Lesson 13; Implicit Differentiation

Monday, September 29, 2025 10:26 AM

Friday from 10–11am in ET314 open office hrs w/ Dr. Swanson (Dept Head of Math)

<u>Explicit Form</u>: $y = f(x)$

<u>Implicit Form</u>: when a function is <u>NOT</u> written in explicit form

<u>ex.</u> ① $y - 2x = 1$ ② $x^2 + y^2 = 2$ ③ $y^2 + y - 1 = x$

To differentiate functions of this kind, we use a technique called <u>implicit differentiation</u>.

We namely use this technique when solving for y is messy.

<u>Ex 1</u>: Use implicit differentiation to find slope of tangent line of $x^2 - y^2 = 4x + 8y$

Remember $y' = \frac{dy}{dx} = \frac{d}{dx}[y]$

$$\frac{d}{dx}[x^2 - y^2] = \frac{d}{dx}[4x + 8y]$$

$$\frac{d}{dx}[x^2] - \frac{d}{dx}[y^2] = \frac{d}{dx}[4x] + \frac{d}{dx}[8y]$$

$$2x \cancelto{1}{\frac{dx}{dx}} - 2y\frac{dy}{dx} = 4\cancelto{1}{\frac{dx}{dx}} + 8\frac{dy}{dx}$$

$$2x - 2y\frac{dy}{dx} = 4 + 8\frac{dy}{dx}$$

Solve for $\frac{dy}{dx}$.

$$2x - 4 = 2y\frac{dy}{dx} + 8\frac{dy}{dx}$$

$$2x - 4 = (2y + 8)\frac{dy}{dx} \Longrightarrow \frac{dy}{dx} = \frac{2x - 4}{2y + 8}$$

Ex 2: Use implicit differentiation to find dy/dx of

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$$yx^2 + e^y = x$$

$$\frac{d}{dx}(yx^2 + e^y) = \frac{d}{dx}(x)$$

$$\underbrace{\frac{d}{dx}(yx^2)}_{\text{Product Rule}} + \frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$\frac{d}{dx}(y)\, x^2 + y \frac{d}{dx}(x^2) + \frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$1 \cdot \frac{dy}{dx} \cdot x^2 + y \cdot 2x \cancel{\frac{dx}{dx}}^{\,1} + e^y \frac{dy}{dx} = 1 \cdot \cancel{\frac{dx}{dx}}^{\,1}$$

$$x^2 \frac{dy}{dx} + 2xy + e^y \frac{dy}{dx} = 1$$

Solve for dy/dx.

$$x^2 \frac{dy}{dx} + e^y \frac{dy}{dx} = 1 - 2xy$$

$$(x^2 + e^y)\frac{dy}{dx} = 1 - 2xy$$

$$\frac{dy}{dx} = \frac{1-2xy}{x^2+e^y}$$

Ex 3: Use implicit to find dy/dx of

$$4\sin(x)\cos(y) = 3$$

$$\frac{d}{dx}\left(4\sin(x)\cos(y)\right) = \frac{d}{dx}(3)$$

$$\frac{d}{dx}(4\sin(x))\cos(y) + 4\sin(x)\frac{d}{dx}(\cos(y)) = \frac{d}{dx}(3)$$

$$4\cos(x)\cancel{\frac{dx}{dx}}^{\,1}\cos(y) + 4\sin(x)(-\sin(y))\frac{dy}{dx} = 0 \quad \cancel{\frac{dx}{dx}}^{\,1} = 0$$

$$4\cos(x)\cos(y) - 4\sin(x)\sin(y)\frac{dy}{dx} = 0$$

Solve for dy/dx.

$$4\cos(x)\cos(y) = 4\sin(x)\sin(y)\frac{dy}{dx}$$

$$\cot(x)\cot(y) = \frac{4\cos(x)\cos(y)}{4\sin(x)\sin(y)} = \frac{dy}{dx}$$

$$\cot(x)\cot(y) = \frac{4\cos(x)\cos(y)}{4\sin(x)\sin(y)} = \frac{dy}{dx}$$

Ex 4: Use implicit to find dy/dx of

$$6\tan(2x+3y) = 11x$$

$$\underbrace{\frac{d}{dx}\left(6\tan(2x+3y)\right)}_{\text{Use chain Rule}} = \frac{d}{dx}(11x)$$

$$6\sec^2(2x+3y)\frac{d}{dx}(2x+3y) = \frac{d}{dx}(11x)$$

$$6\sec^2(2x+3y)\left[\frac{d}{dx}(2x) + \frac{d}{dx}(3y)\right] = \frac{d}{dx}(11x)$$

Remember

$\cos(x) \cdot \sec(x) = 1$

$$2\cancelto{1}{\frac{dx}{dx}} + 3\frac{dy}{dx}$$

$$6\sec^2(2x+3y)\left[2 + 3\frac{dy}{dx}\right] = 11$$

$$\sec^2(2x+3y)\left[2 + 3\frac{dy}{dx}\right] = \frac{11}{6}$$

$$1 = \boxed{\cos^2(2x+3y)\sec^2(2x+3y)}\left[2 + 3\frac{dy}{dx}\right] = \frac{11}{6}\cos^2(2x+3y)$$

$$2 + 3\frac{dy}{dx} = \frac{11}{6}\cos^2(2x+3y)$$

$$3\frac{dy}{dx} = \frac{11}{6}\cos^2(2x+3y) - 2$$

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{11}{6}\cos^2(2x+3y) - 2\right)$$