## Numbers and functions. Introduction to Vojta's analogy

Seminar talk by A. Eremenko, November 23, 1999, Purdue University.
Absolute values. Let $k$ be a field. An absolute value $v$ is a function $k \rightarrow \mathbf{R}$, $x \mapsto|x|_{v}$ with the following properties:

1. $|x|_{v} \geq 0$, and $|x|_{v}=0$ iff $x=0$,
2. $|x y|_{v}=|x|_{v}|y|_{v}$,
3. $|x+y|_{v} \leq|x|_{v}+|y|_{v} \leq 2 \max \left\{|x|_{v},|y|_{v}\right\}$.

If 3 is replaced by the stronger property
$3^{\prime} .|x+y|_{v} \leq \max \left\{|x|_{v},|y|_{v}\right\}$,
then $v$ us called non-archimedian. For an absolute value $v$ we put

$$
v(x)=-\log |x|_{v} .
$$

Then $v: k \rightarrow \mathbf{R} \cup\{-\infty\}$,

$$
\begin{gathered}
v(x y)=v(x)+v(y), \quad \text { and } \\
v(x+y) \geq\left\{\begin{array}{l}
\min \{v(x), v(y)\} \quad \text { in non-archimedian case, and } \\
\min \{v(x), v(y)\}-\log 2 \quad \text { in archimedian case. }
\end{array}\right.
\end{gathered}
$$

Example 1. (Classical). For $k=\mathbf{Q}, x \mapsto|x|_{\infty}$ is the usual, archimedian absolute value. For each prime $p$ we can write every rational number as $x=p^{s} m / n$, where $m$ and $n$ are not divisible by $p$. Then

$$
\begin{equation*}
|x|_{p}=p^{-s}, v_{p}(x)=s \log p \tag{1}
\end{equation*}
$$

is called $p$-adic absolute value. Up to proportionality of $v$-functions, these are the only absolute values in $\mathbf{Q}$ (A. Ostrowski's theorem). We have

$$
\prod_{v}|x|_{v}=1 \quad \text { or } \quad \sum_{v} v(x)=0 \quad \text { for every } x .
$$

This is called the Artin-Whaples Product Formula. For the filed $\mathbf{Q}$ it is equivalent to Euclid's theorem, that every rational number is a product of powers of primes:

$$
x= \pm \prod p_{j}^{s_{j}}, \quad \prod_{v}|x|_{v}=|x|_{\infty} \prod_{v \neq \infty}|x|_{v}=|x|_{\infty} \prod p_{j}^{-s_{j}}=1
$$

Example 2. (Classical). For $k=M(\overline{\mathbf{C}})$, the field of meromorphic functions on the Riemann sphere $\overline{\mathbf{C}}$, and $f \in k,|f|_{p}=e^{-p(f)}$, where $p(f)$ is the multiplicity of zero of $f$ at $p \in \overline{\mathbf{C}}$ (it is negative if $p$ is a pole; $p(0)=\infty$ ). This time the Product Formula is

$$
\sum_{p} p(f)=0
$$

Each absolute value defines a metric and thus a topology on $k$. Equivalent topologies correspond to proportional functions $v$. One also considers completion of a field with respect to an absolute value. Thus in Example 1 we obtain $\mathbf{R}$ as the completion of $\mathbf{Q}$ with respect to $\left|\left.\right|_{\infty}\right.$, and $\mathbf{Q}_{p}$, the $p$-adic numbers fields. In Example 2 we obtain for each $p$ the field of formal Laurent series (with finitely many negative powers) at $p$. Absolute values can be extended to algebraic extensions of a field. In Example 2 one obtains the Puiseaux series in this way.

Example 3. (Vojta). Let $f$ be a meromorphic function in $|z| \leq r$ with $f(0)=1$. Then, for $\theta \in[-\pi, \pi)$

$$
|f|_{r, \theta}=\left|f\left(r e^{i \theta}\right)\right|
$$

is an archimedian absolute value (well, almost; the second part of condition 1 is not satisfied), and for each $p \in \mathbf{C},|p|<r$ :

$$
|f|_{r, p}=\left|\frac{p}{r}\right|^{-p(f)}, \quad v_{r, p}=p(f) \log \left|\frac{p}{r}\right|
$$

is a non-archimedian absolute value (compare with (1)). To write the Product Formula, we have to average over the infinitely many archimedian absolute values:

$$
\sum_{|p|<r} p(f) \log \left|\frac{p}{r}\right|+\frac{1}{2 \pi} \int_{-\pi}^{\pi} \log \left|f\left(r e^{i \theta}\right)\right| d \theta=0
$$

Another name for this is Jensen's formula.
Heights in projective spaces. Let $\mathbf{P}^{n}$ be the $n$-dimensional projective space over $k$, that is the set of non-zero vectors $\left(x_{0}, \ldots, x_{n}\right) \in k^{n+1}$ up to
proportionality. The (logarithmic) height is a function $h: \mathbf{P}^{n} \rightarrow \mathbf{R}$ defined for a point $P=\left(x_{0}: \ldots: x_{n}\right) \in \mathbf{P}^{n}$ by

$$
h(P)=\sum_{v} \log \max _{j}\left|x_{j}\right|_{v}=-\sum_{v} \min _{j} v\left(x_{j}\right),
$$

where the summation extends to a set of absolute values, satisfying the Product Formula. It follows from the Product formula that $h$ is well defined.

Example 1, again. If $k=\mathbf{Q}$, we use $P=\left(m_{0}: \ldots: m_{n}\right)$, where $m_{j}$ are integers without a common factor, and obtain

$$
\begin{aligned}
h(P) & =\sum_{p \neq \infty} \log \max \left\{\left|m_{0}\right|_{p}, \ldots,\left|m_{n}\right|_{p}\right\}+\log \max \left\{\left|m_{0}\right|_{\infty} \ldots,\left|m_{n}\right|_{\infty}\right\} \\
& =\max \left\{\log \left|m_{0}\right|, \ldots, \log \left|m_{n}\right|\right\}
\end{aligned}
$$

because $\max \left\{\left|m_{0}\right|_{p}, \ldots,\left|m_{n}\right|_{p}\right\}=1$ for each prime $p$, because our homogeneous coordinates $m_{j}$ have no common factor.
Example 2, again. If $k=M(\overline{\mathbf{C}})$, the field of rational functions, a point $P \in \mathbf{P}^{n}$ can be written as $P=\left(f_{0}: \ldots: f_{n}\right)$, where $f_{j}$ are polynomials without a common factor, and again only the summand with $p=\infty$ is different from zero:

$$
h(P)=\log \max \left\{\left|f_{0}\right|_{\infty}, \ldots,\left|f_{n}\right|_{\infty}\right\}=\max \left\{\operatorname{deg} f_{0}, \ldots, \operatorname{deg} f_{n}\right\}
$$

In particular, $h(x)$ for $x \in k$ is the degree of a rational function $x$.
Example 3, again. Take $k=\mathbf{C}$ and consider a holomorphic curve $f$ : $B(r) \rightarrow \mathbf{P}^{n}$, where $B(r)=\{z \in \mathbf{C}:|z| \leq r\}$ and $\mathbf{P}^{n}$ is complex projective space. We write $f=\left(f_{0}: \ldots: f_{n}\right)$, where $f_{j}$ are holomorphic functions without common zeros. Then the logarithmic height should be

$$
\begin{aligned}
h_{f}(r) & =" \sum_{v} " \log \max \left\{\left|f_{0}\right|_{v}, \ldots,\left|f_{n}\right|_{v}\right\}=" \sum_{v \in S} " \log \max \left\{\left|f_{0}\right|_{v}, \ldots,\left|f_{n}\right|_{v}\right\} \\
& =\frac{1}{2 \pi} \int_{-\pi}^{\pi} \log \max \left\{\left|f_{0}\left(r e^{i \theta}\right)\right|, \ldots,\left|f_{n}\left(r e^{i \theta}\right)\right|\right\} d \theta
\end{aligned}
$$

where $S=\{v=(r, \theta):|\theta| \leq \pi\}$ is the set of archimedian absolute values.
Another name for this is the Nevanlinna-Cartan characteristic of a holomorphic curve, $T_{f}(r)$. When $n=1$ one obtains the usual Nevanlinna or

Ahlfors characteristic. (They differ by a bounded term, but coincide if $f(0)=1$ ) .

Lang writes:
"The CR note where Cartan announced his Second Main Theorem for the case of hyperplanes was published in 1929, essentially at the same time as Weil's thesis in 1928, where he uses the height which today bears his name and which was defined simultaneously by Siegel [1929]. But no one at the time saw that Cartan's definition of this height was entirely analogous to the definition of the heights in algebraic number theory, and that both were based on the product formula. This gap in understanding is, to me, almost as striking as the gap in understanding between Artin and Hecke in Hamburg about the connection between nonabelian L-series and modular forms. One had to await 40 to 50 years for the connections to be made, conjecturally, by Langlands and Vojta respectively in these two cases. In both cases, some algebraic number theorist's failure to relate properly to analysis (and conversely) contributed to that gap of understanding".

Thue-Siegel-Roth Theorem. Usual formulation: for every algebraic number $x$, and all but finitely many rational numbers $P=m / n$ we have

$$
|x-y| \geq \max \{m, n\}^{-2+\epsilon}
$$

or, using our notation,

$$
-\log |x-y|_{\infty} \leq(2+\epsilon) h(y)
$$

except for finitely many $y$. The following generalization to $p$-adic absolute values is due to Mahler: let $k$ be a number field ( $=$ finite extension of $\mathbf{Q}$ ), $S$ a finite set of absolute values, and $\epsilon>0$. For each $v \in S$ choose $a_{v} \in \overline{\mathbf{Q}}$. Then the inequality

$$
\sum_{v \in S} v^{+}\left(y-a_{v}\right) \leq(2+\epsilon) h(y)
$$

holds for all but finitely many $y \in k$. Notice that "all but finitely many" is the same as "all but a set of bounded height".

Weak form of the Second Main Theorem of Nevanlinna. For every non-constant meromorphic function $f$ in $\mathbf{C}$, for every finite set of constants $\{a\}$, for every $\epsilon>0$ we have

$$
\begin{array}{r}
\sum_{a} \frac{1}{2 \pi} \int_{-\pi}^{\pi} \log ^{+}\left|f\left(r e^{i \theta}\right)-a\right|^{-1} d \theta \\
=\sum_{a} \frac{1}{2 \pi} \int_{-\pi}^{\pi} v_{r, \theta}^{+}(f-a) d \theta \leq(2+\epsilon) h_{f}(r), \tag{2}
\end{array}
$$

where $r \notin E_{f}(\epsilon)$, an exceptional set of finite length. Notice that a "nonconstant meromorphic function" is the same as a "meromorphic function of bounded height".

We call this form of the Second Main Theorem "weak" for two reasons. One is that one can write better "error term", then $\epsilon h_{f}(r)$. But more important omission in (2) is the ramification term $N_{1, f}(r)$, which counts the total multiplicity of $a$-points in $B(r)$. In particular, the Second Main Theorem with the ramification term implies: for three non-proportional entire functions $f_{0}, f_{1}, f_{2}$ without common zeros, and with $f_{0}+f_{1}+f_{2}=0$

$$
\frac{1}{2 \pi} \int_{-\pi}^{\pi} \log \max \left\{\left|f_{0}\right|,\left|f_{1}\right|,\left|f_{2}\right|\right\}\left(r e^{i \theta}\right) d \theta \leq(1+o(1)) \bar{N}\left(r, f_{0} f_{1} f_{2}\right), \quad r \notin E
$$

where $\bar{N}$ is the usual averaged counting function of different zeros, and $E$ is a set of finite length.

Example 1, again. Let us denote by $\bar{n}(m)$ the number of different primes which divide an integer $m$. The following statement is known as the abcconjecture of Masser and Oesterlé: for every $\epsilon>0$ there exists $C(\epsilon)$, such that for any non-zero relatively prime integers $a, b, c$ with $a+b+c=0$

$$
\max \{|a|,|b|,|c|\} \leq C(\epsilon)(\bar{n}(a b c))^{1+\epsilon}
$$

This easily implies that each Fermat's equation has only finitely many solutions, but unlike the Fermat Theorem, $a b c$ is still a conjecture.

Example 2, again. Let us denote by $\bar{n}(f)$ the number of different zeros of a polynomial $f$. The following statement is known as Mason's theorem: for any relatively prime non-proportional polynomials $a, b$, $c$, with $a+b+c=0$

$$
\max \{\operatorname{deg} a, \operatorname{deg} b, \operatorname{deg} c\} \leq \bar{n}(a b c)-1
$$

Lang writes:
"Mason started a trend of thought by discovering an entirely new relation among polynomials, in a very original work as follows..."

HW Exercise: Derive Mason's theorem from the Riemann-Hurwitz formula.

## Appendix (not included in the seminar talk)

A1. The following generalization of the Thue-Siegel-Roth Theorem to several dimensions was obtained by Schmidt. For a hyperplane $H \subset \mathbf{P}^{n}$ given by $a_{0} x_{0}+\ldots+a_{n} x_{n}$ and an absolute value $v$ we define the Weil function

$$
\lambda_{v, H}(P)=-\log \frac{\left|a_{0} x_{0}+\ldots+a_{n} x_{n}\right|_{v}}{\max \left\{\left|x_{0}\right|_{v}, \ldots,\left|x_{n}\right|_{v}\right\}}, P=\left(x_{0}: \ldots: x_{n}\right)
$$

Let $k$ be a number field, $S$ a finite set of absolute values, $H_{1}, \ldots, H_{q}, q \geq n+2$ hyperplanes in general position on $\mathbf{P}^{n}$, and $\epsilon>0$. Then

$$
\sum_{j=1}^{q} \sum_{v \in S} \lambda_{v, H_{j}}(P) \leq(n+1+\epsilon) h(P)
$$

except those $P$ lying in a finite collection of hyperplanes (depending on $\left\{H_{j}\right\}$ and $\epsilon$.)

The corresponding statement for holomorphic curves is called Cartan's Second Main Theorem (in an improved form, due to Vojta): Let $H_{1}, \ldots, H_{q}, q \geq$ $n+2$ be hyperplanes in general position on $\mathbf{P}^{n}$, and $\epsilon>0$. Then

$$
\sum_{j=1}^{q} m\left(r, H_{j}, f\right):=\sum_{j=1}^{q} \frac{1}{2 \pi} \int_{-\pi}^{\pi} \lambda_{(r, \theta), H_{j}}(f) d \theta \leq(n+1+\epsilon) h_{f}(r)
$$

for all non-constant holomorphic curves $f: \mathbf{C} \rightarrow \mathbf{P}^{n}$, except those which lie in a finite collection of hyperplanes (depending on $\left\{H_{j}\right\}$ and $\epsilon$.)

A2. A common use of the Second Main Theorem(s) is to prove Picardtype theorems, that certain equations do not have meromorphic solutions, or have very few of them. Similarly, theorems on diophantine approximation can be used to prove that certain diophantine equations have few solutions. Thus the arithmetic counterpart of the classical Picard's Theorem is the
theorem of Thue and Siegel, about integral points on an affine curve of genus zero. Another example is Borel's Theorem: if $f_{1}, \ldots, f_{q}$ are zero-free entire functions, and

$$
f_{1}+\ldots+f_{q}=0
$$

then some of these functions are proportional. The arithmetic counterpart is due to Schlickewei.

## References

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A3. We summarize what was said (and a part of what was not) in the following table, partially taken from Vojta's book.

Vojta's Analogy

| Function Theory | Number Theory |
| :---: | :---: |
| $f: \mathbf{C} \rightarrow \mathbf{P}^{1}$ | $\{y\} \subset k$ |
| $r$ | $y$ |
| $\theta \in[-\pi, \pi)$ | a finite set $S$ |
| $\log \left\|f\left(r e^{i \theta}\right)-a\right\|^{-1}$ | $v(y-a), v \in S$ |
| $p(f), p \in \mathbf{C}$ | $v(y), v \notin S$ |
| Jensen's Formula | Artin-Whaples Product Formula |
| Proximity function | Weil's function |
| Nevanlinna's characteristic | Height on $k$ |
| Cartan's characteristic | Height on projective space |
| Selberg-Valiron characteristic | Absolute height |
| Nevanlinna's SMT | Thue-Siegel-Roth Theorem |
| Cartan's SMT | Schmidt's Theorem |
| and its improvement by Vojta | and its improvement by Vojta |
| Borel's Theorem | van der Poorten's Theorem |
| Selberg-Valiron SMT | Wirsing's Theorem |
| Picard's Theorem | Thue-Siegel's Theorem |
| Another Picard's Theorem | Faltings Theorem |
| Borel's Theorem | Schlickewei's Theorem |
| Precise error term in SMT | Lang's conjecture |
| SMT with the ramification term | ??? |
| its corollary for 3 functions | $a b c-$ conjecture |
| ?? (some arguments of Osgood?) | Roth's proof |
| ?? Ahlfors' proof of Cartan's SMT | Schmidt's proof |
| Cartan's and Nevanlinna's proofs | ??? |
| (Lemma on log derivative) | ??? |
| Curvature | ??? |
| moving target generalizations | even better analogy? |

