Lesson 15: Derivatives of y=(f(x))^(g(x))

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Lesson 15: Derivatives of $y = f(x)^{g(x)}$

Watch video: Link here

Key takeaways:

- $\ln(x^m) = m\ln(x)$
- Remember take ln of both sides
- Remember at least product rule
- and implicit where $\frac{d}{dx}(\ln(y)) = \frac{1}{y} \cdot \frac{dy}{dx}$

Ex 1: Find $h'(x)$ when

ⓐ $h(x) = x^{\sqrt{x}}$

Rewrite first $\sqrt{x} = x^{1/2}$

$y = x^{x^{1/2}}$ → This is y.

Take ln of both

$\ln(y) = \ln\left[(x)^{x^{1/2}}\right]$

$\ln(y) = x^{1/2}\ln(x)$

$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\left(x^{1/2}\ln(x)\right)$

Product Rule

$(uv)' = uv' + u'v$

$u = x^{1/2}$ $\quad v = \ln(x)$

$u' = \frac{1}{2}x^{-1/2}$ $\quad v' = \frac{1}{x}$

$= \frac{1}{2\sqrt{x}}$

$\frac{1}{y}\frac{dy}{dx} = \left(x^{1/2}\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}}\ln(x)\right)\cancel{\frac{dx}{dx}}^{\,1}$

$$\frac{1}{y}\frac{dy}{dx} = \left(x^{1/2}\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}}\ln(x)\right)\cancel{\frac{dx}{dx}}$$

$$\frac{1}{y}\boxed{\frac{dy}{dx}} = \frac{1}{x^{1/2}} + \frac{1}{2\sqrt{x}}\ln(x)$$

↳ Solve

$$\frac{dy}{dx} = y\left(\frac{1}{x^{1/2}} + \frac{1}{2\sqrt{x}}\ln(x)\right)$$

$$= x^{\sqrt{x}}\left(\frac{1}{x^{1/2}} + \frac{1}{2\sqrt{x}}\ln(x)\right)$$

(b) $h(x) = x^{\sin(x)}$

Rewrite $y = x^{\sin(x)}$ ✱

$$\ln(y) = \ln(x^{\sin(x)})$$

$$\ln(y) = \sin(x)\ln(x)$$

$u = \sin(x)$ $v = \ln(x)$

$u' = \cos(x)$ $v' = \frac{1}{x}$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \sin(x)\cdot\frac{1}{x} + \cos(x)\ln(x)$$

$$\frac{dy}{dx} = y\left(\sin(x)\cdot\frac{1}{x} + \cos(x)\ln(x)\right)$$

$$= x^{\sin(x)}\left(\sin(x)\cdot\frac{1}{x} + \cos(x)\ln(x)\right)$$

(c) $h(x) = 8x^{4x}$

$= 8(x)^{4x}$

Bonus problem: $h(x) = (8x)^{4x}$

Rewrite $y = 8(x)^{4x}$

$$\frac{y}{8} = x^{4x}$$

$$\frac{y}{8} = x$$

$$\ln\left(\frac{y}{8}\right) = \ln(x^{4x})$$

$$\ln\left(\frac{y}{8}\right) = 4x\ln(x)$$

$$\frac{d}{dx}\left(\ln\left(\frac{y}{8}\right)\right) = \frac{d}{dx}(4x\ln(x))$$

$u = 4x \qquad v = \ln(x)$

$u' = 4 \qquad v' = \frac{1}{x}$

$$\frac{1}{y/8} \cdot \frac{1}{8}\frac{dy}{dx} = 4x\left(\frac{1}{x}\right) + 4\ln(x)$$

$$\frac{\cancel{8}}{y} \cdot \frac{1}{\cancel{8}}\frac{dy}{dx} = 4 + 4\ln(x)$$

$$\frac{1}{y}\frac{dy}{dx} = 4 + 4\ln(x)$$

$$\frac{dy}{dx} = y(4 + 4\ln(x))$$

$$\frac{dy}{dx} = 8x^{4x}(4 + 4\ln(x))$$

(d) $h(x) = (2x)^{\ln(2x)}$

Rewrite $y = (2x)^{\ln(2x)}$

$$\ln(y) = \ln\left((2x)^{\ln(2x)}\right)$$

$$\ln(y) = \ln(2x) \cdot \ln(2x)$$

$$\ln(y) = [\ln(2x)]^2$$

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\left[[\ln(2x)]^2\right] \rightarrow \text{Chain Rule}$$

$$= 2[\ln(2x)]\frac{d}{dx}(\ln(2x))$$

$$= 2[\ln(2x)]\,\frac{d}{dx}(\ln(2x))$$

$$= 2\ln(2x)\cdot\frac{1}{2x}\cdot\frac{d}{dx}(2x)$$

$$= 2\ln(2x)\cdot\frac{1}{\cancel{2}x}\cdot\cancel{2}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{2\ln(2x)}{x}$$

$$\frac{dy}{dx} = y\left(\frac{2\ln(2x)}{x}\right) = (2x)^{\ln(2x)}\left(\frac{2\ln(2x)}{x}\right)$$