of a Riemann's surface, the portion of either of which lies on one side of the cut being connected with the opposite portion of the other. In the resulting polygon the side CD must overlap itself. This process is known as the *polar attachment* of a circle, and may be repeated any number of times. If the same process be applied to increase the angle C, which, with the surface of the polygon, lies upon the convex side of AB, the portion of a plane exterior to a circle having the same radius as AB is to be employed. To cover such cases, the term circle, as in the Theory of Functions, will here be used to denote alike the portion of a plane within or without the bounding circumference. To increase two angles each by 2π, the process of *diagonal attachment* may be used. An entire plane is placed upon the polygon,

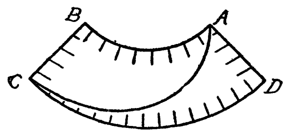

FIG. 5

the two are then cut along a common line between the vertices of the two angles (Fig. 5), and finally are connected in the manner before described. A third process, known as *lateral attachment*, increases each of two adjacent angles by π. Along the intervening side a circle is placed which has the same radius and which continues the surface of the polygon across this side. The connecting side is then erased so that the two figures form a continuous surface. Fig. 6 gives the result of such

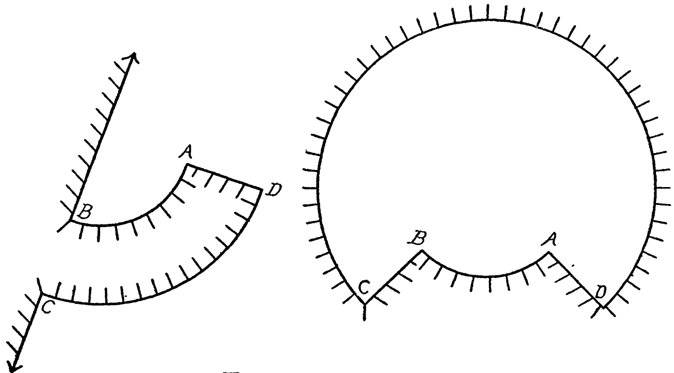

FIG. 6

attachments on the sides CD and BC of Fig. 4. Two successive attachments on the same side are together equivalent to a single diagonal attachment of an entire plane between the two extremities of the side. It is to be observed that this attachment is not applicable to a side which overlaps itself. A fourth process, known as *transversal attachment*, adds to the polygon a circular ring, and is sufficiently explained by Fig. 7. The attachment can only be made to two sides,

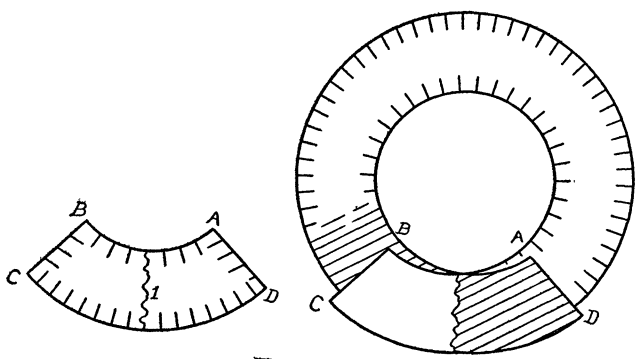

FIG 7

which are arcs of non-intersecting circles, and leaves the angles of the polygon unaltered. The various attachments which have been described are not always possible, nor, when separately possible, are they always compatible one with another, but a glance at the polygon is usually sufficient to determine what system of attachments is applicable. It is therefore unnecessary to discuss the limitations upon their use further than to say that no cut can cross itself or any other cut. Whenever we have occasion to employ these attachments, they will be indicated merely by drawing the cuts and placing beside each cut a number to show how many attachments are to be made upon it. In the case of lateral attachment on any side, the number will be placed adjacent to the side.

§22. We may now return from our digression and take up the case of four singular points. The exponents in this case are

$$\begin{pmatrix} \frac{1}{2}+m_1 & \frac{1}{2}+m_2 & \frac{1}{2}+m_3 & \frac{n_\infty}{2} \\ 0 & 0 & 0 & -\frac{n}{2} \end{pmatrix}$$

and the differential equation takes the form

$$\frac{d^2y}{dx^2}+\left(\frac{\frac{1}{2}-m_1}{x-e_1}+\frac{\frac{1}{2}-m_2}{x-e_2}+\frac{\frac{1}{2}-m_3}{x-e_3}\right)\frac{dy}{dx} + \left(\frac{-n_\infty\, nx+h}{4\,(x-e_1)(x-e_2)(x-e_3)}\right)y=0. \qquad [17]$$

If m_4 is used to designate the integral component of $\lambda_\infty=\frac{n_\infty+n_1}{2}$, it is easy to prove that *the sum of the four m is equal to the degree n of the polynomial.*

The polygon corresponding to this equation, whether it consist of one or many leaves, is in general a curvilinear quadrilateral bounded by two arcs of concentric circles and by two straight lines which cut the circles at right angles. By geometrical considerations, which will here be only briefly outlined, it can be shown that there are eleven types of reduced polygons of this character and no more. These are shown in Plate I. The apparent form of some of these types can be altered by the substitution $\bar{\eta}=\frac{1}{\eta}$, which exchanges y_1 and y_2, but for our purpose the original and the transformed polygon are obviously equivalent. In the case of types 2, 4 and 6, both forms of the polygon are presented.

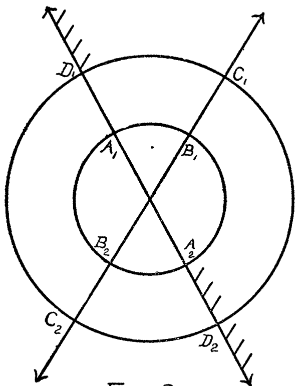

Fig. 8

The construction of these eleven types is based upon Fig. 8, which consists simply of two concentric circles cut by two straight lines through their common center. The vertices of the polygon must be selected, one from each of the four pairs of intersections A_1, A_2; B_1, B_2; C_1, C_2; D_1, D_2. We will first suppose that no side of the reduced polygon overlaps itself. If neither of the rectilinear sides passes through 0 or ∞, the one must be either B_1C_1 or B_2C_2 and the other D_1A_1 or D_2A_2. The boundary of the polygon has therefore the form represented in type 1. To show that the polygon itself must lie with reference to the boundary as represented in our diagram, it suffices to observe that if it were on the other side of the boundary it would contain the whole of the circle of which AB is an arc, and would therefore be reducible by a lateral detachment of this circle. These considerations, however, as yet only determine the angles to within multiples of 2π. But any other polygon, bounded in the same manner as the first polygon of the plate, would contain at least two angles which would exceed the corresponding angles of the latter by multiples of 2π, and would therefore permit of the diagonal detachment of one or more planes. Type 1 therefore represents the only type of reduced polygon which has no side which overlaps itself or passes through 0 or ∞. In the discussion of subsequent types similar reasoning will show, after the boundary of the polygon has been determined and also the side of the boundary upon which the polygon lies, that there is only one reduced polygon which meets the requirements. This will hereafter be assumed without further remark.

We proceed next to determine the reduced polygons which have but a single side which passes through either the origin or infinity. It is immaterial through which point the side is assumed to pass, since the points may be exchanged by the substitution $\bar{\eta} = \frac{1}{\eta}$. This side may therefore be taken as $D_1 \infty A_2$ or $D_1 \infty A_1$, and we may also suppose that the adjacent surface of the polygon is the border shaded in Fig. 8. The second rectilinear side must be either B_1C_1 or B_2C_2. If, now, the first side terminates in A_2, this vertex must be connected with B_1 and B_2 respectively by the arcs A_2B_1 and $A_2B_1B_2$, because otherwise the polygon would contain the whole of the circle lying within the circumference $A_2B_2B_1$ and would consequently be reducible. Completing, finally, the polygon by the addition of a fourth side, we obtain types 2, 3 and 4. The polygon in which the fourth side is the arc $C_2C_1D_1$ is excluded, because it would necessitate a winding point at D_1 and would therefore be reducible by lateral detachment along this side. If, on the other hand, the first rectilinear side terminates in A_1, the second cannot be B_2C_2. For if it were, the whole of the half-plane adjacent to the former side would be contained in the polygon and could be detached laterally. We have therefore only to connect $D_1 \infty A_1$ with B_1C_1, and this can be done in two ways, as shown in types 5 and 6.

If both the rectilinear sides of the polygon pass through the origin or infinity, we may distinguish the following cases:

(1). One rectilinear side $D_1 \infty A_2$ passes through ∞ and the other, $B_1B_2C_2$ or $B_2B_1C_1$, through the origin (Types 7 (a) and 7 (b)).

(2). Both sides pass through the origin or through ∞, say the origin. We have then to connect two such segments as $D_1A_1A_2$ and $B_1B_2C_2$ (Type 8).

(3). One side $D_1 \infty A_1$ passes through the origin and infinity, and the other only through the origin. The segment $B_1B_2C_2$ must be selected as the second side, since otherwise the polygon could be reduced by the lateral detachment of the half-plane adjacent to the former side (Type 9).

(4). Each side passes through the origin and infinity. With $D_1 \infty A_1$ must be associated the segment $B_1 \infty C_2$, since otherwise a half-plane could be removed (Type 10).

It remains now to consider the possible forms of a reduced polygon, one or more of whose sides overlap. Examples of such polygons can be obtained from two of the preceding types, namely, Types 4 and 7(b) by prolonging the opposite arcs each by a semi-circumference. We shall, however, still consider the polygons to be of the same type. The only other types in which the arcs can be produced till they overlap are the 1st and 3d, but these polygons will then be reducible either by transversal or by polar detachment. There are therefore no other reduced polygons in which the sides overlap because the polygon winds in ring-form between the concentric arcs. We have therefore only to consider the cases in which the overlapping is effected in some other way. Since the surface overlaps at the same time as the side, it must wind around one or both of the vertices opposite to the side. If the side be rectilinear, it is easy to see (compare Fig. 9)

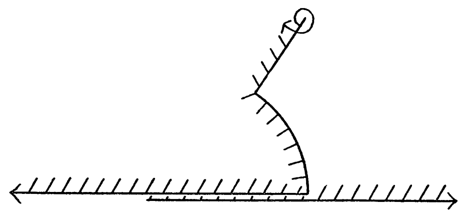

FIG. 9

that the surface makes a complete winding around one of the two vertices. The polygon can therefore be reduced by polar detachment. The same is true if a complete winding takes place around a vertex opposite to a circular side. There remains therefore only the case where there is a partial winding around both vertices, so that the angles here are $\frac{3}{2}\pi$. The only reduced polygon of this character which can be constructed is presented in Type 11. Our list of reduced polygons is therefore now complete.

Each reduced polygon gives rise by attachment to a system of polygons. Many of the polygons thus constructed can, however, be constructed from two or more distinct types. We will, for example, obtain the same form of polygon by a lateral attachment on BC in Type 3 as by a diagonal attachment between B and D in Type 4; or again, by a lateral attachment on DC in Type 8 as by a lateral attachment on BC in Type 7 (a).

§23. We are now prepared to construct for any given values of the m a polygon which corresponds to the differential equation [17], and to trace the successive changes in form, when the parameter h of the equation is continuously varied. A complete determination of the polygon depends, of course, upon the anharmonic ratio of the singular points as well as upon the accessory parameter. A two-fold variation in the form of the polygon is accordingly possible. Either the ratio of the radii of the two concentric arcs or the inclination of the two rectilinear sides may be continuously altered. We shall, however, take account only of such changes of form as affect the type of the reduced polygon and the corresponding system of attachments. With this understanding it will be first found that by continuous geometrical deformation a series of different forms is obtained, which succeed one another in definite order, and subsequently it will be shown that a variation of the parameter h alone gives rise to the series thus obtained. We may start with any polygon having the angles $(m_i + \frac{1}{2})\pi$, for from it all other forms of polygons with the same angles will be subsequently obtained. We will first consider the case in which some one of the m, say m_4, is equal to or greater than the sum of all the others.

$$\text{I.} \quad m_4 \geqq m_1 + m_2 + m_3.$$

To bring the m to a form corresponding to the system of attachments to be employed in the construction of the polygon, we shall avail ourselves of one of the four following arithmetical reductions, in which s, t, x and z denote integers, positive or zero.

(1) $m_4 = 2s + 2t + x + z$
$m_1 = x$
$m_2 = 2t$
$m_3 = z.$

(2) $m_4 = 2s + 2t + x + z + 1$
$m_1 = x$
$m_2 = 2t + 1$
$m_3 = z.$

(3) $m_4 = 2s + 2t + x + z + 1$
$m_1 = x$
$m_2 = 2t$
$m_3 = z.$

(4) $m_4 = 2s + 2t + x + z + 2$
$m_1 = x$
$m_2 = 2t + 1$
$m_3 = z.$

The first two reductions are to be employed when the polynomial is of even degree; the last two, when the polynomial is of odd degree. In all four cases the form of the reduction shows that after the selection of a suitable reduced polygon, a system of attachments may be employed consisting of t diagonal attachments between E_2 and E_4, x and z lateral attachments on the sides E_4E_1 and E_4E_3 respectively, and s polar attachments from E_4 to one of the two opposite sides, say E_2E_3. In the first case we must select the first type of reduced polygon, in the remaining three cases types 7, 4 and 3 respectively, A, B, C and D being taken in each case as the vertices E_∞, E_1, E_2, E_3.

All possible changes in the form of the polygon for case 1 are shown in Plates II and III. As before pointed out, the essential features of the polygon are modified only by transition through a rectilinear form. It suffices therefore to indicate in our figures these successive transitions. The rectilinear forms are marked in the plates with even numbers, the intermediate stages with odd numbers. The passage to a rectilinear form is effected, of course, by withdrawing the center of the concentric arcs to ∞. With the exceptions to be hereafter noted, the successive transitions can be effected only in the order in which they are given in the plates.

The two plates, taken together, are divided into four sections, each of which illustrates a cycle of changes which is to be repeated as many times as possible. In the first cycle a polar attachment is transferred from E_2E_3 to E_2E_1, as is seen by a comparison of the first and last figures of the cycle. The lateral and diagonal attachments remain, however, unaltered. The corresponding index numbers have been inserted only in the first and last polygons, it being understood that in each intermediate polygon there is an equal number of lateral, as of polar, attachments. The second figure may be obtained from the first by withdrawing the center of the circular sides to ∞. The only new form which is possible when it reappears in the finite plane is that represented in Fig. 3 (a) or a similar figure in which E_1E_2 is the inner and $E_\infty E_3$ the outer arc. These two figures are, however, equivalent by virtue of the substitution $\bar{\eta} = \frac{1}{\eta}$. Figure 3 (b) is of the same form as 3 (a), one of the s polar attachments being explicitly represented. If, now, in this figure the center of the concentric arcs is carried to the right along the side E_2E_3 to ∞—if carried in the opposite direction, we return to Fig. 2—the vertices E_3 and E_2 both pass to ∞, but E_1 must remain in the finite plane, since otherwise the polygon would degenerate into a triangle. We thus arrive at Fig. 4. The passage thence to Fig. 6 requires no comment. The seventh and eighth polygons have been omitted, inasmuch as they can readily be supplied by the reader, being similar in structure to the fifth and fourth polygons respectively, but with an interchange in the roles of E_1 and E_3. Omissions of like character will likewise be made in subsequent cycles. This cycle is to be repeated s times, that is, until all the polar attachments have been transferred to E_1E_2. It may then be applied once more, until the reduced polygon 3 (a) is reached, when it will be found impossible to proceed further. The polygon thus obtained is the initial figure of the second section.

The second cycle removes a diagonal attachment and replaces it by a pair of polar attachments to E_1E_2. At the same time the number of lateral attachments on $E_\infty E_3$ is diminished and the number on E_3E_2 increased, each by two. The successive changes require no particular comment, until we reach the two polygons 5. These (as later other pairs of polygons) are numbered alike to call attention to the fact that, although constructed from two different types of reduced polygon, they are identical in form. On leaving this figure, two alternative courses are open, either to proceed as in the plate to the ninth polygon or to pass from the one to the other by means of 5 (b) and 5 (c) (see adjoining Fig. 10)

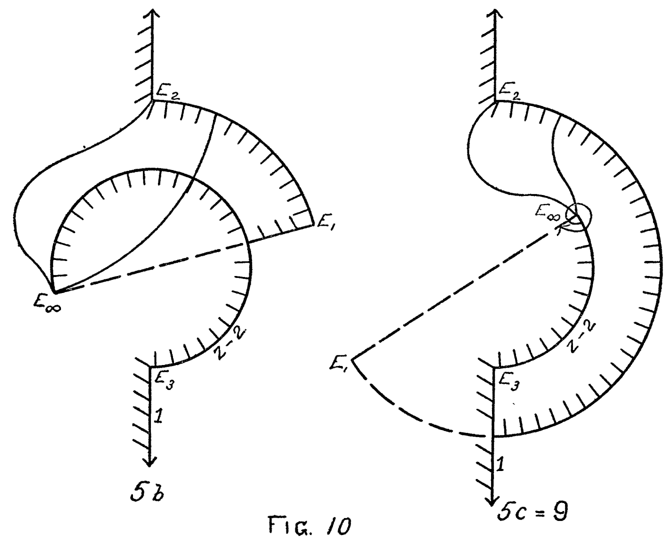

FIG. 10

without the insertion of any rectilinear form. Either succession of changes is geometrically possible, and a decision between them cannot here be made. Presumably it is dependent upon the position of the singular points upon the axis. The cycle can be repeated until either all the diagonal attachments or all the lateral attachments on $E_\infty E_3$ have been removed. The former is the case when $\frac{z}{2} \geqq t$, that is, $m_3 \geqq m_2$; the latter, when $m_3 \leqq m_2$.

The third cycle is applicable only when $m_3 \geqq m_2$, and the effect of its repeated application is to remove the remaining lateral attachments. The changes for the first half of the cycle are the same as in Figs. 1 to 5 of cycle 2. We then insert two new figures, numbered 5 (a) and 5 (b), and thence proceed as in polygons 10 to 13 of cycle 2 to the ninth and final figure of the cycle. Each half cycle removes a lateral attachment on $E_\infty E_3$ and replaces it, the one by a polar attachment from E_∞ to $E_1 E_2$, the other by a polar attachment from E_3 to the same side. According as the number of lateral attachments to be removed is odd or even, the reduced polygon with which we conclude the last application of this cycle will have the form given in 5 or in 9. Each of these polygons contains part of a circular ring included between the sides $E_\infty E_3$

and E_1E_2. Since all the lateral attachments have been removed from these sides, they can be indefinitely prolonged, thus adding an indefinite number of circular rings to the figure. This, as will be later shown analytically, is always the final outcome of an indefinite increase of the parameter.

When $m_3 \leqq m_2$, the prolongation of the two circular sides begins immediately upon conclusion of the second cycle, the reduced polygon being then either 5 or 13 of cycle 2. Owing, however, to the presence of diagonal attachments between E_∞ and E_2, this will not result at once in the addition of circular rings to the polygon. The last section of Plate III shows the effect of a prolongation of each of the circular sides for a complete circumference, a diagonal attachment being of necessity replaced by two polar attachments, the one from E_∞ to the side E_1E_2, the other from E_2 to the side $E_\infty E_3$. By a repetition of this process the diagonal attachments will be removed, but at any time before this has been accomplished another change in the form of the polygon may be made. By passage through a rectilinear form, the circular sides to which the polar attachments are made may be converted into the rectilinear sides. But if this is done, to continue the transformation of the polygon, it will be necessary to re-exchange the circular and rectilinear sides either by retracing our steps or by completing the series of changes as indicated in Fig. 11* of the text. The final outcome will be the same whether these changes be included or not. The diagonal attachments will eventually be all replaced by polar attachments, and the further prolongation of the circular sides $E_\infty E_3$ and E_1E_2 will thereafter result in the addition of circular rings ad infinitum.

We have now traced all possible changes in the form of the polygon upon the hypothesis that the parameter is varied continuously in one direction. It remains to consider what changes the polygon will undergo when the parameter is varied in the opposite direction. As already stated, the only difference between the first polygon of cycle 2 and the first of cycle 1 is that the polar attachments are all made to E_1E_2 in the one case and to E_2E_3 in the other. By a change of subscripts, the subsequent cycles will therefore apply equally well to either side of the first cycle. A variation of the parameter in the opposite direction will also ultimately result in the addition of circular rings, which, however, will be included between the sides $E_\infty E_1$ and E_2E_3.

*** The transition from 3 (a) to 3 (b) is effected by increasing in the former polygon the radius of the inner arc E_3E_2 until it exceeds the radius of $E_\infty E_1$. The two rectilinear sides then overlap, as in polar attachment.**

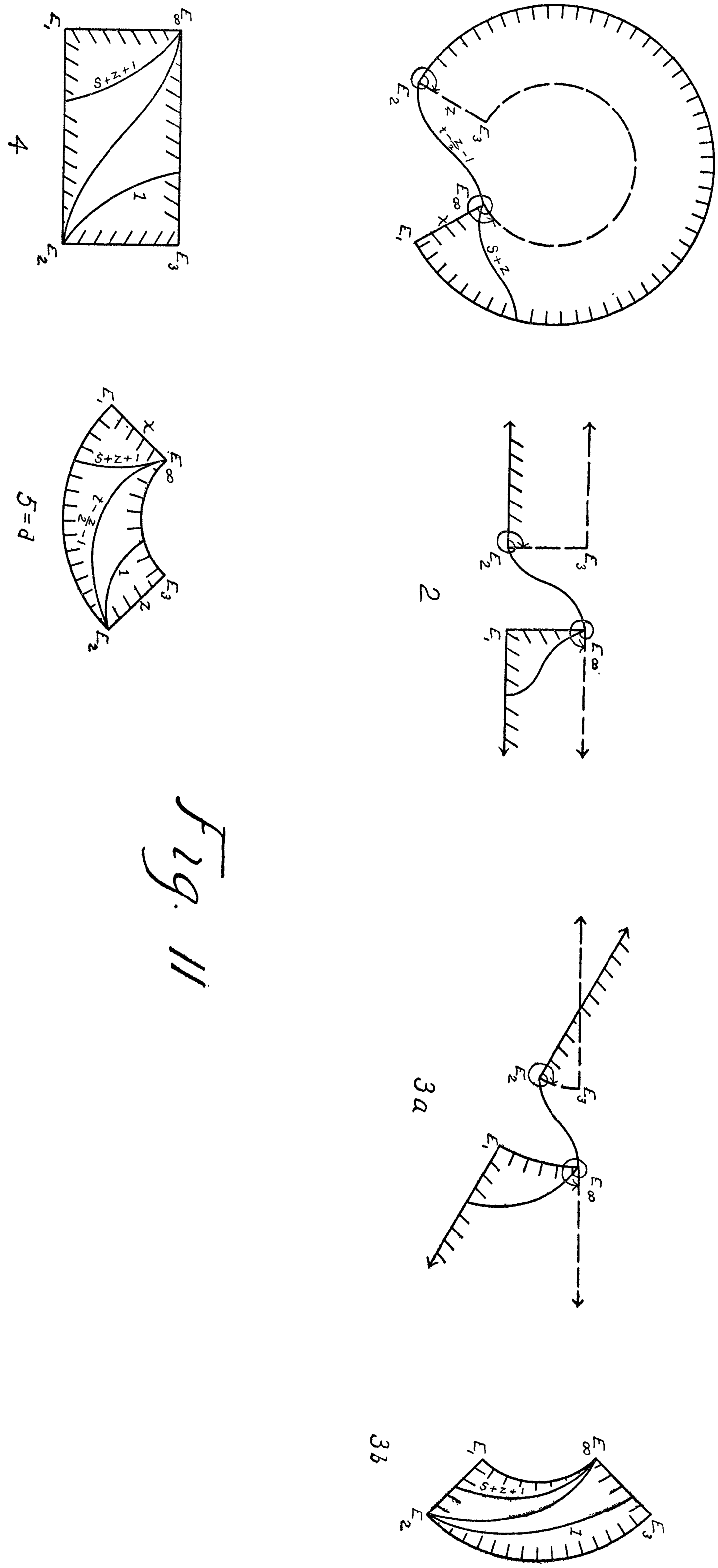

Fig. 11

This completes the discussion for case 1. The second case differs from the first in no essential feature. Plate IV gives the first cycle of changes. By its repeated application the polar attachments, as before, are transferred from E_2E_3 to E_1E_2. In the last application of the cycle it will be found that the seventh polygon is identical in structure with the fifth of cycle 2, case 1, except that there is no lateral attachment upon E_2E_3. From this point on, the discussion is the same as in case 1. In the subsequent figures there will be an even or an odd number of attachments on this side according as the number of such attachments was in the former case odd or even.

The changes for case 3 are shown in Plates V and VI and for case 4 in Plate VII. The successive cycles are in every way similar to those of the first two cases, and case 4* is related to case 3 precisely as case 2 to case 1.

§24. When no one of the m is greater than the sum of the remaining three, we may, without loss of generality, assume that

$$\text{II.} \quad m_4 + m_2 \geqq m_3 + m_1, \qquad m_4 + m_1 \geqq m_2 + m_3.$$

One of the four following arithmetical reductions may then be made, t, z and x being non-negative integers and y a positive integer.

$$(5). \quad m_4 = 2t + z + x, \quad m_1 = x + y, \quad m_2 = 2t + y, \quad m_3 = z.$$

$$(6). \quad m_4 = 2t + z + x + 1, \quad m_1 = x + y, \quad m_2 = 2t + y + 1, \quad m_3 = z.$$

$$(7). \quad m_4 = 2t + z + x + 1, \quad m_1 = x + y, \quad m_2 = 2t + y, \quad m_3 = z.$$

$$(8). \quad m_4 = 2t + z + x + 2, \quad m_1 = x + y, \quad m_2 = 2t + y + 1, \quad m_3 = z.$$

In the first two of these four cases the polynomial is of even degree; in the last two, of odd degree. The same reduced polygons may be selected as in the corresponding cases of I, namely, the first, seventh, fourth and third types, B, C, D and A being taken as before for the vertices E_1, E_2, E_3 and E_∞. The

*The seventh polygon of cycle 1 of this case, after the polar attachments have all been transferred to E_1E_2, is converted into one similar to 5 (b) or 9 of cycle 2, case 3, by enlarging the radius of the inner arc until it exceeds that of the other arc.

polygons are then completed by t diagonal attachments between E_∞ and E_2 and by x, y and z lateral attachments along the sides $E_\infty E_1$, E_1E_2 and $E_\infty E_3$. The polygon for case 8 can also be built up from Type 4, since the first lateral attachment on BC in Type 3 is equivalent to a diagonal attachment between B and D in Type 4, and the polygon will therefore differ essentially from that for case 7 only in the selection of the vertices E_i.

The first cycle of changes for cases 5–7 is shown in Plate VIII. The changes in case 8 are similar to those in case 7. With each repetition of the cycle the number of lateral attachments on each of two opposite sides, E_1E_2 and E_3E_∞, is diminished by a unit, while the number on each of the other sides is increased a like amount. The cycle is to be repeated until all the lateral attachments have been removed from one of the first two sides. This side is then free for polar attachment, as was also E_2E_3 at the outset. The cycle is therefore to be both preceded and followed by other cycles in exactly the same manner as was cycle 1 in the corresponding cases of I. We may therefore limit our attention altogether to the present cycle of changes in the polygon, this being the only one of a new character.

§25. We have now seen for each case all possible changes in the form of the polygon. It remains to prove that when the parameter is continuously varied, the polygon will pass through the series of changes which have been described. For this it will evidently suffice to show that an indefinite increase or decrease of the parameter will result in the addition of an indefinite number of circular rings included in the one case between E_3E_∞ and E_1E_2, in the other, between $E_\infty E_1$ and E_2E_3. To demonstrate this we first reduce equation [17] by the substitution

$$y = (x - e_1)^{\frac{1}{2}m_1 - \frac{1}{4}} \ldots . (x - e_3)^{\frac{1}{2}m_3 - \frac{1}{4}}\,\bar{y}$$

to the form

$$\frac{d^2y}{dx^2} + \left(R(x) + \frac{h}{4(x - e_1)(x - e_2)(x - e_3)}\right)y = 0, \qquad [18]$$

in which $R(x)$ is a rational fraction that is finite except at the singular points. If h is then taken sufficiently large, the coefficient of y will have for any given value of x the same sign as $\dfrac{h}{(x - e_1)(x - e_2)(x - e_3)}$. For large positive values of h the sign will therefore be positive in the segments e_1e_2 and $e_3\infty$, for large negative values in the segments ∞e_1 and e_2e_3. The coefficient can, moreover, be made greater than any given positive constant a. Now it is well known that every real solution of the equation $y'' + ay = 0$ has an infinite number of real roots which cumulate in both directions in the vicinity of the point at ∞. But, by a theorem of Sturm,* if G' and G'' are two functions of x which are finite and continuous for any interval of the axis of x, and if G' is algebraically less than G'', then between any two successive roots of a real solution of $y'' + G'y = 0$, which are situated in this interval, there must lie at least one root of every real solution of $y'' + G''y = 0$. It follows that *when h is indefinitely increased, any real solution of* [17] *will have an infinite number of roots in the segment $e_3\infty$, and when h is indefinitely decreased, an infinite number of roots in the segment ∞e_1.* Like results must also hold for the segments e_1e_2 and e_2e_3, respectively, since by a linear substitution the singular points e_2 and ∞ can be interchanged and at the same time the value of h is multiplied by a negative constant.† Thus, whether h is indefinitely increased or diminished, every real solution will have an indefinitely large number of roots in alternate segments of the axis. Furthermore these segments cannot be hyperbolic segments, because in such segments the two factors of our polynomial product are real solutions and its degree would then be infinite. In the elliptic segments the two solutions will have the form $AP_i^0 \pm \sqrt{-1}\, BP^{m_i+\frac{1}{2}}$, in which P_i^0 and $P^{m_i+\frac{1}{2}}$ will each have an infinite number of zeros, the zeros of P_i^0 alternating with those of $P^{m_i+\frac{1}{2}}$. Hence as x traverses either elliptic segment, the argument of η, which is equal to $2\tan^{-1}\dfrac{BP^{m_i+\frac{1}{2}}}{AP_i^0}\,\dfrac{B}{A}$, will increase without limit. It follows that when h approaches ∞, an indefinite number of complete circumferences will eventually be added to the circular sides of the polygon. Since the angles of the polygon remain unaltered, this can be done only by the successive addition of circular rings.

§26. Our figures may now be applied to a study of the polynomial product. First consider cases 1 and 2 in which the product is of even degree. Upon examination of the rectilinear polygons it will be found either that the vertices all lie in the finite plane or that two of the vertices E_1, E_2, E_3 are situated at ∞. For the critical values of the parameter the square root of the polyno-

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* Lionville, tome I, p. 135.

† See my article in the Bulletin of the American Mathematical Society, June, 1898, p. 432.

mial product can therefore be expressed in one of the four following forms:

$$(1).\ P_{\frac{n}{2}},$$
$$(2).\ (x-e_2)^{m_2+\frac{1}{2}}(x-e_3)^{m_3+\frac{1}{2}}P_{\frac{n}{2}-m_2-m_3-1},$$
$$(3).\ (x-e_1)^{m_1+\frac{1}{2}}(x-e_3)^{m_3+\frac{1}{2}}P_{\frac{n}{2}-m_1-m_3-1},$$
$$(4).\ (x-e_1)^{m_1+\frac{1}{2}}(x-e_2)^{m_2+\frac{1}{2}}P_{\frac{n}{2}-m_1-m_2-1}.$$

The polynomials P thus introduced fall into four distinct classes, and those which belong to the same class are solutions of differential equations with common exponent-differences. According to Heine's formula [8] the number of polynomials in the several classes must be equal to

$$(1).\ \frac{n}{2}+1,\quad (2).\ \frac{n}{2}-m_2-m_3,\quad (3).\ \frac{n}{2}-m_1-m_3,\quad (4).\ \frac{n}{2}-m_1-m_2,$$

and the total number will be

$$2n+1-2\,[m_1+m_2+m_3]=4s+4t+2x+2z+\begin{cases}1, \text{ case } 1;\\ 3, \text{ case } 2.\end{cases}$$

The number of rectilinear polygons will not, however, necessarily be so great, inasmuch as they correspond only to real values of the accessory parameter, that is, to polynomials with real coefficients. The lower limit to the number of such polygons can be obtained by a count of the minimum number of rectilinear polygons included between the two polygons with series of ring attachments, and it will be found to be $4s+2x+2z\begin{cases}+1, \text{ case } 1\\ -1, \text{ case } 2\end{cases}$. *Our geometrical investigation furnishes*, therefore, for the cases under consideration, *a supplement to Heine's theorem.* The missing polynomials belong to the first and third classes.

An inspection of the plates also shows that the polynomials of the several classes recur in each cycle in a definite order. The first cycle is, however, the only one in which all four classes are included. The order in which they there recur is for case 1 the same as that in which they were above enumerated; in case 2 they recur in opposite order.

As before explained, the changes in the distribution of the real roots of the polynomial product which result from a continuous change of the parameter h can easily be traced by comparing successively each rectilinear polygon with the polygons which immediately precede and follow it. In the two cases before us, as also in all cases to be hereafter examined, each passage of the polygon through a rectilinear form exchanges the rectilinear with the circular sides. *As therefore the accessory parameter passes successively through the critical values, each segment of the axis will be alternately elliptic and hyperbolic.*

The successive changes in the position of the roots may be advantageously shown by a graphical representation such as was introduced by Klein in his discussion of Hermite's equation. For this purpose the values of h are plotted as ordinates and the roots of the corresponding polynomials as abscissas. The resulting curve $F(P_n, h) = 0$ shows at a glance the dependence of the roots upon the parameter h. Specimen sections of the curve, which correspond to the first applications of the various cycles, are given in the first half of Plate IX for case 1. Horizontal lines which represent the critical values of the parameter are added and numbered to correspond with the rectilinear polygons in Plates II and III. These, together with the vertical lines $x = e_1, e_2, e_3$, divide the plane into rectangles, in which alternately the two solutions are elliptic and hyperbolic To each successive repetition of cycle 1 corresponds a branch of the curve similar to that drawn in the plate, but the number of oscillations between e_2 and e_3 which corresponds to the number of polar attachments on the rectilinear side $E_2 E_3$ is every time diminished by a unit, and the number of oscillations between e_1 and e_2 is increased by a unit. In like manner the number of oscillations between e_1 and e_2 is increased by two units with each successive repetition of cycle 2 or 3. The dotted portions of the curve correspond to the series of polygons in cycles 2 and 4 whose presence cannot definitely be affirmed. The ovals are to be included in the curve only when the numbers which they enclose are odd. Below the first section of the plate are to be added sections similar in structure to those above it, but the roles of the segments $e_1 e_2$ and $e_2 e_3$ must be interchanged. The first section of the curve for case 2 is indicated at the foot of Plate IV. The second and third sections are similar to the corresponding sections of case 1 but with the insertion of an oval in the lowest rectangular space between $x = e_2$ and $x = e_3$.

The curve given by Klein for Hermite's equation is comprised under case 1, when the degree of the polynomial-product is even, and is the special case in which there is but a single cycle of changes. The curve therefore consists entirely of sections similar to that given for the first cycle, but without the ovals.

§27. *Cases* 3 *and* 4.

The theory of these two cases, for which the polynomial-product is of odd degree, runs parallel to that of the first two cases and may therefore be very briefly indicated. In each rectilinear polygon either all three of the vertices E_1, E_2, E_3 lie at ∞ or only one. For the critical values of the parameter the square root of the polynomial has accordingly one of the four following forms:

$$(1).\ (x-e_3)^{m_3+\frac{1}{2}} P_{\frac{n-1}{2}-m_3},$$
$$(2).\ (x-e_2)^{m_2+\frac{1}{2}} P_{\frac{n-1}{2}-m_2},$$
$$(3).\ (x-e_1)^{m_1+\frac{1}{2}} P_{\frac{n-1}{2}-m_1},$$
$$(4).\ (x-e_1)^{m_1+\frac{1}{2}}(x-e_2)^{m_2+\frac{1}{2}}(x-e_3)^{m_3+\frac{1}{2}} P_{\frac{n-3}{2}-m_1-m_2-m_3}$$

We have again four classes of the polynomials, and by [8] the total number in each class is as follows:

$$(1).\ \frac{n+1}{2}-m_3, \qquad (2).\ \frac{n+1}{2}-m_2,$$
$$(3).\ \frac{n+1}{2}-m_1, \qquad (4).\ \frac{n-1}{2}-m_1-m_2-m_3;$$

in all, $2n+1-2[m_1+m_2+m_3]=4s+4t+2x+2z+\begin{cases}3, \text{ case } 3\\ 1, \text{ case } 4\end{cases}$. A count of the total number of polygons will show that at least $4s+2x+2z+\begin{cases}3, \text{ case } 3\\ +1, \text{ case } 4\end{cases}$ are real. The missing polynomials belong to the first and third classes. In each cycle there is again a definite order in which the classes recur, the order in which they were just enumerated being that for the first cycle of case 3. Specimen sections of the curve $F(P_n, h)=0$ are drawn for case 3 in the second half of Plate IX and for the first cycle of case 4 at the foot of Plate VII. The curves differ mainly from those of the first two cases in that the principal branches for the separate cycles are no longer closed curves, but form one continuous curve which traverses the entire plane. The curve given by Klein for Hermite's equation, when the degree of the polynomial-product is odd, is included under case 3 and consists entirely of sections similar to the first of the plate, but with the omission of the ovals.

§28. *II. Cases* 5–8.

An inspection of Plate VIII shows that the nature of the cycle there represented varies greatly in the several cases. In the fifth case the vertices of the polygon remain throughout the entire cycle in the finite plane. There is therefore but a single class of rectilinear polygons, and all the corresponding polynomials have the form $P_{\frac{n}{2}}$. The cycles which precede and follow that given in the plate introduce three other classes of polynomials. If $z > y$, that is, if $m_4 + m_3 > m_2 + m_1$, we have the same four classes of polynomials as in cases 1 and 2, and their total number

$$2n - 2[m_1 + m_2 + m_3] + 1 \text{ is equal to } 4t + 2z + 2x + 1.$$

Of these all except possibly $4t$ must be real. The missing polynomials belong to the first and third classes. If, on the other hand, $z < y$, the fourth class of polynomials must be replaced by one for which the square root of the polynomial product has the form $(x - e_3)^{m_3 + \frac{1}{2}} P_{\frac{n}{2} - m_3 - m_4 - 1}$. The reduction of the degree of this expression below $\frac{n}{2}$ is due to the coincidence of $2(m_4 + \frac{1}{2})$ roots of the polynomial-product with the singular point ∞. Such a reduction can take place only if $n - m_4 - 1$, which is the negative of $\frac{n_\infty}{2}$ or the second exponent for ∞, is positive, and the necessary condition for this is easily seen to be the condition which is common to cases 5–8, viz. $m_4 < m_1 + m_2 + m_3$. The total number of polynomials in the four classes is

$$2n - 3m_3 - m_1 - m_2 - m_4 + 1 = 4t + 2x + 2y + 1.$$

Of these all except $4t$ must certainly be real. All the unreal polynomials belong again to the first and third classes.

Case 6. Two alternatives are apparently possible in the first cycle, between which we cannot here decide. The cycle may, namely, be concluded, as in the plate, without the insertion of any rectilinear polygon whatever, or at its close a series of changes similar to the series given in figures 5b to 9 of cycle 2, case 1, may be added. The case differs essentially from the preceding in this, and only in this, cycle.

Cases 7–8. In the first cycle each vertex of the polygon in turn recedes

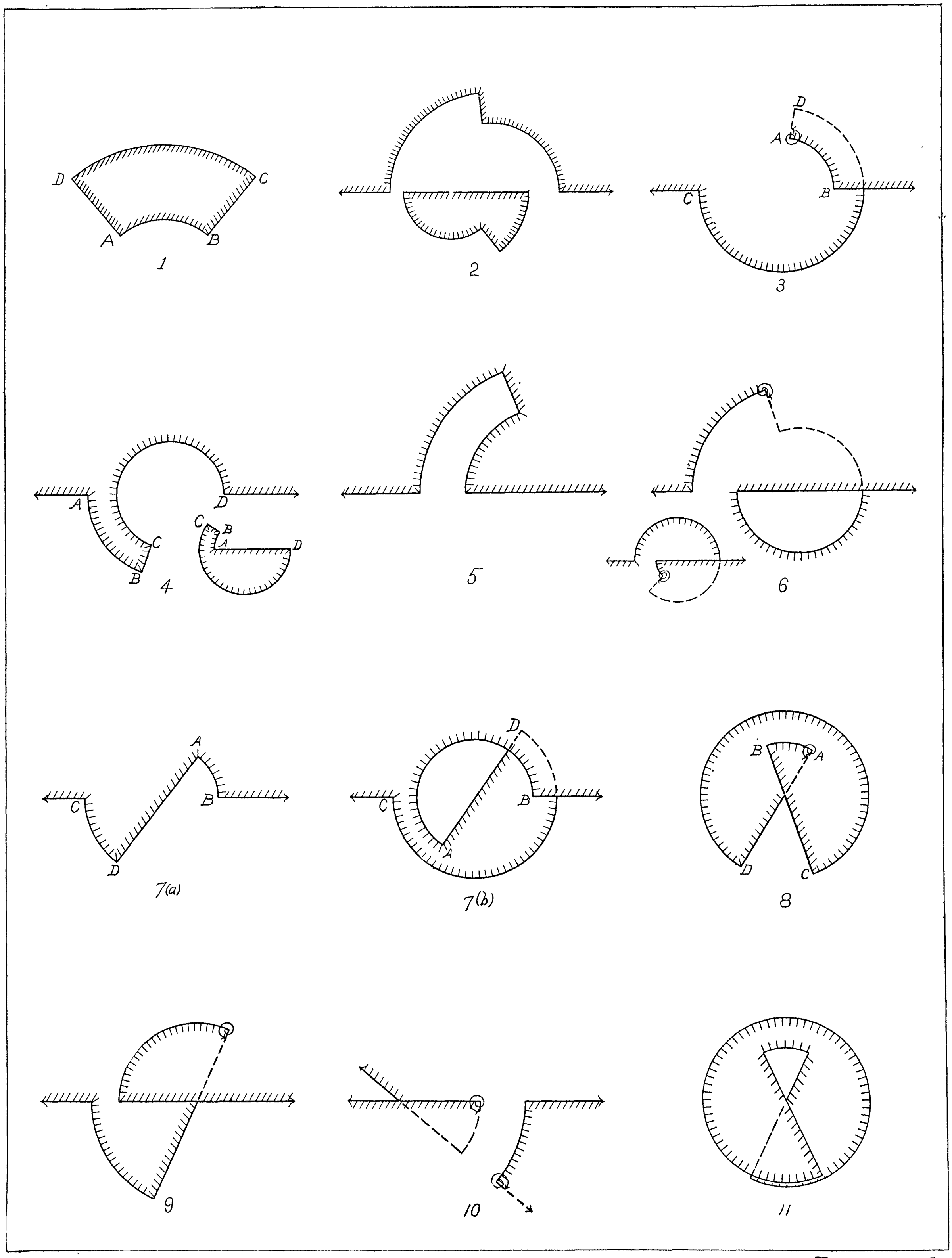


PLATE I

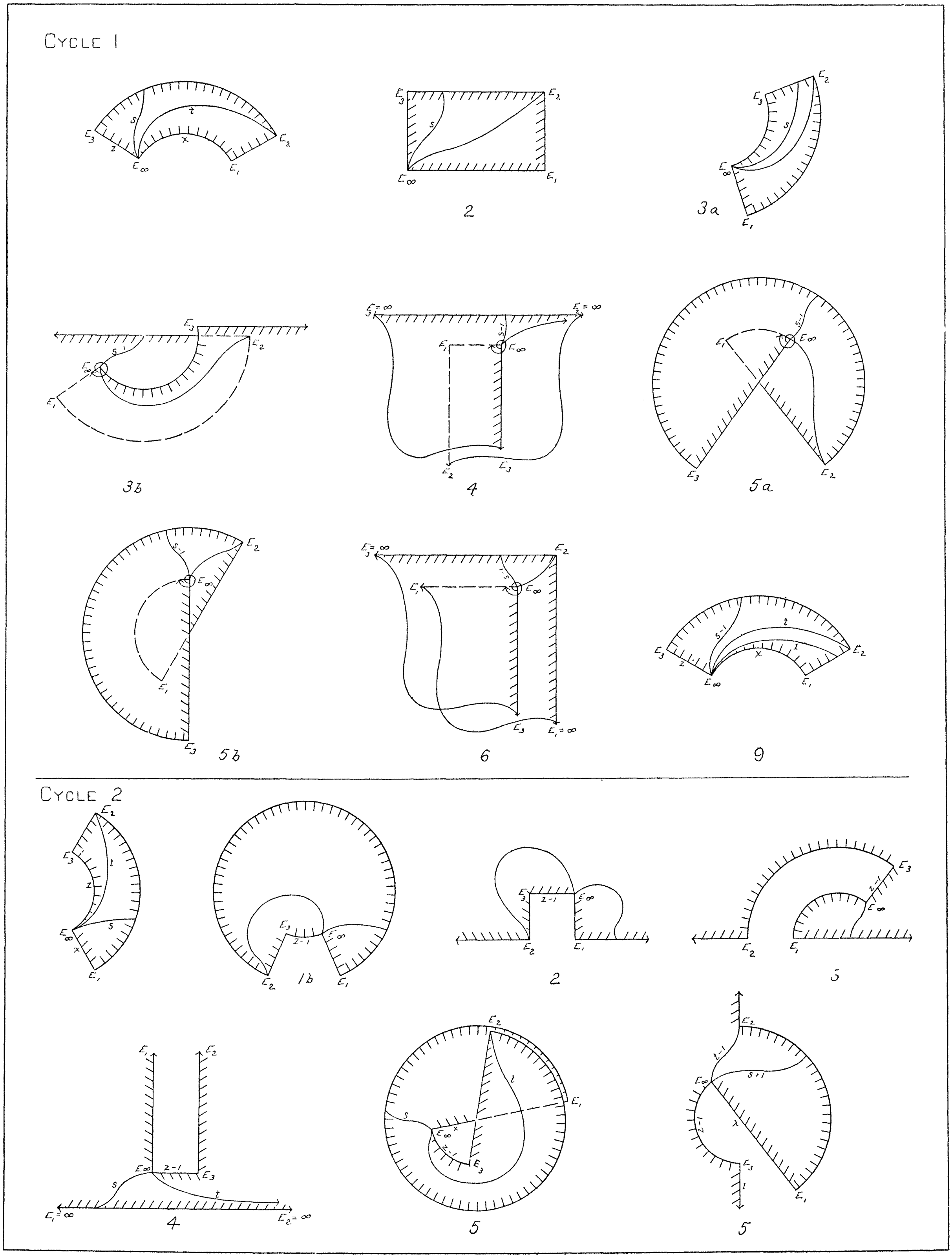


PLATE II

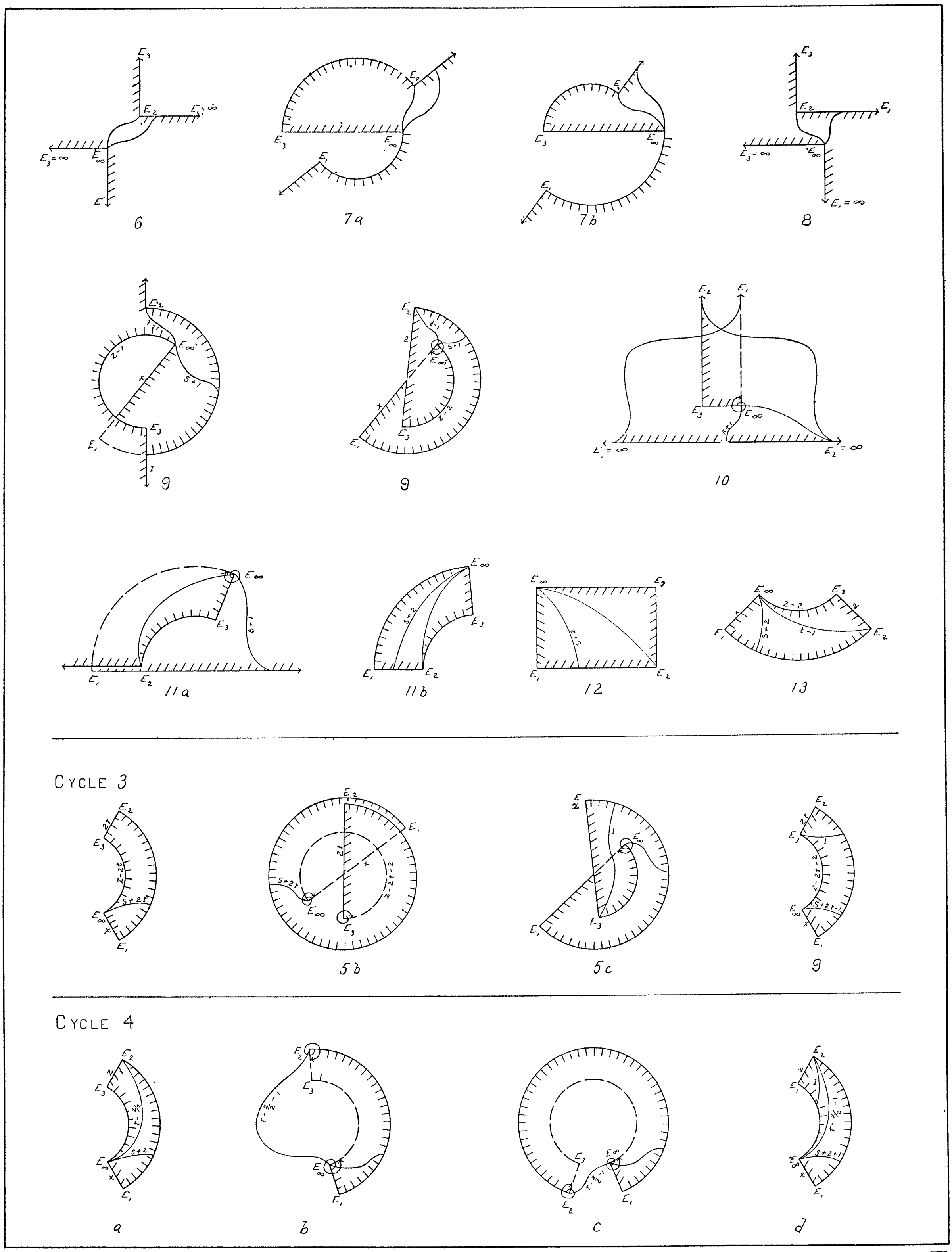


PLATE III

CYCLE I

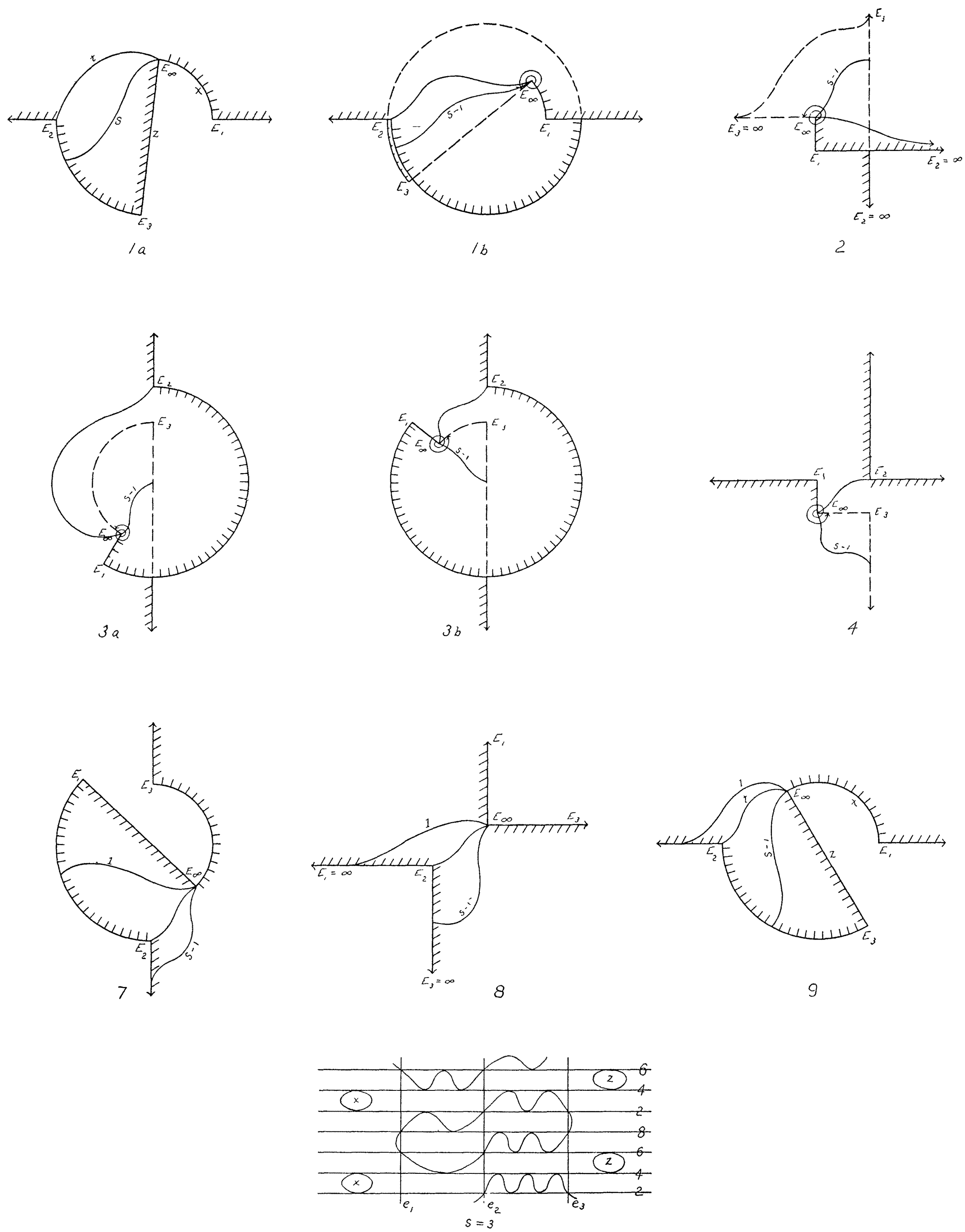
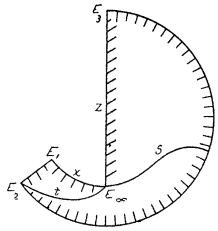
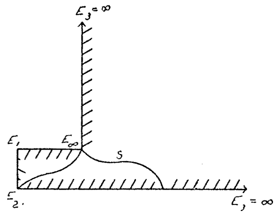


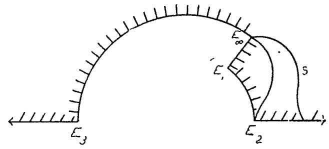
PLATE IV

Cycle 1

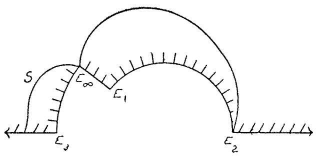




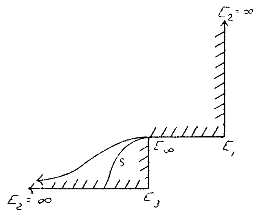
2



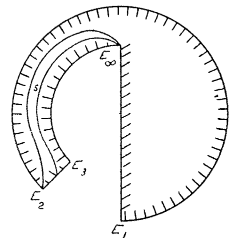
3 a



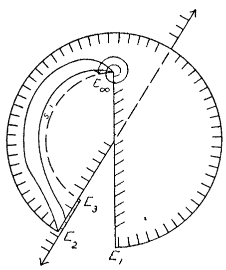
3 b



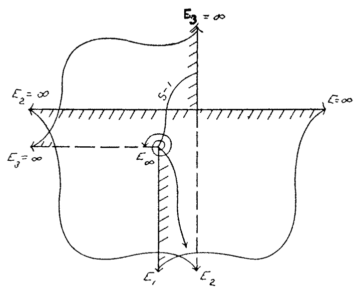
4



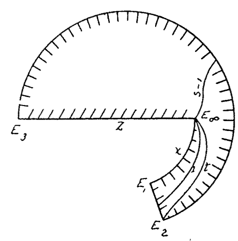
7 a



7 b

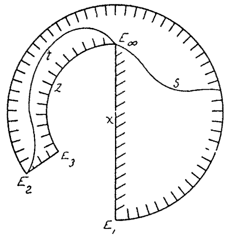


8

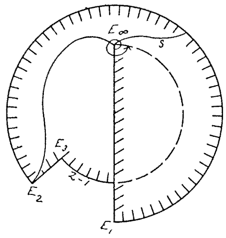


9

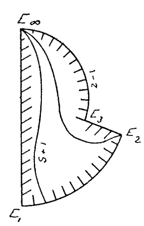
Cycle 2



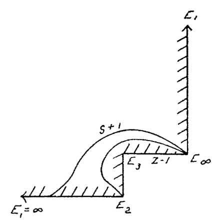
1 a



1 b



1 c



2

Plate V

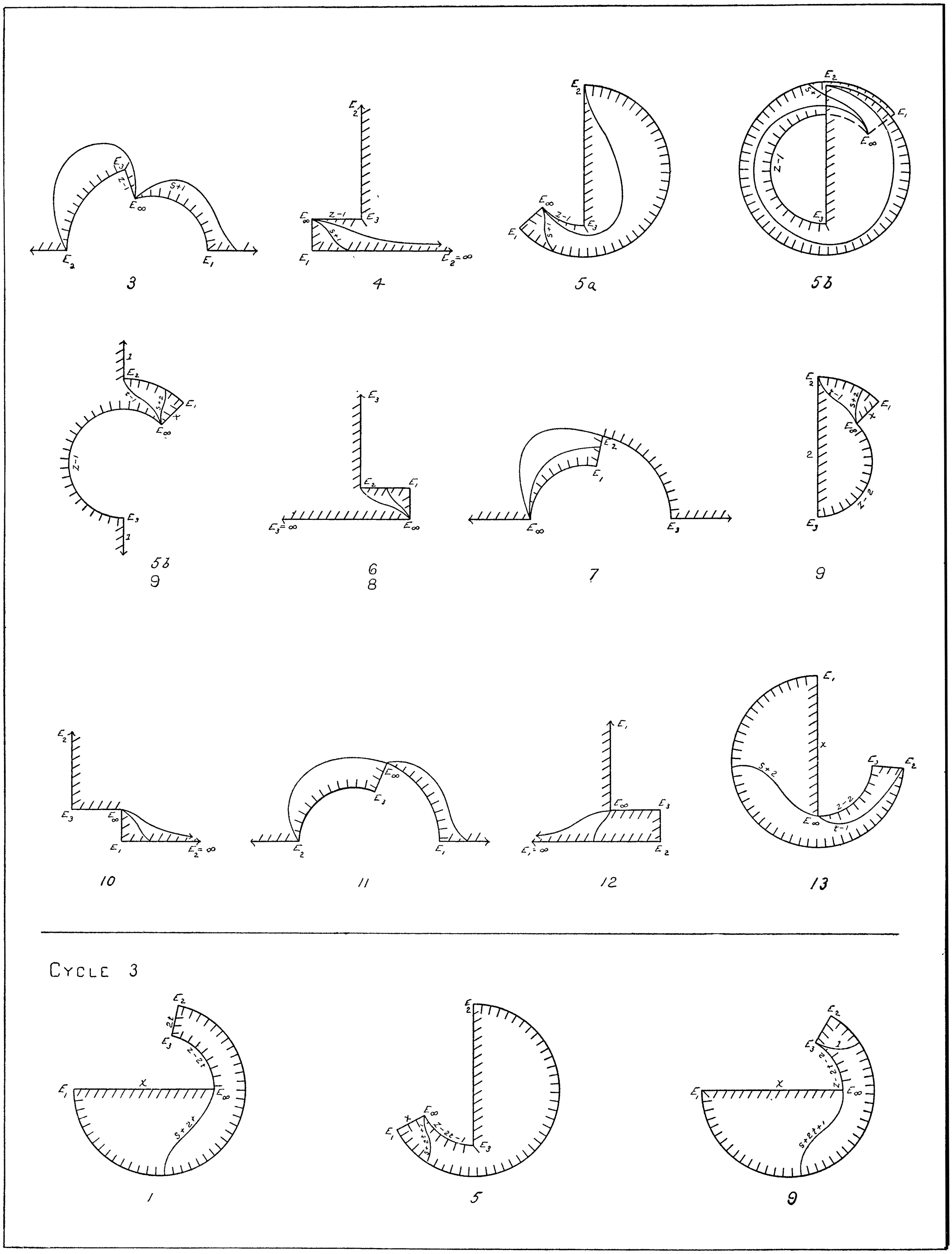


PLATE VI

Cycle I

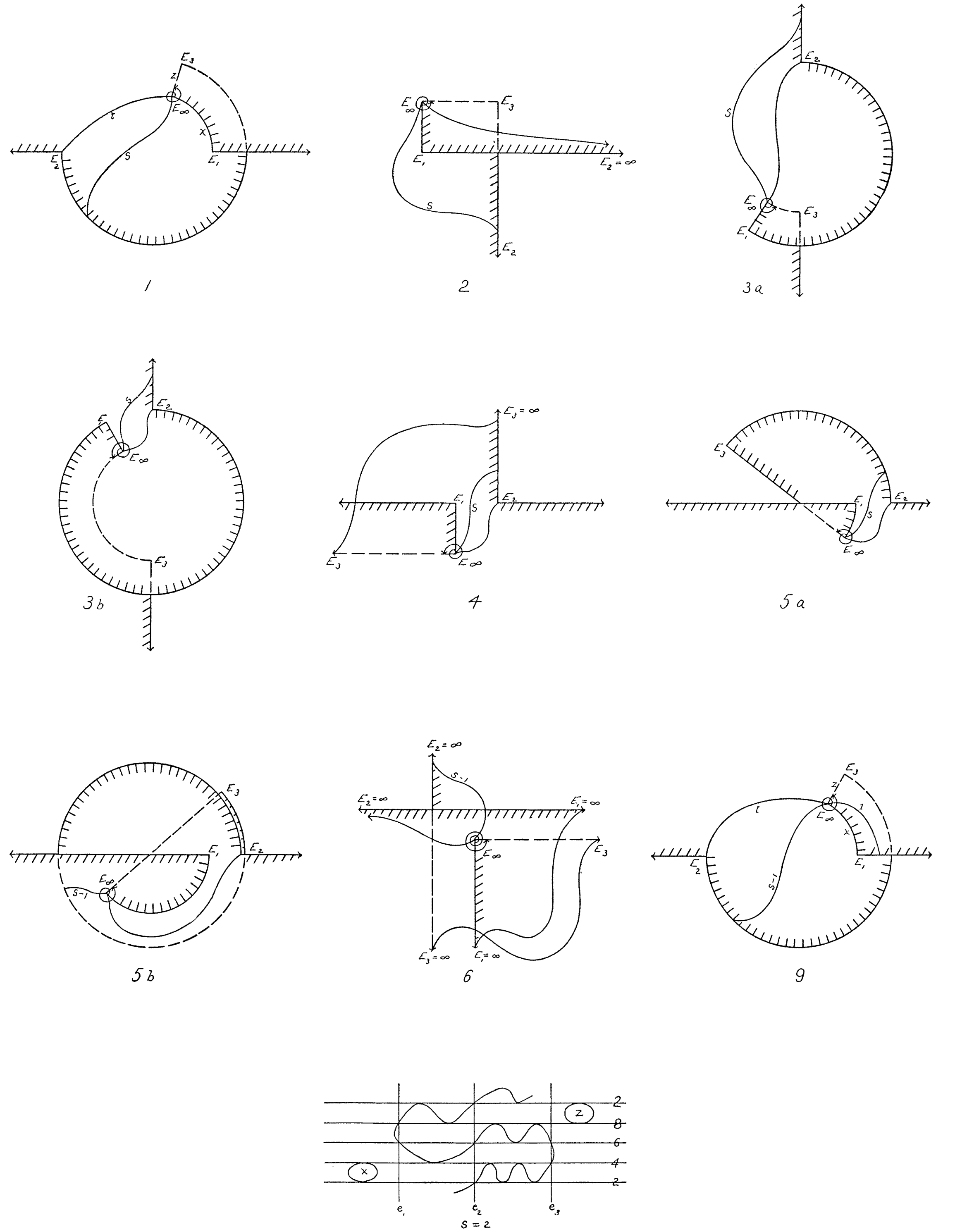
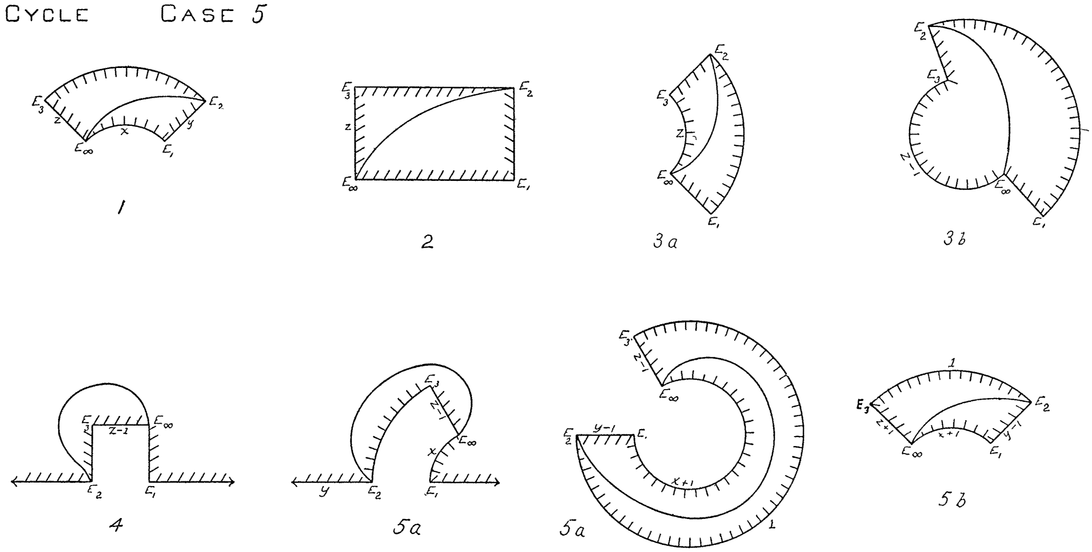
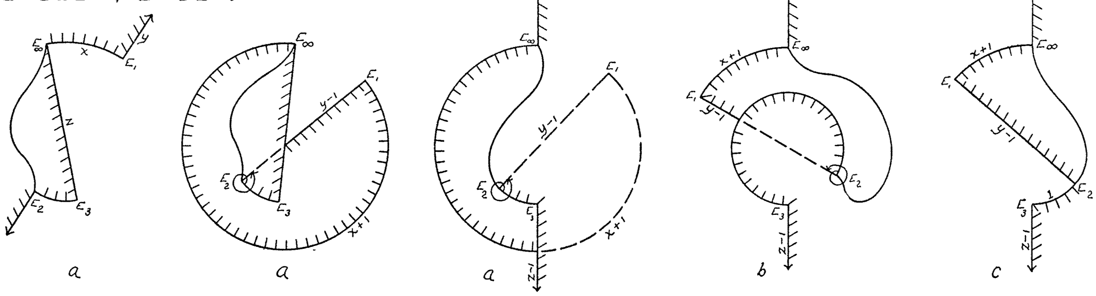


Plate VII

Cycle Case 5



Cycle I, Case 6



Cycle I, Case 7

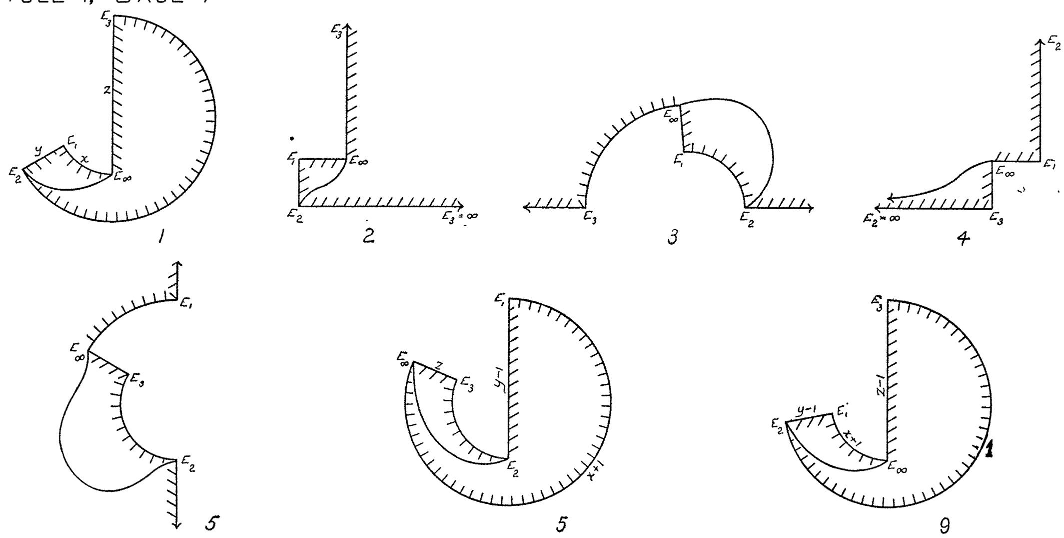


Plate VIII

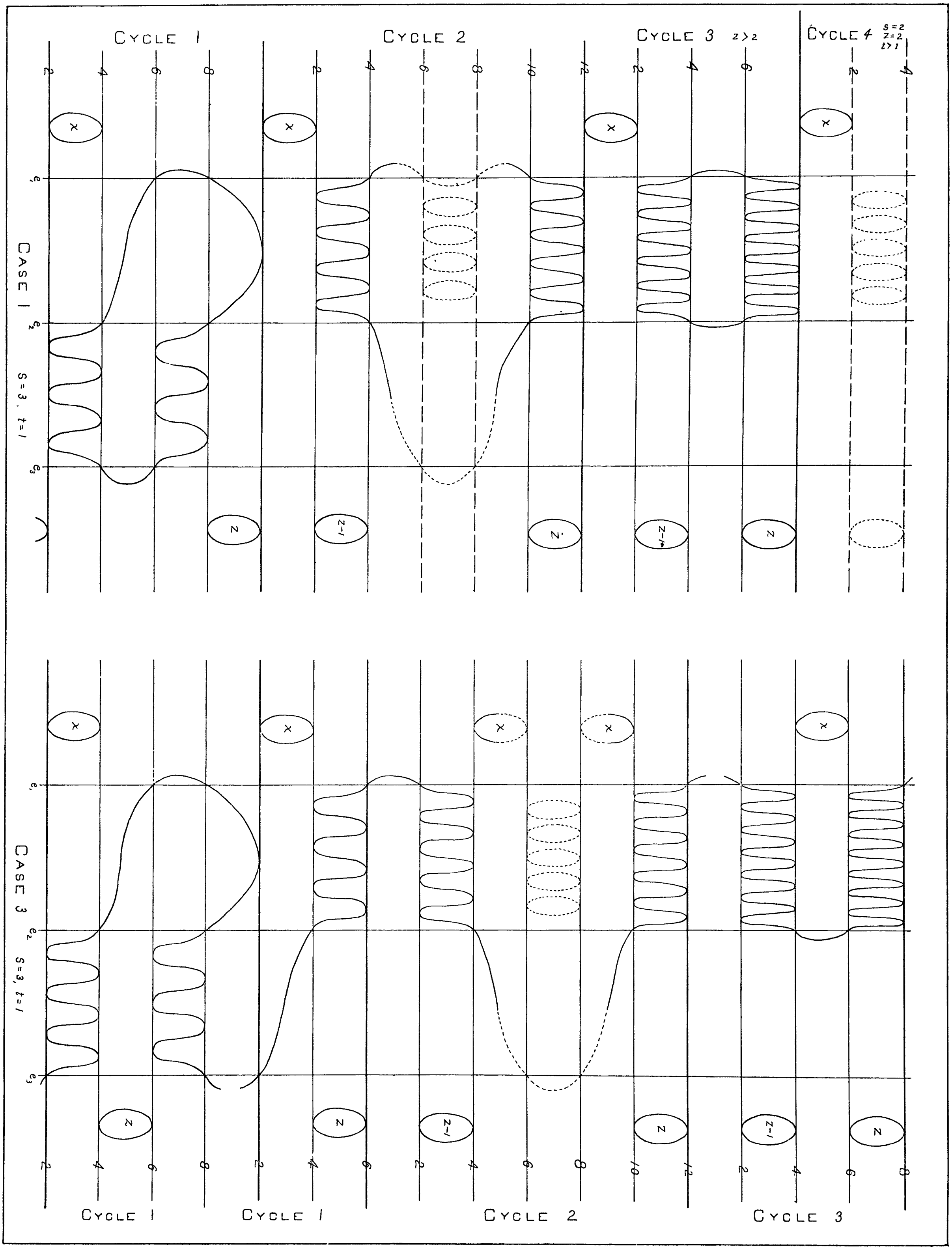

CYCLE 1
CYCLE 2
CYCLE 3 z > 2
CYCLE 4 s = 2 z = 2 t > 1
CASE 1
s = 3, t = 1
CASE 3
s = 3, t = 1
CYCLE 1
CYCLE 1
CYCLE 2
CYCLE 3

PLATE IX

to ∞. Accordingly, for critical values of the parameter, the square root of the polynomial-product will have the form:

(1). $(x - e_3)^{m_3 + \frac{1}{2}} P_{\frac{n-1}{2} - m_3}$, (2). $(x - e_2)^{m_2 - \frac{1}{2}} P_{\frac{n-1}{2} - m_2}$,

(3). $(x - e_1)^{m_1 + \frac{1}{2}} P_{\frac{n-1}{2} - m_1}$, (4). $P_{\frac{n-1}{2} - m_4 = -\frac{n_\infty}{2}}$.

Four classes of polynomials are thereby distinguished, and their total number is

$$2n - [m_1 + m_2 + m_3 + m_4] + 2 = 4t + 2x + 2y + 2z + \begin{cases} 3, \text{ case } 7, \\ 5, \text{ case } 8. \end{cases}$$

Of these only $\begin{cases} 4t, \text{ case } 7, \\ 4(t+1), \text{ case } 8, \end{cases}$ belonging to the 1st and 3d classes can be imaginary.

It is noteworthy that in all 8 cases, with the single exception of case 6, the maximum number of imaginary polynomials is either $4t$ or $4(t+1)$, depending solely upon the number of diagonal attachments in the initial polygon.

WESLEYAN UNIVERSITY, *Aug.* 1898.