Lesson 22: Graphing Functions

Saturday, November 1, 2025 9:17 AM

Now we are going to assemble what we learned to analyze and sketch rational functions

Ex 1: Analyze and sketch the graph of $f(x) = \dfrac{x^2-x}{x-3}$.

① **Domain of f:** When does f(x) DNE?

Fractions are undefined when denominator = 0

$x-3=0$

$x=3 \Rightarrow$ Domain: $(-\infty, 3) \cup (3, \infty)$

② **x-intercept:** Set y=0. Solve for x.

$0 = \dfrac{x^2-x}{x-3}$

$0(x-3) = x^2-x$

$0 = x^2-x$

$0 = x(x-1)$

$0 = x \quad | \quad 0 = x-1$

$ \quad x = 1$

x-intercepts: $(0,0), (1,0)$

③ **y-intercept:** Set x=0. Solve for y.

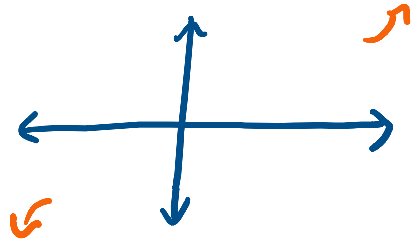
$y = \dfrac{0^2-0}{0-3} = \dfrac{0}{-3} = 0.$ y-intercept: $(0,0)$

④ **End Behavior:**

ⓐ $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x^2-x}{x-3} \approx \lim\limits_{x\to\infty} \dfrac{x^2}{x} = \lim\limits_{x\to\infty} x = \infty$

$\lim_{x\to\infty} \qquad \lim_{x\to\infty} \quad x-3 \qquad \lim_{x\to\infty} \quad x$

ⓑ $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{x^2-x}{x-3} \approx \lim\limits_{x\to-\infty} \dfrac{x^2}{x} = \lim\limits_{x\to-\infty} x = -\infty$



⑤ Asymptotes:

ⓐ Vertical Asymptote

① Check that $f(x)$ is simplify. ✓

② Set denominator to 0. Solve.

$x-3=0$

$x=3 \Rightarrow VA: x=3$

ⓑ Horizontal Asymptote: Use ④

Since $\lim\limits_{x\to\pm\infty} f(x) \neq L$ where L is a constant, then there is no HA.

ⓒ Slant Asymptote

① Check that difference between the powers of the leading terms of numerator and denominator is equal to 1.

② If so, use sythetic division or long division.

$$\begin{array}{r|rrr} 3 & 1 & -1 & 0 \\ & \downarrow & 3 & 6 \\ \hline & 1 & 2 & 6 \end{array} \Rightarrow f(x) = x+2 + \frac{6}{x-3}$$

Slant Asymptote @ $y = x+2$

$-1 \quad 2 \quad 6$

Slant Asymptote @ $y = x + 2$

⑥ Critical #'s $f'(x) = 0$ and $f'(x)$ DNE.

$u = x^2 - x \qquad v = x - 3$

$u' = 2x - 1 \qquad v' = 1$

	$2x$	-1
x	$2x^2$	$-x$
-3	$-6x$	3

$$f' = \frac{u'v - uv'}{v^2}$$

$$= \frac{(2x-1)(x-3) - (x^2 - x)}{(x-3)^2}$$

$$= \frac{2x^2 - 7x + 3 - x^2 + x}{(x-3)^2}$$

$$= \frac{x^2 - 6x + 3}{(x-3)^2} = 0$$

$x^2 - 6x + 3 = 0 \longleftarrow$ Use Quadratic Formula

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36 - 12}}{2}$$

$$= \frac{6 \pm \sqrt{24}}{2} = \frac{6 \pm 2\sqrt{6}}{2} = 3 \pm \sqrt{6}$$

$f'(x) = \dfrac{x^2 - 6x + 3}{(x-3)^2}$ DNE when $(x-3)^2 = 0$

$x = 3$

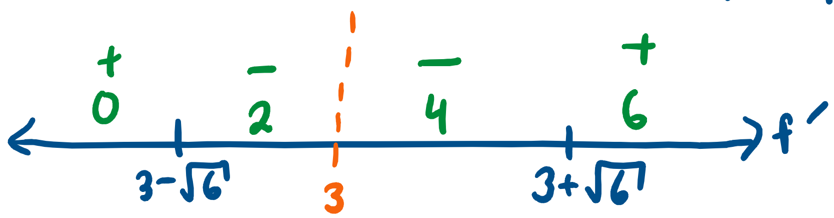
Recall domain of f is $(-\infty, 3) \cup (3, \infty)$. So $x = 3$ can't be a critical #.

Critical #s: $x = 3 \pm \sqrt{6}$

7 Increasing/Decreasing

$$f' = \frac{x^2 - 6x + 3}{(x-3)^2}$$

Note the denominator is always positive.

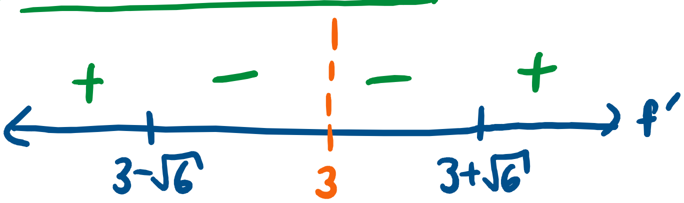


b/c f isn't defined @ $x = 3$

Increasing: $(-\infty, 3-\sqrt{6}) \cup (3+\sqrt{6}, \infty)$

Decreasing: $(3-\sqrt{6}, 3) \cup (3, 3+\sqrt{6})$

8 Relative Extrema: Use 7.



By First Derivative Test,

Rel max: $x = 3 - \sqrt{6}$

Rel min: $x = 3 + \sqrt{6}$

9 Concavity: $f''(x) = 0$ and $f''(x)$ DNE

$$f' = \frac{x^2 - 6x + 3}{(x-3)^2}$$

$u = x^2 - 6x + 3$ $\qquad$ $v = (x-3)^2$

$u' = 2x - 6$ $\qquad$ $v' = 2(x-3)$

$= 2(x-3)$

$$f''(x) = \frac{u'v - uv'}{v^2}$$

$$f''(x) = \frac{u'v - uv'}{v^2}$$

$$= \frac{2(x-3)(x-3)^2 - 2(x-3)(x^2-6x+3)}{[(x-3)^2]^2}$$

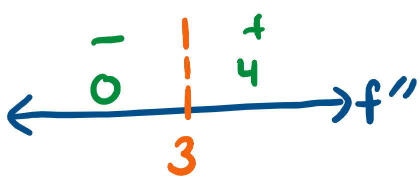
$$= \frac{2\cancel{(x-3)}[(x-3)^2 - (x^2-6x+3)]}{(x-3)^{\cancel{4}\,3}}$$

$$= \frac{2[\cancel{x^2} - \cancel{6x} + 9 - \cancel{x^2} + \cancel{6x} - 3]}{(x-3)^3}$$

$$= \frac{2 \cdot 6}{(x-3)^3} = \frac{12}{(x-3)^2} = 0 \Rightarrow f''(x) \neq 0 \text{ b/c numerator } \neq 0$$

$f''(x)$ DNE when $(x-3)^3 = 0$
$x = 3$

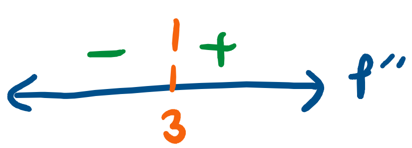
But we know $f(x)$ DNE @ $x = 3$



Concave Up: $(3, \infty)$

Concave Down: $(-\infty, 3)$

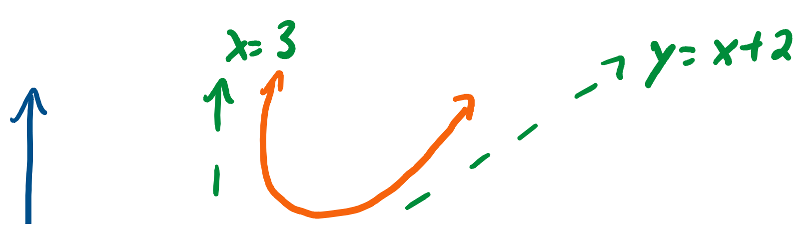
⑩ Inflection Pts: Use ⑨ and check for sign change.

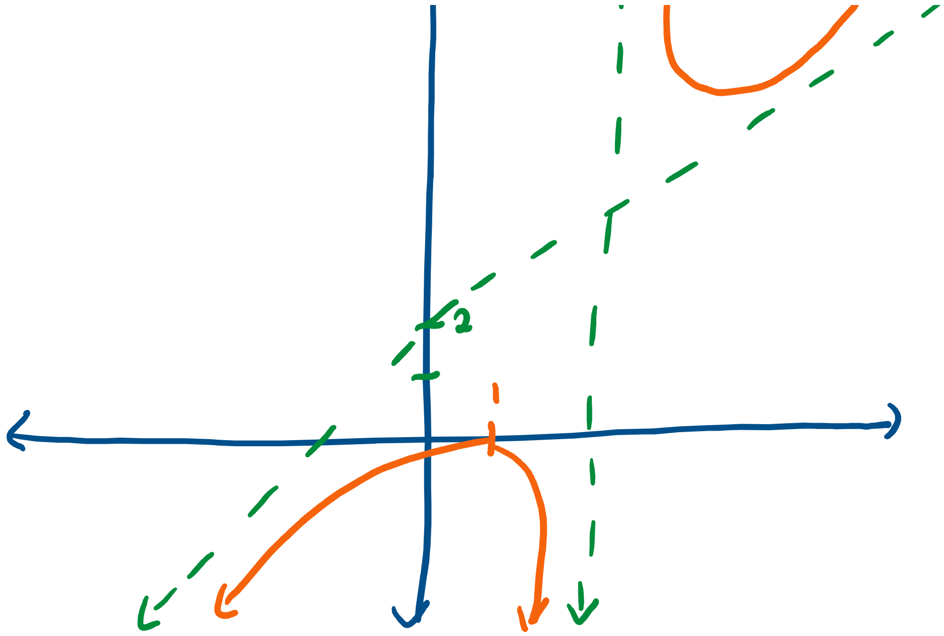


Is there a change? Yes.

BUT $f(x)$ isn't defined @ $x=3$, so no inflection pt.

⑪ Graph:

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