FA25_MA165_L23+24_Handout

Friday, October 24, 2025 9:07 AM

FA25_MA
165_L23+...

MA 165 LESSONS 23+24: OPTIMIZATION

<u>Optimization Problems:</u> Often this includes finding the maximum or minimum value of some function.

- e.g. The minimum time to make a certain journey,
- e.g. The minimum cost for constructing some object,
- e.g. The maximum profit to gain for a business, and so on.

<u>How do we solve an optimization problem?</u>

- Determine a function (**known as objective function)** that we need to maximize or minimize.

- Determine if there are some constraints on the variables. (The equations that describe the constraints are called **the constraint equations**.)
    - o If there are constraint equations, rewrite the **objective function** as a function of only one variable.

- Then we can solve for absolute maximum or minimum like we did before.
    - o Using either the First Derivative Test or Second Derivative Test.

Recipe for Solving an Optimization Problem

Step 1: Identify what quantity you are trying to optimize.

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

Step 5: Identify the domain for the function you found in Step 4.

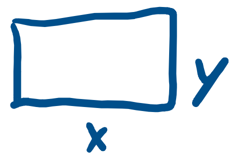
Step 6: Find the absolute extrema of the variable to be optimized on this domain.

Step 7: Reread the question and be sure you have answered exactly what was asked.

Example 1: A carpenter is building a rectangular room with a fixed perimeter of 54 feet. What are the dimensions of the largest room that can be built? What is its area?

Step 1: Identify what quantity you are trying to optimize. Area, A

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$A = xy$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$54 = P = 2x + 2y \iff 27 = x + y$$

Step 5: Identify the domain for the function you found in Step 4.

no length $\Rightarrow x=0, y=0$
no width $\Rightarrow x=27, y=27$
} Domain of x & y : $(0, 27)$

Step 6: Find the absolute extrema of the variable to be optimized on this domain.

Solve eqn from 4 for y.

$y = 27 - x$

Plug y into Area

$A = x(27-x)$

$= 27x - x^2$

Find A' and set $=0$.

$A' = 27 - 2x = 0$

$27 = 2x$

$x = \frac{27}{2}$

Check $x = 27/2$ gives abs max.

By 2^{nd} derivative Test

$A'' = -2$

$A''\left(\frac{27}{2}\right) = -2 < 0$

$\Rightarrow$ abs max

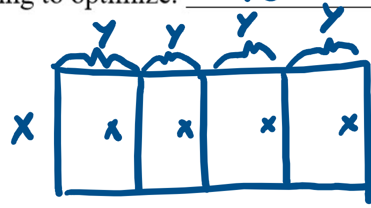
Step 7: Reread the question and be sure you have answered exactly what was asked.

$x = \frac{27}{2} \Rightarrow y = 27 - x = 27/2$ } Area $= \frac{27}{2} \times \frac{27}{2} = \frac{729}{4}$

Example 2: A rancher plans to make four identical and adjacent rectangular pens against a barn, each with an area of 25 m^2. What are the dimensions of each pen that minimize the amount of fence that must be used?

Step 1: Identify what quantity you are trying to optimize. Perimeter, P

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$P = 8y + 5x$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$A = xy = 25$

$y = 25/x$

So $P = 8y + 5x$

$= 8\left(\frac{25}{x}\right) + 5x = 200x^{-1} + 5x$

Step 5: Identify the domain for the function you found in Step 4.

no area when x or y equals 0.

So domain of x and y $(0, \infty)$

Step 6: Find the absolute extrema of the variable to be optimized on this domain.

$P'(x) = -200x^{-2} + 5 = 0$

$5 = \frac{200}{x^2}$

$5x^2 = 200$

$x^2 = 40$

$x = 2\sqrt{10}$

Check $x = 2\sqrt{10}$ is an abs min.

$P'' = 400x^{-3}$

$P''(2\sqrt{10}) = \frac{400}{(2\sqrt{10})^3} > 0$

$\Rightarrow$ abs min

Step 7: Reread the question and be sure you have answered exactly what was asked.

Dimensions: If $x = 2\sqrt{10}$

$y = 25/2\sqrt{10}$

So $2\sqrt{10}$ by $\frac{25}{2\sqrt{10}}$

Example 3: An open-top box with a square base is to have a volume of 8 cubic feet. Find the dimensions of the box that can be made with the least amount of material.

Step 1: Identify what quantity you are trying to optimize. Surface Area, A

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.

Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$A = x^2 + 4xy$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$8 = V = x^2 y \Rightarrow y = \frac{8}{x^2}$$

Step 5: Identify the domain for the function you found in Step 4.

no length, no width $\}$ $x=0, y=0$ $\Rightarrow$ Domain of x and y : $(0,\infty)$

Step 6: Find the absolute extrema of the variable to be optimized on this domain.

Plug $y=\frac{8}{x^2}$ into Area

$$A = x^2 + 4x\left(\frac{8}{x^2}\right)$$
$$= x^2 + \frac{32}{x}$$
$$= x^2 + 32x^{-1}$$

Find A' and set $=0$.

$$A' = 2x - 32x^{-2} = 0$$
$$2x - \frac{32}{x^2} = 0$$
$$2x = \frac{32}{x^2}$$
$$x^3 = 16$$
$$x = \sqrt[3]{16}$$

Check $x = \sqrt[3]{16}$ gives abs min

By 2nd Derivative Test,

$$A'' = 2 + 64x^{-3}$$
$$A''(\sqrt[3]{16}) > 0$$
$\Rightarrow$ abs min

Step 7: Reread the question and be sure you have answered exactly what was asked.

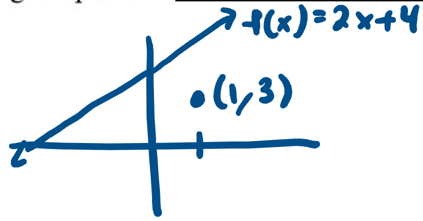
$$x = \sqrt[3]{16} \Rightarrow y = \frac{8}{x^2} = \frac{8}{16^{2/3}}$$

Dimensions $\sqrt[3]{16} \times \sqrt[3]{16} \times \frac{8}{16^{2/3}}$

Example 4: Find the point on the graph of $f(x) = 2x + 4$ that is the closest to the point $(1,3)$.

Step 1: Identify what quantity you are trying to optimize. Distance D

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$D = (x-x_1)^2 + (y-y_1)^2 = (x-1)^2 + (y-3)^2$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$y = 2x+4$$

Step 5: Identify the domain for the function you found in Step 4.

Domain: $(-\infty, \infty)$

Step 6: Find the absolute extrema of the variable to be optimized on this domain.

$$D = (x-1)^2 + (2x+4-3)^2$$
$$= (x-1)^2 + (2x+1)^2$$

$$D' = 2(x-1) + 2(2x+1)\cdot 2 = 0$$
$$x-1+2(2x+1) = 0$$
$$x-1+4x+2 = 0$$
$$5x+1 = 0$$
$$x = -1/5$$

Check $x=0$ gives abs min. By 2nd Derivative Test,

$$D'' = 5$$
$$D''(-\tfrac{1}{5}) = 5 > 0$$
$$\Rightarrow \text{abs min}$$

Step 7: Reread the question and be sure you have answered exactly what was asked.

Pt: $y = 2x+4$

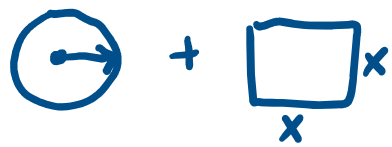
$y = -\frac{2}{5} + 4 = \frac{18}{5}$

$\left(-\frac{1}{5}, \frac{18}{5}\right)$

Example 5: A piece of wire of length 40 is cut, and the resulting two pieces are formed to make a circle and a square. Where should the wire be cut to minimize the combined area of the circle and the square?

Step 1: Identify what quantity you are trying to optimize. Area, A

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$A = \pi r^2 + x^2$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

$$2\pi r + 4x = 40 \iff \pi r + 2x = 20$$

Step 5: Identify the domain for the function you found in Step 4.

$r=0 \Rightarrow 2x=20$, $x=10$

$x=0 \Rightarrow \pi r = 20 \Rightarrow r = 20/\pi$

Domain of r: $(0, 20/\pi)$

Domain of x: $(0, 10)$

Step 6: Find the absolute extrema of the variable to be optimized on this domain.

Solve $\pi r + 2x = 20$ for x.

$$20 - \pi r = 2x$$
$$10 - \frac{\pi}{2} r = x$$

Plug into A.

$$A = \pi r^2 + \left(10 - \frac{\pi}{2} r\right)^2$$

Find A' and set $=0$.

$$A' = 2\pi r + 2\left(10 - \frac{\pi}{2} r\right)\left(-\frac{\pi}{2}\right) = 0$$

$$A' = 2\pi r - \pi\left(10 - \frac{\pi}{2} r\right) = 0$$
$$2r - \left(10 - \frac{\pi}{2} r\right) = 0$$
$$2r - 10 - \frac{\pi}{2} r = 0$$
$$\frac{4r}{2} - \frac{\pi}{2} r = 10$$
$$\frac{4-\pi}{2} r = 10$$
$$r = \frac{20}{4-\pi}$$

Where to cut? Well what's circumference of the circle.

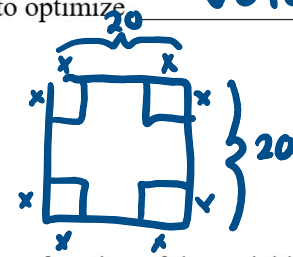
$$C = 2\pi r$$
$$= \frac{2\pi \cdot 20}{4-\pi}$$

Example 6: From a thin piece of cardboard 20 in by 20 in, square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?

Step 1: Identify what quantity you are trying to optimize. Volume, V

Step 1: Identify what quantity you are trying to optimize. Volume, V

Step 2: Draw a picture (if applicable), corresponding to the problem, and label it with your variables.



Step 3: Express the variable to be optimized as a function of the variables you used in Step 2.

$$V = x(20-2x)^2$$

Step 4: Find relations among the variables from Step 2 and express the variable to be optimized a function of just one of the variables from Step 2.

N/A

Step 5: Identify the domain for the function you found in Step 4.

Domain of x: $(0, 10)$

Step 6: Find the absolute extrema of the variable to be optimized on this domain.

$$V' = (20-2x)^2 + x(2)(20-2x)(-2)$$
$$= (20-2x)[20-2x-4x]$$
$$= (20-2x)(20-6x)$$

$20-2x=0$ $\quad$ $20-6x=0$

$\cancel{x=10}$ $\quad$ $x = 20/6$

Not in domain.

So only check $x = 10/3$ gives abs max.

$$V'' = -2(20-6x) - 6(20-2x)$$
$$V''\left(\tfrac{20}{6}\right) = -6\left(20-2\left(\tfrac{10}{3}\right)\right) < 0$$

$\Rightarrow$ abs

Step 7: $x = \frac{10}{3}$ $\quad$ $w = 20-2x = 40/3$ $\quad$ Dimensions $10/3 \times 40/3 \times 40/3$ $\quad$ $V = \frac{16000}{27}$