Start
10/8

MA 165 LESSONS 17-18: RELATED RATES

Example 1: If x and y are both functions of t and $x + y^3 = 2$.

(a) Find $\frac{dy}{dt}$ when $\boxed{\frac{dx}{dt} = -2}$ and $\boxed{y = 1}$

$$\frac{d}{dt}(x + y^3) = \frac{d}{dt}(2)$$

$$1 \cdot \frac{dx}{dt} + 3y^2\frac{dy}{dt} = 0$$

$$\frac{dx}{dt} + 3y^2\frac{dy}{dt} = 0$$

$$-2 + 3(1)^2\frac{dy}{dt} = 0$$

$$3\frac{dy}{dt} = 2$$

$$\frac{dy}{dt} = \frac{2}{3}$$

(b) Find $\frac{dx}{dt}$ when $\frac{dy}{dt} = 3$ and $x = 1$.

$$\frac{dx}{dt} + 3y^2\frac{dy}{dt} = 0$$

$$\frac{dx}{dt} + 3y^2(3) = 0$$

$$\frac{dx}{dt} = -9y^2$$

$$\frac{dx}{dt} = -9(1)^2 = -9$$

well $x + y^3 = 2$

Plug $x = 1$ into

$1 + y^3 = 2$

$y^3 = 1$

$y = 1$

Recipe for Solving a Related Rates Problem

Step 1: Draw a good picture. Label all constant values and give variable names to any changing quantities.

Step 2: Determine what information you **KNOW** and what you **WANT** to find.

Step 3: Find an equation relating the relevant variables. This usually involves a formula from geometry, similar triangles, the Pythagorean Theorem, or a formula from trigonometry. **Use your picture!**

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

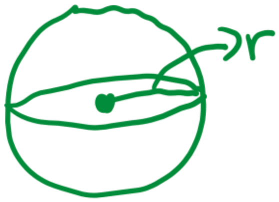
Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you **WANT**. Do **NOT** substitute before this step!

Some Useful Formulas

Right Triangle *Pythagorean Theorem:* $a^2 + b^2 = c^2$	Triangle $A = \frac{1}{2}bh$ $P = a + b + c$	Trapezoid $A = \frac{1}{2}(a + b)h$
Rectangular Box $V = lwh$ $S = 2(hl + lw + hw)$	Rectangle $A = lw$ $P = 2l + 2w$	Circle $A = \pi r^2$ $C = 2\pi r$
Right Circular Cylinder $V = \pi r^2 h$ $SA = 2\pi rh$	Sphere $V = \frac{4}{3}\pi r^3$ $S = 4\pi r^2$	Cone $V = \frac{1}{3}\pi r^2 h$ $SA = \pi rl + \pi r^2$

Example 2: A spherical balloon is being deflated at a constant rate of 20 cubic cm per second. How fast is the radius of the balloon changing at the instant when the balloon's radius is 12 cm?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dV}{dt} = -20 \frac{cm^3}{s}$ **WANT:** $\left.\frac{dr}{dt}\right|_{r=12}$

Step 3: Find an equation relating to the relevant variables.

$$V = \frac{4}{3}\pi r^3$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = \frac{4}{3}\pi(3)r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT**.**

$$-20 = 4\pi(12)^2 \frac{dr}{dt}$$

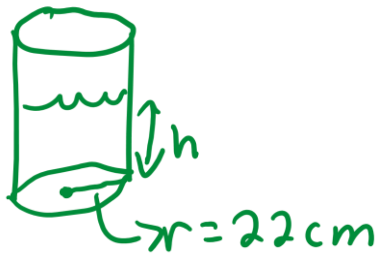
$$\frac{-\cancel{20}^{\,5}}{4\pi(144)} = \frac{dr}{dt}$$

$$\frac{-5}{144\pi} \frac{cm}{s} = \frac{dr}{dt}$$

Example 3: A cylindrical tank standing upright (with one circular base on the ground) has a radius of 22 cm for the base. How fast does the water level in the tank drop when the water is being drained at 28 cm^3/sec? Note: The formula right circular cylinder is

$$V = \pi r^2 h.$$

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $r = 22\text{cm}$

$\frac{dV}{dt} = -28 \frac{\text{cm}^3}{\text{s}}$

WANT: $\frac{dh}{dt}$

Step 3: Find an equation relating to the relevant variables.

$$V = \pi r^2 h \iff V = \pi(22)^2 h$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\pi (22)^2 h\right)$$

$$\frac{dV}{dt} = (22)^2\pi \frac{dh}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT**.**

$$-28 = (22)^2 \pi \frac{dh}{dt}$$

$$-\cancel{4}\cdot 7 = \cancel{2^2}\cdot 11^2 \pi \frac{dh}{dt}$$

$$\frac{-7}{121\pi}\frac{\text{cm}}{\text{s}} = \frac{dh}{dt}$$

cone

<u>Example 4:</u> Sand falls from an overhead bin and accumulates in a conical pile with a radius that is always four times its height. Suppose the height of the pile increases at a rate of 3 cm/s when the pile is 15 cm high. At what rate is the sand leaving the bin at that instant? Note: The formula of conical pile (aka cone) is $V = \frac{\pi}{3} r^2 h$.

h

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.

Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $r = 4h$

$\frac{dh}{dt} = 3 \frac{cm}{s}$

WANT: $\left.\frac{dV}{dt}\right|_{h=16\text{ cm}}$

Step 3: Find an equation relating to the relevant variables.

$V = \frac{\pi}{3} r^2 h$ Plug in $r = 4h$ $V = \frac{\pi}{3}(4h)^2 h = \frac{16}{3}\pi h^3$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{16}{3}\pi h^3\right)$

$\frac{dV}{dt} = \frac{16}{\cancel{3}}\pi(\cancel{3})h^2 \frac{dh}{dt}$

$\frac{dV}{dt} = 16\pi h^2 \frac{dh}{dt}$

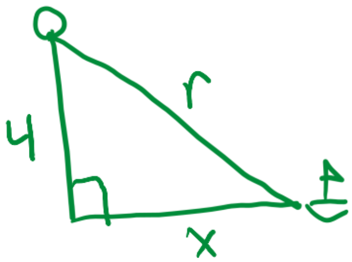
Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$\frac{dV}{dt} = 16\pi(15)^2 \cdot 3$

$= 10800 \frac{cm^3}{s}$

Example 5: A rope passing through a capstan on a dock is attached to a boat offshore at water level. The rope is pulled in at a constant rate of 5 ft/s, and the capstan is 4 ft vertically above the water. How fast is the boat traveling when it is 10 ft from the dock?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



r-rope
x - distance from capstan to boat

Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dr}{dt} = 5\ \frac{ft}{s}$ **WANT:** $\left.\frac{dx}{dt}\right|_{x=10ft}$

Step 3: Find an equation relating to the relevant variables.

$$x^2 + 4^2 = r^2 \iff x^2 + 16 = r^2$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(x^2 + 16) = \frac{d}{dt}(r^2)$$

$$2x\frac{dx}{dt} = 2r\frac{dr}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$2(10)\frac{dx}{dt} = 2r(5)$$

$$\frac{dx}{dt} = \frac{10r}{20} = \frac{r}{2}$$

$$\frac{dx}{dt} = \frac{\sqrt{116}}{2}\ \frac{ft}{s}$$

To find r use x=10 and plug into eqn,

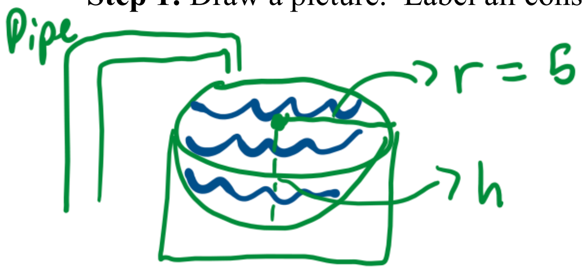
$$x^2 + 16 = r^2$$

$$10^2 + 16 = r^2$$

$$\sqrt{116} = r$$

Example 6: A hemispherical tank with a radius of 5 m is filled from an inflow pipe at a rate of 4 m^3/min. How fast is the water level rising when the water level is 3 m from the bottom of the tank? (Hint: The volume of a cap of thickness h sliced from a sphere of radius r is $\frac{\pi h^2(3r-h)}{3}$).

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dV}{dt} = 4 \frac{m^3}{min}$ $r = 5$

WANT: $\left.\frac{dh}{dt}\right|_{h=3\ m}$

Step 3: Find an equation relating to the relevant variables.

$$V = \pi \frac{h^2(3r-h)}{3}$$

Plug $r=5$

$$V = \frac{\pi h^2}{3}(3(5)-h) = \frac{\pi}{3} h^2 (15-h)$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

Rewrite

$$V = \frac{\pi}{3}[15h^2 - h^3]$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{\pi}{3}[15h^2 - h^3]\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3}(30h - 3h^2)\frac{dh}{dt}$$

$$\frac{dV}{dt} = \pi(10h - h^2)\frac{dh}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$4 = \pi(10(3) - 3^2)\frac{dh}{dt}$$

$$4 = \pi(30 - 9)\frac{dh}{dt}$$

$$4 = 21\pi \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{21\pi} \frac{m}{min}$$

<u>Example 7:</u> A plane is flying directly away from you at 500 mph at an altitude of 3 miles.

(a) How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.

Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: **WANT:**

Step 3: Find an equation relating the relevant variables.

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT**.**

Example 7: A plane is flying directly away from you at 500 mph at an altitude of 3 miles.
(b) How fast is the angle of elevation changing when it is $\pi/3$?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.

Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: **WANT:**

Step 3: Find an equation relating the relevant variables.

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

Example 8: A ladder 5 meters long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.3 m/sec. How fast is the top sliding down the wall when the foot of the ladder is 3 m from the wall?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.

Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: **WANT:**

Step 3: Find an equation relating the relevant variables.

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT**.**

<u>Example 9:</u> A streetlight fastened to the top of a 20-ft high pole. If a 5-ft tall woman walks away from the pole in a straight line over level ground at a rate of 6 ft/s, how fast is the length of her shadow changing when she is 18 ft away from the pole?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.

Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: **WANT:**

Step 3: Find an equation relating the relevant variables.

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

<u>Example 10:</u> A lighthouse is 375 m from a straight shoreline. Its light rotates 5 times per minute. How fast is the light spot moving along the shore when it hits a point 175 m away from the point nearest the lighthouse?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.

Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: **WANT:**

Step 3: Find an equation relating the relevant variables.

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT**.**