Lesson 20: Mean Value Theorem + What derivatives tell us (Part 1)

Friday, October 17, 2025 10:24 AM

Mean Value Theorem: Suppose $f(x)$ is a function that is both

① continuous on $[a,b]$ ② differentiable on (a,b)

Then there is a c such that $a<c<b$ and

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Ex 1: Find all c which satisfy the MVT for

$$f(x) = 2x^3 - 3x + 1 \quad \text{on } [-2,2]$$

Note this polynomial so ① and ② are satisfied.

So $f'(x) = 6x^2 - 3$

$$f'(c) = \frac{f(2)-f(-2)}{2-(-2)}$$

$$6c^2 - 3 = \frac{2(2)^3 - 3(2) + 1 - [2(-2)^3 - 3(-2) + 1]}{4}$$

$$6c^2 - 3 = \frac{11-(-9)}{4} = 5$$

$$6c^2 - 3 = 5$$

$$6c^2 = 8$$

$$c^2 = 8/6 = 4/3$$

Check is $\pm\sqrt{4/3}$ in $[-2,2]$

$$c = \pm\sqrt{4/3}$$

$c^2 = 8/6 = 4/3$ Check is $\pm\sqrt{4/3}$ in $[-2,2]$

$c = \pm\sqrt{4/3}$ ✓

Answer: $c = \pm\sqrt{4/3}$

Rolle's Theorem (Special Case of MVT)

Suppose $f(x)$ is a function that satisfies all of these

① continuous on $[a,b]$ ② differentiable on (a,b)

③ $f(a) = f(b)$

Then there is a c such that $a<c<b$ and $f'(c)=0$.

From the Rolle's Theorem,

Fact: If $f'(x)=0$ for all x in (a,b), then $f(x)$ is constant on (a,b).

Ex 2: Find all c using Rolle's Theorem where

$$h(x) = \boxed{e^{x^2}} \text{ on } [-a,a]$$

We know by the graph of e and x^2 that it is continuous and differentiable everywhere. So what remains to check is if $h(-a)=h(a)$.

$$h(-a) = e^{(-a)^2} = e^{a^2} = h(a) \checkmark$$

By Rolle's Theorem, we have a c for $h'(c)=0$.

$$h'(x) = e^{x^2}(2x) = 0$$

$e^{x^2} = 0$ NEVER

$2x = 0$

$x = 0$ Check if $0 \in [-a,a]$ ✓

Answer: $c = 0$

1^{st} Derivative Test

Let c be a critical # of $f(x)$ $[f'(c)=0]$ that is continuous on an open interval, I, containing c.

① $\xleftarrow{\quad + \quad \underset{c}{|} \quad - \quad}$ f' ex. $\Rightarrow$ relative/local max @ $x = c$

② $\xleftarrow{\quad - \quad \underset{c}{|} \quad + \quad}$ f' ex. $\Rightarrow$ relative/local min @ $x = c$.

Ex 3: Given $f'(x) = (3x+6)(x-5)$. Find the local extrema of $f(x)$.

$$f'(x) = (3x+6)(x-5) = 0$$

$3x+6=0$ $\qquad$ $x-5=0$

$3x=-6$ $\qquad$ $x=5$

$x=-2$

$+ \quad - \quad +$

$-3 \quad 0 \quad 6$

$\longleftarrow \underset{-2}{|} \longrightarrow \underset{5}{|} \longrightarrow$ f'

max $\quad$ min

$f'(-3) = (3(-3)+6)(-3-5)$

$= - \cdot - = +$

$f'(0) = (3(0)+6)(0-5)$

$= + \cdot - = -$

$f'(6) = (3(6)+6)(6-5)$

$= + \cdot + = +$

Local max @ $x=-2$ $\qquad$ Local min @ $x=5$

2^{nd} Derivative Test

Let $f(x)$ be a function such that $f'(c)=0$ and $f''(x)$ exists on an open interval containing c.

① If $f''(c) > 0$ then $f(x)$ has a relative min @ $x=c$.

② If $f''(c) < 0$ then $f(x)$ has a relative max @ $x=c$.

① If $f''(c) > 0$ then $f(x)$ has a relative min @ $x = c$.

② If $f''(c) < 0$ then $f(x)$ has a relative max @ $x = c$.

Note: If $f''(c) = 0$, the 2nd Derivative Test doesn't apply. BUT that doesn't mean we don't have a relative extrema. It means you have to use the 1st Derivative Test.

Ex 4: Find all relative extrema of $f(x) = x^4$.

ⓐ with 2nd derivative test, if possible

ⓑ with 1st derivative test

ⓐ $f'(x) = 4x^3 = 0$

$x = 0 \rightarrow$ critical pt

$f''(x) = 12x^2$

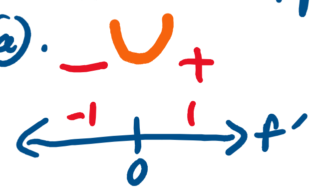
Plug critical pt $(x=0)$

$f''(0) = 12(0)^2 = 0$

2nd Derivative Test Fails

ⓑ Found critical pt $(x=0)$ in ⓐ.

$f'(x) = 4x^3$



Relative min at $x = 0$

Point find y w/ $f(x) = x^4$

$(0, 0)$