Start 10/8

MA 165 LESSONS 17-18: RELATED RATES

Example 1: If x and y are both functions of t and $x + y^3 = 2$.

(a) Find $\frac{dy}{dt}$ when $\boxed{\frac{dx}{dt} = -2}$ and $\boxed{y = 1}$

$\frac{d}{dt}(x + y^3) = \frac{d}{dt}(2)$

$1 \cdot \frac{dx}{dt} + 3y^2 \frac{dy}{dt} = 0$

$\frac{dx}{dt} + 3y^2 \frac{dy}{dt} = 0$

$-2 + 3(1)^2 \frac{dy}{dt} = 0$

$3 \frac{dy}{dt} = 2$

$\frac{dy}{dt} = \frac{2}{3}$

(b) Find $\frac{dx}{dt}$ when $\frac{dy}{dt} = 3$ and $x = 1$.

well $x + y^3 = 2$

Plug $x = 1$ into

$1 + y^3 = 2$

$y^3 = 1$

$y = 1$

$\frac{dx}{dt} + 3y^2 \frac{dy}{dt} = 0$

$\frac{dx}{dt} + 3y^2(3) = 0$

$\frac{dx}{dt} = -9y^2$

$\frac{dx}{dt} = -9(1)^2 = -9$

Recipe for Solving a Related Rates Problem

Step 1: Draw a good picture. Label all constant values and give variable names to any changing quantities.

Step 2: Determine what information you **KNOW** and what you **WANT** to find.

Step 3: Find an equation relating the relevant variables. This usually involves a formula from geometry, similar triangles, the Pythagorean Theorem, or a formula from trigonometry. **Use your picture!**

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

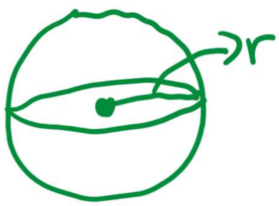
Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you **WANT**. Do **NOT** substitute before this step!

Some Useful Formulas

Right Triangle *Pythagorean Theorem:* $a^2 + b^2 = c^2$	Triangle $A = \frac{1}{2}bh$ $P = a + b + c$	Trapezoid $A = \frac{1}{2}(a+b)h$
Rectangular Box $V = lwh$ $S = 2(hl + lw + hw)$	Rectangle $A = lw$ $P = 2l + 2w$	Circle $A = \pi r^2$ $C = 2\pi r$
Right Circular Cylinder $V = \pi r^2 h$ $SA = 2\pi rh$	Sphere $V = \frac{4}{3}\pi r^3$ $S = 4\pi r^2$	Cone $V = \frac{1}{3}\pi r^2 h$ $SA = \pi rl + \pi r^2$

<u>Example 2:</u> A spherical balloon is being deflated at a constant rate of 20 cubic cm per second. How fast is the radius of the balloon changing at the instant when the balloon's radius is 12 cm?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dV}{dt} = -20\ \frac{cm^3}{s}$ **WANT:** $\left.\frac{dr}{dt}\right|_{r=12}$

Step 3: Find an equation relating to the relevant variables.

$$V = \frac{4}{3}\pi r^3$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = \frac{4}{3}\pi(3)r^2\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT**.**

$$-20 = 4\pi(12)^2\frac{dr}{dt}$$

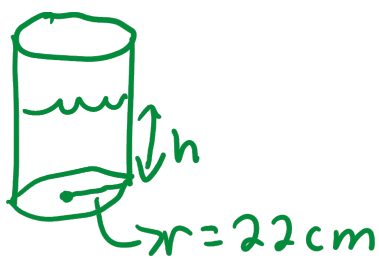
$$\frac{-\overset{5}{\cancel{20}}}{\cancel{4}\pi(144)} = \frac{dr}{dt}$$

$$\frac{-5}{144\pi}\ \frac{cm}{s} = \frac{dr}{dt}$$

Example 3: A cylindrical tank standing upright (with one circular base on the ground) has a radius of 22 cm for the base. How fast does the water level in the tank drop when the water is being drained at 28 cm^3/sec? Note: The formula right circular cylinder is

$$V = \pi r^2 h.$$

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $r = 22\text{cm}$, $\frac{dV}{dt} = -28\ \frac{\text{cm}^3}{\text{s}}$ **WANT:** $\frac{dh}{dt}$

Step 3: Find an equation relating to the relevant variables.

$$V = \pi r^2 h \iff V = \pi (22)^2 h$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\pi (22)^2 h\right)$$

$$\frac{dV}{dt} = (22)^2 \pi \frac{dh}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT**.**

$$-28 = (22)^2 \pi \frac{dh}{dt}$$

$$-\cancel{4} \cdot 7 = \cancel{2^2} \cdot 11^2 \pi \frac{dh}{dt}$$

$$\frac{-7}{121\pi}\ \frac{\text{cm}}{\text{s}} = \frac{dh}{dt}$$

cone

Example 4: Sand falls from an overhead bin and accumulates in a conical pile with a radius that is always four times its height. Suppose the height of the pile increases at a rate of 3 cm/s when the pile is 15 cm high. At what rate is the sand leaving the bin at that instant? Note: The formula of conical pile (aka cone) is $V = \frac{\pi}{3} r^2 h$.

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.

Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $r = 4h$, $\frac{dh}{dt} = 3\,\frac{cm}{s}$

WANT: $\left.\frac{dV}{dt}\right|_{h=16\text{ cm}}$

Step 3: Find an equation relating to the relevant variables.

$V = \frac{\pi}{3} r^2 h$ Plug in $r = 4h$ $V = \frac{\pi}{3}(4h)^2 h = \frac{16}{3}\pi h^3$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{16}{3}\pi h^3\right)$$

$$\frac{dV}{dt} = \frac{16}{\cancel{3}}\pi(\cancel{3})h^2\frac{dh}{dt}$$

$$\frac{dV}{dt} = 16\pi h^2 \frac{dh}{dt}$$

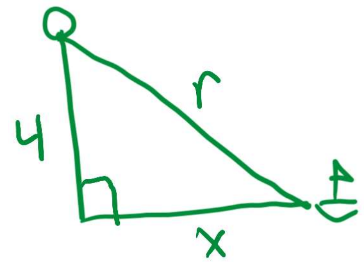
Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$\frac{dV}{dt} = 16\pi(15)^2 \cdot 3$$

$$= 10800\ \frac{cm^3}{s}$$

Example 5: A rope passing through a capstan on a dock is attached to a boat offshore at water level. The rope is pulled in at a constant rate of 5 ft/s, and the capstan is 4 ft vertically above the water. How fast is the boat traveling when it is 10 ft from the dock?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



r – rope

x – distance from capstan to boat

Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dr}{dt} = 5\,\frac{ft}{s}$ **WANT:** $\left.\frac{dx}{dt}\right|_{x=10ft}$

Step 3: Find an equation relating to the relevant variables.

$$x^2 + 4^2 = r^2 \iff x^2 + 16 = r^2$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(x^2 + 16) = \frac{d}{dt}(r^2)$$

$$2x\,\frac{dx}{dt} = 2r\,\frac{dr}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$2(10)\frac{dx}{dt} = 2r(5)$$

$$\frac{dx}{dt} = \frac{10r}{20} = \frac{r}{2}$$

$$\frac{dx}{dt} = \frac{\sqrt{116}}{2}\ \frac{ft}{s}$$

To find r use $x = 10$ and plug into eqn,

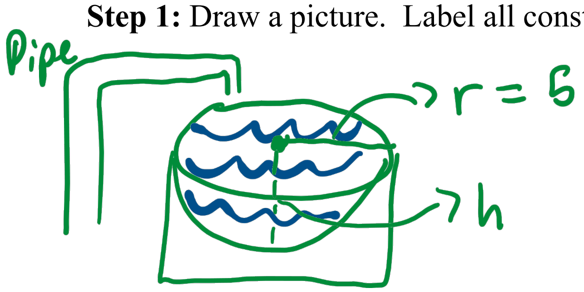
$$x^2 + 16 = r^2$$

$$10^2 + 16 = r^2$$

$$\sqrt{116} = r$$

Example 6: A hemispherical tank with a radius of 5 m is filled from an inflow pipe at a rate of 4 m^3/min. How fast is the water level rising when the water level is 3 m from the bottom of the tank? (Hint: The volume of a cap of thickness h sliced from a sphere of radius r is $\frac{\pi h^2(3r-h)}{3}$).

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dV}{dt} = 4 \frac{m^3}{min}$ $\quad r=5$

WANT: $\left.\frac{dh}{dt}\right|_{h=3\text{ m}}$

Step 3: Find an equation relating to the relevant variables.

$V = \pi \frac{h^2(3r-h)}{3}$ Plug $r=5$ $\quad V = \frac{\pi h^2}{3}(3(5)-h) = \frac{\pi}{3}h^2(15-h)$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t.*

Rewrite

$V = \frac{\pi}{3}[15h^2 - h^3]$

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{\pi}{3}[15h^2-h^3]\right)$

$\frac{dV}{dt} = \frac{\pi}{3}(30h - 3h^2)\frac{dh}{dt}$

$\frac{dV}{dt} = \pi(10h - h^2)\frac{dh}{dt}$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT**.**

$4 = \pi(10(3) - 3^2)\frac{dh}{dt}$

$4 = \pi(30-9)\frac{dh}{dt}$

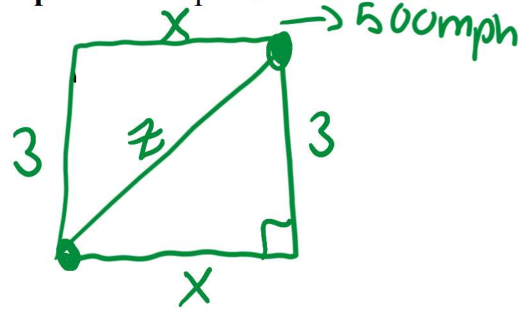
$4 = 21\pi \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{4}{21\pi} \frac{m}{min}$

Example 7: A plane is flying directly away from you at 500 mph at an altitude of 3 miles.

(a) How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dx}{dt} = 500\text{mph}$ **WANT:** $\left.\frac{dz}{dt}\right|_{x=4}$

Step 3: Find an equation relating the relevant variables.

$$x^2 + 3^2 = z^2 \iff x^2 + 9 = z^2$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(x^2+9) = \frac{d}{dt}(z^2)$$

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$\cancel{2}(4)(500) = \cancel{2}z\frac{dz}{dt}$$

$$\frac{4\cdot 500}{z} = \frac{dz}{dt}$$

$$400\frac{m}{h} = \frac{4\cdot \cancel{500}^{100}}{\cancel{5}} = \frac{dz}{dt}$$

$x^2+9=z^2$ plug $x=4$

$4^2+9=z^2$

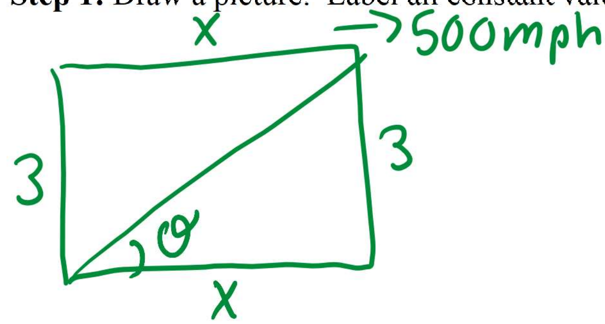
$25=z^2$

$z=5$

Example 7: A plane is flying directly away from you at 500 mph at an altitude of 3 miles.

(b) How fast is the angle of elevation changing when it is $\pi/3$?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dx}{dt} = 500\text{ mph}$ **WANT:** $\left.\frac{d\theta}{dt}\right|_{\theta=\pi/3}$

Step 3: Find an equation relating the relevant variables.

$$\tan\theta = \frac{3}{x} \iff \tan\theta = 3x^{-1}$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}(3x^{-1})$$

$$\sec^2\theta\,\frac{d\theta}{dt} = -3x^{-2}\,\frac{dx}{dt}$$

$$\sec^2\theta\,\frac{d\theta}{dt} = -\frac{3}{x^2}\,\frac{dx}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$\sec^2\left(\frac{\pi}{3}\right)\cdot\frac{d\theta}{dt} = -\frac{3}{x^2}\cdot 500$$

$$\frac{d\theta}{dt} = -\frac{3\cdot 500\cdot\cos^2(\pi/3)}{x^2}$$

$$\frac{d\theta}{dt} = \frac{-3\cdot 500\cdot 1/4}{x^2}$$

$$= -\frac{3\cdot 125}{x^2} = -\frac{\cancel{3}\cdot 125}{\cancel{3}}$$

$$= -125\ \text{rad/h}$$

$$\tan\theta = \frac{3}{x} \text{ @ } \theta = \frac{\pi}{3}$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{3}{x}$$

$$\frac{\sqrt{3}/2}{1/2} = \frac{3}{x}$$

$$\sqrt{3} = \frac{3}{x}$$

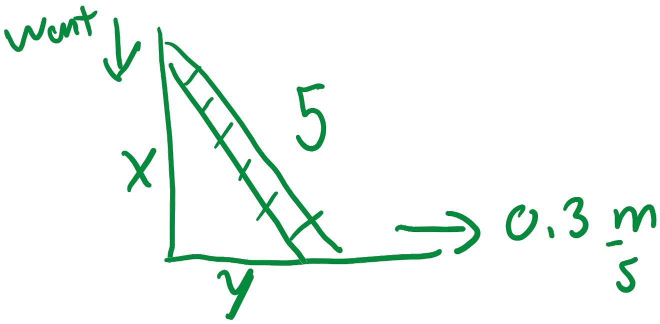
$$\sqrt{3}\,x = 3$$

$$x = 3/\sqrt{3}\cdot\frac{\sqrt{3}}{\sqrt{3}}$$

$$= 3\sqrt{3}/3 = \sqrt{3}$$

Example 8: A ladder 5 meters long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.3 m/sec. How fast is the top sliding down the wall when the foot of the ladder is 3 m from the wall?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dy}{dt} = 0.3 \, \frac{m}{s}$ **WANT:** $\left.\frac{dx}{dt}\right|_{y=3}$

Step 3: Find an equation relating the relevant variables.

$$x^2 + y^2 = 5^2 = 25$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(25)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$2x\frac{dx}{dt} + 2(3)(0.3) = 0$$

$$\frac{dx}{dt} = \frac{-\cancel{2}(3)(0.3)}{\cancel{2}x}$$

$$\frac{dx}{dt} = -\frac{0.9}{x} = -\frac{0.9}{4} \, \frac{m}{s}$$

$x^2 + y^2 = 25$ @ $y = 3$

$$x^2 + 3^2 = 25$$

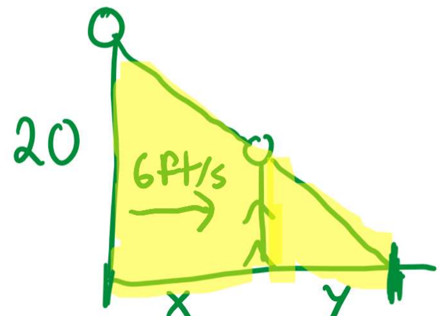
$$x^2 + 9 = 25$$

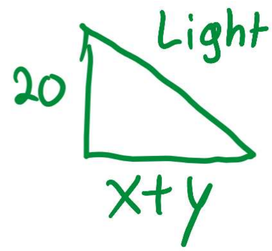
$$x^2 = 16$$

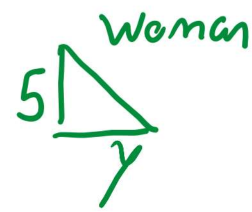
$$x = 4$$

Example 9: A streetlight fastened to the top of a 20-ft high pole. If a 5-ft tall woman walks away from the pole in a straight line over level ground at a rate of 6 ft/s, how fast is the length of her shadow changing when she is 18 ft away from the pole?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.







Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{dx}{dt} = 6\frac{ft}{s}$

WANT: $\left.\frac{dy}{dt}\right|_{x=18}$

Step 3: Find an equation relating the relevant variables.

Similar Triangle

$$\frac{20}{x+y} = \frac{5}{y} \iff \begin{array}{l} 20y = 5(x+y) \\ 20y = 5x + 5y \end{array} \quad \Bigg| \quad \begin{array}{l} 15y = 5x \\ y = \frac{1}{3}x \end{array}$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(y) = \frac{d}{dt}\left(\frac{1}{3}x\right)$$

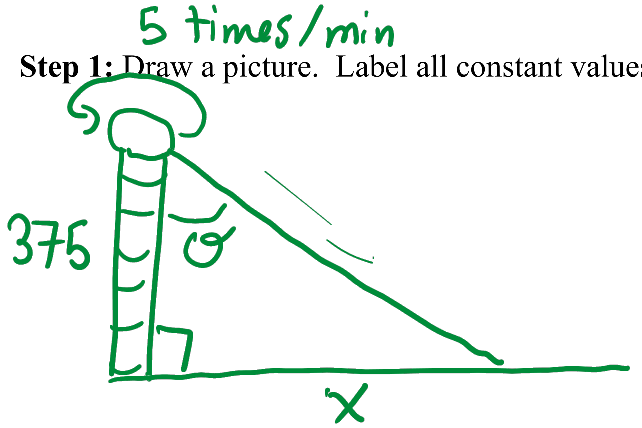
$$\frac{dy}{dt} = \frac{1}{3}\frac{dx}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

$$\frac{dy}{dt} = \frac{1}{3}(6) = 2\ \frac{ft}{s}$$

Example 10: A lighthouse is 375 m from a straight shoreline. Its light rotates 5 times per minute. How fast is the light spot moving along the shore when it hits a point 175 m away from the point nearest the lighthouse?

Step 1: Draw a picture. Label all constant values and give variable names to any changing quantities.



Step 2: Determine what information you **KNOW** and what you WANT to find.

KNOW: $\frac{d\theta}{dt} = 5\ \frac{\text{rev}}{\text{min}} = 5\ \frac{\cancel{\text{rev}}}{\text{min}} \cdot 2\pi\ \frac{\text{rad}}{\cancel{\text{rev}}}$ **WANT:** $\left.\frac{dx}{dt}\right|_{x=175}$

$= 10\pi$ rad/min

Step 3: Find an equation relating the relevant variables.

$$\tan\theta = \frac{x}{375}$$

Step 4: Use implicit differentiation to differentiate the equation with respect to time *t*.

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{x}{375}\right)$$

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{375}\frac{dx}{dt}$$

Step 5: Substitute in what you **KNOW** from **Step 2** and any information that your equation in **Step 3** can give you and solve for the quantity you WANT.

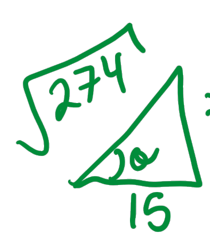
$$\sec^2\theta\,(10\pi) = \frac{1}{375}\frac{dx}{dt}$$

$$\frac{10\pi(375)}{\cos^2\theta} = \frac{dx}{dt}$$

$$\frac{10\pi(375)}{(15/\sqrt{274})^2}\ \frac{m}{s} = \frac{dx}{dt}$$

$\tan\theta = \frac{x}{375}$ $\quad x = 175$

$\tan\theta = \frac{175}{375} = \frac{7}{15}$



$\Rightarrow \cos\theta = \frac{15}{\sqrt{274}}$