FA25_MA165_Study_Guide_Exam_1

Friday, September 19, 2025 8:55 PM

FA25_MA1
65_Study...

Please show **all** your work! Answers without supporting work will not be given credit. Write answers in spaces provided.

Name:____

1. You are supposed to know and understand the basics about the exponential function, and about the logarithmic function as the inverse of the exponential function. You should be able to solve the equations involving them.

Example Problems

(a) Solve the following equations.

i. $\ln(x) + \ln(x-3) = 0$

$\ln(x \cdot (x-3)) = 0$

$\cancel{e}^{\cancel{\ln}(x(x-3))} = e^0$

$x(x-3) = 1$

$x^2 - 3x = 1$

$x^2 - 3x - 1 = 0$

$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2(1)}$

$= \frac{3 \pm \sqrt{9+4}}{2}$

No neg b/c $x = \frac{3 - \sqrt{13}}{4}$ gives us for $\ln(x-3)$ $\ln(-)$ which can't happen

x = $\frac{3 + \sqrt{13}}{2}$

ii. $7^{2x+1} = 21$

$\cancel{\log_7}(\cancel{7}^{2x+1}) = \log_7(21)$

$2x+1 = \log_7(21)$

$2x = \log_7(21) - 1$

$2x = \log_7(3) + 1 - 1$

Note: $\log_7(21) = \log_7(3 \cdot 7)$

$= \log_7(3) + \log_7(7)$

$= \log_7(3) + 1$

x = $\frac{\log_7(3)}{2}$ [Note the same as $\ln(3)/[2\ln(7)]$]

iii. $3^{\ln(x)} = 5$

$\cancel{\log_3}(\cancel{3}^{\ln(x)}) = \log_3(5)$

$\ln(x) = \log_3(5)$

$e^{\ln(x)} = e^{\log_3(5)}$

$x = e^{\log_3(5)}$

x = $e^{\log_3(5)}$ [Note the same as $5^{1/\ln(3)}$]

iv. $e^{2x^2-7x-15}=1$

$\ln(e^{2x^2-7x-15})=\ln(1)$

$2x^2-7x-15=0$

$2x^2-10x+3x-15=0$

$2x(x-5)+3(x-5)=0$

$(2x+3)(x-5)=0$

$-30=2(-15)$

$-10 \quad 3$

x = $-\frac{3}{2}, 5$

(b) Find the exact values.

i. $\frac{log_{15}(9)+\log_{15}(25)}{e^{3\ln(5)+2\ln(7)}}$

Num: $\log_{15}(9)+\log_{15}(25)=\log_{15}(9\cdot 25)=\log_{15}(3^2\cdot 5^2)=\log_{15}((3\cdot 5)^2)=\log_{15}(15^2)$
$=2$

Deno: $e^{3\ln(5)+2\ln(7)}=e^{\ln(5^3)+\ln(7^2)}=e^{\ln(5^3\cdot 7^2)}=5^3\cdot 7^2$

$\frac{log_{15}(9)+\log_{15}(25)}{e^{3ln(5)+2ln(7)}} = \frac{2}{5^3\cdot 7^2}$

ii. $\log_b \frac{\sqrt{x}}{\sqrt[3]{z}}$ when $\log_b(x)=2.4$ and $\log_b(z)=3.6$

$\log_b(\sqrt{x})-\log_b(\sqrt[3]{z})=\frac{1}{2}\log_b(x)-\frac{1}{3}\log_b(z)$

$=\frac{1}{2}(2.4)-\frac{1}{3}(3.6)$

$=1.2-1.2$

$=0$

$\log_b \frac{\sqrt{x}}{\sqrt[3]{z}}$ = 0

2. You are supposed to know the domain and range of each of the inverse trigonmetric functions $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$. You are also supposed to know the basic identities involving the inverse trigonometric functions.

Example Problems

(a) Find the exact values for the following:

i. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \theta$

Also remember $\sin^{-1}\theta$ is defined when $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$\sin\theta = -\frac{\sqrt{3}}{2}$

$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

S	A
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→ since we want negative

$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) =$ $-\pi/3$

ii. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \theta$

Also remember $\cos^{-1}\theta$ is defined when $0 < x < \pi$

$\cos\theta = -\frac{\sqrt{2}}{2}$

$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\pi - \frac{\pi}{4} = \frac{3\pi}{4}$

since we want neg ←

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$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) =$ $\frac{3\pi}{4}$

iii. $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \theta$

$\tan\theta = \frac{\sqrt{3}}{3} = \frac{\sqrt{3}\sqrt{3}}{3\sqrt{3}} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1/2}{\sqrt{3}/2}$

When $\sin\theta = 1/2$ and $\cos\theta = \sqrt{3}/2$ is

$\theta = \frac{\pi}{6}$

$\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) =$ $\frac{\pi}{6}$

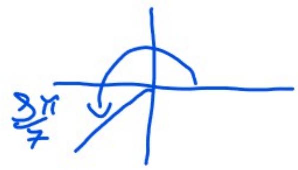
iv. $\sin^{-1}\left(\sin\left(\frac{8\pi}{7}\right)\right) = \frac{8\pi}{7}$ but wait $\sin^{-1}\theta$ exist at $-\frac{\pi}{2} < x < \frac{\pi}{2}$

and $\frac{8\pi}{7} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Note $\sin\left(\frac{8\pi}{7}\right) = -\sin\left(\frac{8\pi}{7} - \pi\right)$

$= -\sin\left(\frac{\pi}{7}\right)$

Since sine is odd $\leftarrow = \sin(-\pi/7)$



$\sin^{-1}\left(\sin\left(\frac{8\pi}{7}\right)\right) =$ $\sin^{-1}\left(\sin\left(-\frac{\pi}{7}\right)\right) = -\frac{\pi}{7}$

v. $\sin^{-1}\left(\underbrace{\cos\left(\frac{3\pi}{5}\right)}\right)$

Rewrite to be sin

$\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$

$\cos\left(\frac{3\pi}{5}\right) = \sin\left(\frac{\pi}{2} - \frac{3\pi}{5}\right)$

$= \sin\left(\frac{5\pi}{10} - \frac{6\pi}{10}\right)$

$= \sin\left(-\frac{\pi}{10}\right)$

$\sin^{-1}\left(\sin\left(-\frac{\pi}{10}\right)\right) = -\frac{\pi}{10}$

Since $-\frac{\pi}{2} < -\frac{\pi}{10} < \frac{\pi}{2}$

$\sin^{-1}\left(\cos\left(\frac{3\pi}{5}\right)\right) =$ $-\pi/10$

vi. $\sin^{-1}\left(\underbrace{\cos\left(\frac{4\pi}{7}\right)}\right)$

Rewrite to be sin

$\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$

$\cos\left(\frac{4\pi}{7}\right) = \sin\left(\frac{\pi}{2} - \frac{4\pi}{7}\right)$

$= \sin\left(\frac{7\pi}{14} - \frac{8\pi}{14}\right)$

$= \sin\left(-\frac{\pi}{14}\right)$

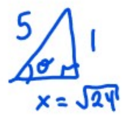
$\sin^{-1}\left(\sin\left(-\frac{\pi}{14}\right)\right) = -\frac{\pi}{14}$

Sin $-\frac{\pi}{2} < -\frac{\pi}{14} < \frac{\pi}{2}$

$\sin^{-1}\left(\cos\left(\frac{4\pi}{7}\right)\right) =$ $-\pi/14$

vii. $\cos\left(\underbrace{\sin^{-1}\left(\frac{1}{5}\right)}_{=\theta}\right) = \cos\theta = \frac{adj}{hyp} = \frac{\sqrt{24}}{5}$

$\sin\theta = \frac{1}{5}$



$x^2 + 1^2 = 5^2$

$x^2 + 1 = 25$

$x^2 = 24 \rightarrow x = \sqrt{24}$

$\cos\left(\sin^{-1}\left(\frac{1}{5}\right)\right) = \underline{\sqrt{24}/5 \;\left[\text{or } 2\sqrt{6}/5\right]}$

viii. $\sin\left(\underbrace{\cos^{-1}\left(-\frac{1}{5}\right)}_{\theta}\right) = \sin\theta = \frac{opp}{hyp} = \frac{\sqrt{24}}{5}$

$\cos\theta = \frac{-1}{5}$

$x^2 + (-1)^2 = 5^2$

$x^2 + 1 = 25 \rightarrow x^2 = 24$

$x = \sqrt{24}$

$\sin\left(\cos^{-1}\left(-\frac{1}{5}\right)\right) = \underline{\sqrt{24}/5 \;\left[\text{or } 2\sqrt{6}/5\right]}$

ix. $\tan\left(\underbrace{\sin^{-1}\left(\frac{1}{2}\right)}_{\theta}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{\sin(\pi/6)}{\cos(\pi/6)} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

$\tan\left(\sin^{-1}\left(\frac{1}{2}\right)\right) = \underline{1/\sqrt{3} \;\left[\text{or } \sqrt{3}/3\right]}$

x. $\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}\left(\frac{1}{3}\right)$

Identity: For $x \in [-1, 1]$

$$\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$$

Since $\frac{1}{3} \in [-1, 1]$ then

$$\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}\left(\frac{1}{3}\right) = \pi/2$$

$\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}\left(\frac{1}{3}\right) =$ $\pi/2$

xi. $\tan^{-1}\left(\tan\left(\frac{4\pi}{5}\right)\right) + \tan^{-1}\left(\tan\left(\frac{6\pi}{5}\right)\right)$

Domain of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\tan(x) = \tan(x - \pi)$

$$\tan\left(\frac{4\pi}{5}\right) = \tan\left(\frac{4\pi}{5} - \pi\right) = \tan\left(-\frac{\pi}{5}\right)$$

$$\tan\left(\frac{6\pi}{5}\right) = \tan\left(\frac{6\pi}{5} - \pi\right) = \tan\left(\frac{\pi}{5}\right)$$

So $\tan^{-1}\left(\tan\left(-\frac{\pi}{5}\right)\right) + \tan^{-1}\left(\tan\left(\frac{\pi}{5}\right)\right)$

$$= -\frac{\pi}{5} + \frac{\pi}{5} = 0$$

$\tan^{-1}\left(\tan\left(\frac{4\pi}{5}\right)\right) + \tan^{-1}\left(\tan\left(\frac{6\pi}{5}\right)\right) =$ 0

3. You are supposed to be able to solve the equations involving the trigonometric functions and find solutions on the given interval, using the basic formulas of the trigonometric functions (e.g., double angle foula for sine and cosine, $\sin^2(x) + \cos^2(x) = 1$, etc.).

Example Problems

(a) Find the values of x on the interval $[0, 2\pi]$ which satisfy the equation.

i. $\cos^2(x) = \cos(2x)$

$\cos(2x) = \cos^2 x - \sin^2 x$

So $\cos^2(x) = \cos(2x)$

$\cancel{\cos^2}(x) = \cancel{\cos^2} x - \sin^2 x$

$0 = -\sin^2 x$

$\sin x = 0$

$x = 0, \pi, 2\pi$

x = $0, \pi, 2\pi$

ii. $2\sin^2(x) = \cos(2x)$

$\cos(2x) = 1 - 2\sin^2(x)$

So $2\sin^2(x) = 1 - 2\sin^2(x)$

$4\sin^2(x) = 1$

$\sin^2(x) = \frac{1}{4}$

$\sin(x) = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\sin x = -\frac{1}{2} \Rightarrow x = \frac{7\pi}{6}, \frac{11\pi}{6}$

x = $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

(b) Find the values of x on the interval $[0, 2\pi)$ which satisfy the equation

i. $\sin(x) = \sin(2x)$

$\sin(2x) = 2\sin(x)\cos(x)$

So $\sin(x) = 2\sin(x)\cos(x)$

$0 = 2\sin(x)\cos(x) - \sin(x)$

$= \sin(x)[2\cos(x) - 1]$

$\sin(x) = 0$

$x = 0, \pi$

$2\cos(x) - 1 = 0$

$\cos(x) = 1/2$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

x = $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

ii. $\sin(x) = \cos(2x)$

$\cos(2x) = 1 - 2\sin^2(x)$

So, $\sin(x) = 1 - 2\sin^2(x)$

$2\sin^2(x) + \sin(x) - 1 = 0$

$(2\sin(x) - 1)(\sin(x) + 1) = 0$

$2\sin(x) - 1 = 0$

$\sin(x) = 1/2$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\sin(x) = -1$

$x = 3\pi/2$

x $=$ $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

iii. $-\sin^2(x) = \cos(2x)$

$\cos(2x) = 1 - 2\sin^2(x)$

So $-\sin^2(x) = 1 - 2\sin^2(x)$

$\sin^2(x) = 1$

$\sin(x) = \pm 1$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

x $=$ $\pi/2, 3\pi/2$

(c) Find the values of x on the interval $[0, 2\pi]$ which satisfy the equation

$$\tan^2(x) - 3 = 0$$

$\tan^2(x) = 3$

$\tan(x) = \sqrt{3} = \frac{\sqrt{3}}{1} = \frac{\sqrt{3}/2}{1/2}$

$\frac{\sin(x)}{\cos(x)} = \frac{\sqrt{3}/2}{1/2}$

$x = \frac{\pi}{3}, \frac{4\pi}{3}$

$\tan(x) = -\sqrt{3}$

$= \frac{-\sqrt{3}/2}{1/2}$ or $\frac{\sqrt{3}/2}{-1/2}$

$\Downarrow$ $\Downarrow$

$x = \frac{2\pi}{3}$ $x = \frac{5\pi}{3}$

x $=$ $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

4. You are supposed to express the trigonometric functions, using a right-triangle relationship, as an algebraic function of x, when the angle is given in terms of x via the inverse trigonometric functions.

Example Problems

(a) Express the following as an algebraic function of x.

i. $\cos\left(\underbrace{\sin^{-1}\left(\frac{x}{3}\right)}_{\theta}\right)$

$\sin\theta = x/3$

$y = \sqrt{9-x^2}$

Find y.

$x^2+y^2=3^2$

$y^2 = 9-x^2$

$y=\sqrt{9-x^2}$

$\cos\theta = \frac{adj}{hyp} = \frac{\sqrt{9-x^2}}{3}$

$\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) = \frac{\sqrt{9-x^2}}{3}$

ii. $\sin\left(\underbrace{\cos^{-1}\left(\frac{x}{5}\right)}_{\theta}\right)$

$\cos\theta = x/5$

Find y.

$x^2+y^2=5^2$

$y^2 = 25-x^2$

$y=\sqrt{25-x^2}$

$\sin\theta = \frac{opp}{hyp} = \frac{\sqrt{25-x^2}}{5}$

$y=\sqrt{25-x^2}$

$\sin\left(\cos^{-1}\left(\frac{x}{5}\right)\right) = \frac{\sqrt{25-x^2}}{5}$

iii. $\cos\left(\underbrace{\sin^{-1}(7x)}_{\theta}\right)$

$\sin\theta = 7x = \frac{7x}{1}$

Find y.

$(7x)^2+y^2=1^2$

$49x^2+y^2=1$

$y^2=1-49x^2$

$y=\sqrt{1-49x^2}$

$\cos\theta = \frac{adj}{hyp} = \frac{\sqrt{1-49x^2}}{1}$

$y = \sqrt{1-49x^2}$

$\cos\left(\sin^{-1}(7x)\right) = \sqrt{1-49x^2}$

iv. $\tan\left(\sin^{-1}(7x)\right)$

Find y.

$\tan\theta = \frac{opp}{}$

iv. $\tan\left(\underbrace{\sin^{-1}(7x)}_{\theta}\right)$

$\sin\theta = 7x = \frac{7x}{1}$

y
$\parallel$
$\sqrt{1-49x^2}$

Find y.

$y^2 + (7x)^2 = 1^2$

$y^2 + 49x^2 = 1$

$y^2 = 1 - 49x^2$

$y = \sqrt{1-49x^2}$

$\tan\theta = \frac{opp}{adj}$

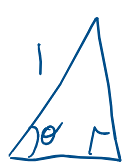
$= \frac{7x}{\sqrt{1-49x^2}}$

$\tan\left(\sin^{-1}(7x)\right) = 7x/\sqrt{1-49x^2}$

v. $\sin\left(2\cos^{-1}(3x)\right)$

Let $\theta = \cos^{-1}(3x)$

$\cos\theta = \frac{3x}{1}$



$y = \sqrt{1-9x^2}$

$3x$

Let find y.

$(3x)^2 + y^2 = 1^2$

$9x^2 + y^2 = 1$

$y^2 = 1 - 9x^2$

$y = \sqrt{1-9x^2}$

Note

$\sin(2\theta) = 2\sin\theta\cos\theta$

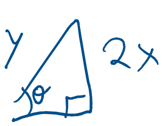
$= 2\left(\sqrt{1-9x^2}\right)(3x)$

$\sin\left(2\cos^{-1}(3x)\right) = 6x\sqrt{1-9x^2}$

vi. $\sin\left(\underbrace{\tan^{-1}(2x)}_{\theta}\right)$

$\tan(\theta) = \frac{2x}{1}$

$\sqrt{1+4x^2}$
$\parallel$
y



Find y.

$1^2 + (2x)^2 = y^2$

$1 + 4x^2 = y^2$

$y = \sqrt{1+4x^2}$

$\sin\theta = \frac{opp}{hyp}$

$= \frac{2x}{\sqrt{1+4x^2}}$

$\sin\left(\tan^{-1}(2x)\right) = \frac{2x}{\sqrt{1+4x^2}}$

vii. $\cot\left(\underbrace{\sin^{-1}\left(\frac{x}{5}\right)}_{\theta}\right)$

$\sin\theta = x/5$

Find y.

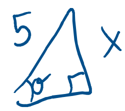
$y^2 + x^2 = 5^2$

$y^2 = 25 - x^2$

$y = \sqrt{25-x^2}$

$\cot\theta = \frac{adj}{opp}$

$= \frac{\sqrt{25-x^2}}{x}$



$y = \sqrt{25-x^2}$

$\cot\left(\sin^{-1}\left(\frac{x}{5}\right)\right) = \frac{\sqrt{25-x^2}}{x}$

5. You should know the condition (one-to-one) for a function to have its inverse. Given a function which is one-to-one, you are supposed to be able to find the formula of the inverse function, its domain and range, and draw the graph of the inverse function.

Example Problems

(a) Find the formula, and state the domain and range of the inverse of the function

$$y = f(x) = \ln\left(\frac{1}{3}e^{2x} - 3\right)$$

Remember range of ln is $(-\infty, \infty)$. So

Range of f = Domain of f^{-1} $= (-\infty, \infty)$

Inverse:

$x = \ln\left(\frac{1}{3}e^{2y} - 3\right)$

$e^x = e^{\ln\left(\frac{1}{3}e^{2y}-3\right)}$

$e^x = \frac{1}{3}e^{2y} - 3$

$e^x + 3 = \frac{1}{3}e^{2y}$

$3(e^x + 3) = e^{2y}$

$\ln(3e^x + 9) = 2y$

$\frac{1}{2}\ln(3e^x + 9) = y$

Domain of f: $\ln(?) \Rightarrow ? > 0$

$\frac{1}{3}e^{2x} - 3 > 0$

$\frac{1}{3}e^{2x} > 3$

$e^{2x} > 9$

$\ln(e^{2x}) > \ln(9)$

$2x > \ln(9)$

$x > \frac{\ln(9)}{2} = \ln(9^{1/2}) = \ln(3)$

$(\ln(3), \infty)$

Remember domain of f is range of f^{-1}. So range of f^{-1}: $(\ln(3), \infty)$

$f^{-1}(x) = \frac{1}{2}\ln(3e^x + 9)$

Domain of $f^{-1}(x)$: $(-\infty, \infty)$

Range of $f^{-1}(x)$: $(\ln(3), \infty)$

(b) Consider the function $y = f(x) = \frac{x+5}{x-2}$ over the domain $(2, \infty)$. Find the formula and domain of its inverse function.

Inverse

$y = \frac{x+5}{x-2}$

Solve for x.

$(x-2)y = x+5$

$xy - 2y = x+5$

$xy - x = 2y+5$

$x(y-1) = 2y+5$

$x = \frac{2y+5}{y-1}$

So $f^{-1} = \frac{2x+5}{x-1}$

Since domain of f is $(2,\infty)$

We need

$x > 2$

$\frac{2y+5}{y-1} > 2$

$\frac{2y+5}{y-1} - 2 > 0$

$\frac{2y+5}{y-1} - \frac{2(y-1)}{y-1} > 0$

$\frac{2y+5-2y+2}{y-1} > 0$

$\frac{7}{y-1} > 0$

$\Rightarrow y-1 > 0$

$y > 1$

$\Rightarrow (1,\infty)$

$f^{-1}(x) = \frac{2x+5}{x-1}$

Domain of $f^{-1}(x)$: $(1,\infty)$

(c) Consider the function $y = f(x) = \frac{2x-1}{x+5}$ over the domain $(0, \infty)$. Find the formula and domain of its inverse function.

Inverse

$y = \frac{2x-1}{x+5}$

Solve for x

$(x+5)y = 2x-1$

$xy + 5y = 2x-1$

$xy - 2x = -5y-1$

$x(y-2) = -5y-1$

$x = \frac{-5y-1}{y-2}$

$= \frac{-(5y+1)}{-(2-y)}$

$= \frac{5y+1}{2-y} \Rightarrow f^{-1} = \frac{5x+1}{2-x}$

Since domain of f is $(0,\infty)$. We need

$x > 0$

$\frac{5y+1}{2-y} > 0$

$5y+1 > 0$ and $2-y > 0$

$5y > -1$

$y > -1/5$

$2 > y$

$\underbrace{y > -1/5 \quad 2 > y}_{2 > y > -1/5} \Rightarrow (-1/5, 2)$

$f^{-1}(x) = \frac{5x+1}{2-x}$

Domain of $f^{-1}(x)$: $(-1/5, 2)$

(d) Consider the function $y = f(x) = \frac{2x+1}{x+1}$ over the domain $(-1, 0)$. Find the formula and domain of its inverse function.

Inverse

$y = \frac{2x+1}{x+1}$

Solve for x

$(x+1)y = 2x+1$

$xy + y = 2x+1$

$xy - 2x = 1 - y$

$x(y-2) = 1-y$

$x = \frac{1-y}{y-2} = \frac{y-1}{2-y}$

Since domain of f is (-1,0).
We need $x<0$ and $x>-1$

$\frac{y-1}{2-y} < 0$

$y-1<0$ and $2-y>0$

$y<1$ and $2>y$

$y<2$

$\frac{y-1}{2-y} > -1$

$y-1 > -1(2-y)$

$y-1 > -2+y$

$-1 > -2$

True everywhere

Combine where both are true. $\Rightarrow y<1$

$f^{-1}(x) \quad = \quad (x-1)/(2-x)$

Domain of $f^{-1}(x)$: $(-\infty, 1)$

6. You are supposed to be able to compute the (right/left hand side) limit, understanding its proper meaning, and using the Squeeze Theorem. You are also supposed to be able to determine the exact value of the limit who has an indeterminate form (e.g., $0/0$, $\pm\infty/\pm\infty$, $\infty - \infty$) using some proper technique.

Example Problems

(a) Compute the following limits:

i. $\lim_{x \to 5^-} \frac{x^2 - 3x - 10}{|x-5|}$

implies $|x-5| = -(x-5)$

$$\lim_{x\to 5} \frac{x^2-3x-10}{-(x-5)} = \lim_{x\to 5} \frac{(x-5)(x+2)}{-(x-5)} = \lim_{x\to 5} -(x+2) = -7$$

$\lim_{x \to 5^-} \frac{x^2 - 3x - 10}{|x-5|} \quad = \quad -7$

ii. $\lim_{x\to(\pi/2)^-} \tan(x) = \lim_{x \to \frac{\pi}{2}^-} \frac{\sin(x)}{\cos(x)}$

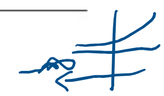
So $\sin\left(\frac{\pi}{2}\right) = 1$ no matter if $\frac{\pi}{2}^+$ or $\frac{\pi}{2}^-$

$\cos\left(\frac{\pi}{2}\right) = 0^+$ (slightly positive b/c left of $\frac{\pi}{2}$ cosine is still positive)

S | A
T | C

So $\frac{1}{0^+} = \infty$

$\lim_{x\to(\pi/2)^-} \tan(x) =$ ∞



iii. $\lim_{x\to(\pi/2)^+} e^{\tan(x)} = \exp\left[\lim_{x\to(\pi/2)^+} \tan(x)\right]$

$e^x = \exp[x]$

$= \exp\left[\lim_{x\to(\frac{\pi}{2})^+} \frac{\sin(x)}{\cos(x)}\right] = \exp\left[\frac{1}{0^-}\right] = \exp[-\infty] = 0$

So $\sin\left(\frac{\pi}{2}\right) = 1$ no matter if $\frac{\pi}{2}^+$ or $\frac{\pi}{2}^-$

S | A
T | C

$\cos\left(\frac{\pi}{2}\right) = 0^-$ (slightly negative b/c left of $\frac{\pi}{2}$ cosine is negative)

$\lim_{x\to(\pi/2)^+} e^{\tan(x)} =$ 0

iv. $\lim_{x\to 2} \left(\frac{x^2+5x-14}{x^2-6x+8}\right) = \lim_{x\to 2} \frac{(x+7)\cancel{(x-2)}}{(x-4)\cancel{(x-2)}} = \lim_{x\to 2} \frac{x+7}{x-4} = \frac{2+7}{2-4} = \frac{9}{-2}$

$\lim_{x\to 2} \left(\frac{x^2+5x-14}{x^2-6x+8}\right) =$ $-9/2$

v. $\lim_{x\to 0}\left(\frac{5}{x^2-x}+\frac{5}{x}\right) = \lim_{x\to 0}\left(\frac{5x}{x(x^2-x)} + \frac{5(x^2-x)}{x(x^2-x)}\right)$

$= \lim_{x\to 0}\left(\frac{\cancel{5x}+5x^2-\cancel{5x}}{x(x^2-x)}\right)$

$= \lim_{x\to 0}\frac{5\cancel{x^2}}{\cancel{x}\cdot\cancel{x}(x-1)} = \lim_{x\to 0}\frac{5}{x-1} = -5$

$\lim_{x\to 0}\left(\frac{5}{x^2-x}+\frac{5}{x}\right) = \underline{\quad -5 \quad}$

vi. $\lim_{x\to\infty}(\sqrt{9x^2-x}-3x)$

$= \lim_{x\to\infty}\frac{(\sqrt{9x^2-x}-3x)}{1}\cdot\frac{(\sqrt{9x^2-x}+3x)}{(\sqrt{9x^2-x}+3x)}$

$= \lim_{x\to\infty}\frac{9x^2-x-(3x)^2}{\sqrt{9x^2-x}+3x}$

$= \lim_{x\to\infty}\frac{9x^2-x-9x^2}{\sqrt{9x^2-x}+3x}$

$= \lim_{x\to\infty}\frac{-x}{\sqrt{9x^2-x}+3x}$

$= \lim_{x\to\infty}\frac{\frac{1}{x}(-x)}{\frac{1}{x}[\sqrt{9x^2-x}+3x]}$

$= \lim_{x\to\infty}\frac{-1}{\sqrt{\frac{9x^2-x}{x^2}}+3}$

$= \lim_{x\to\infty}\frac{-1}{\sqrt{9-\frac{1}{x}}+3} = \frac{-1}{\sqrt{9}+3} = \frac{-1}{6}$

$\lim_{x\to\infty}(\sqrt{9x^2-x}-3x) = \underline{\quad -1/6 \quad}$

vii. $\lim_{x\to-\infty}(\sqrt{x^2+3x}+x)$

$= \lim_{x\to-\infty}\frac{\sqrt{x^2+3x}+x}{1}\cdot\left(\frac{\sqrt{x^2+3x}-x}{\sqrt{x^2+3x}-x}\right)$

$= \lim_{x\to-\infty}\frac{\cancel{x^2}+3x-\cancel{x^2}}{\sqrt{x^2+3x}-x}$

$= \lim_{x\to-\infty}\frac{3x}{\sqrt{x^2+3x}-x}$

$= \lim_{x\to-\infty}\frac{\frac{1}{x}(3x)}{\frac{1}{x}(\sqrt{x^2+3x}-x)}$

$= \lim_{x\to-\infty}\frac{3}{-\sqrt{\frac{x^2+3x}{x^2}}-1}$

$= \lim_{x\to-\infty}\frac{3}{-\sqrt{1+\frac{3}{x}}-1} = \frac{3}{-1-1}$

$\frac{1}{x} = \frac{1}{-\sqrt{x^2}}$ b/c $x\to-\infty$

$\lim_{x\to-\infty}(\sqrt{x^2+3x}+x) = \underline{\quad -3/2 \quad}$

15

viii. $\lim_{x\to 0}\frac{|3x-4|-|5x+4|}{x}$

viii. $\lim_{x\to 0} \frac{|3x-4|-|5x+4|}{x}$

$|3x-4|$: $x \to 0 \Rightarrow |3x-4| = -(3x-4)$

$|5x+4|$: $x \to 0 \Rightarrow |5x+4| = (5x+4)$

$$\lim_{x\to 0} \frac{-(3x-4)-(5x+4)}{x} = \lim_{x\to 0} \frac{-3x+4-5x-4}{x}$$

$$= \lim_{x\to 0} \frac{-8x}{x} = \lim_{x\to 0} -8 = -8$$

$\lim_{x\to 0} \frac{|3x-4|-|5x+4|}{x}$ = -8

ix. $\lim_{x\to 0} \frac{\sin(5x)}{2x}$

$$= \lim_{x\to 0} \frac{\sin(5x)}{5x} \cdot \frac{5}{2}$$

$$= \underbrace{\left(\lim_{x\to 0} \frac{\sin(5x)}{5x}\right)}_{=1} \left(\lim_{x\to 0} \frac{5}{2}\right) = \frac{5}{2}$$

$\lim_{x\to 0} \frac{\sin(5x)}{2x}$ = $5/2$

x. $\lim_{x\to\infty} \frac{\sin(x)}{\ln(x)}$

$\sin(x) \to$ 

BUT $\ln(x) \to \infty$ when $x \to \infty$

and $\frac{1}{\ln(x)} \to 0$ when $x \to \infty$

So $\sin(x)$ doesn't matter

$\lim_{x\to\infty} \frac{\sin(x)}{\ln(x)}$ = $0.$

xi. $\lim\limits_{x\to\infty} \frac{\cos(3x)}{\ln(x)}$

$\cos(x) \rightarrow$ 

BUT $\ln(x) \to \infty$ when $x \to \infty$

and $\frac{1}{\ln(x)} \to 0$ when $x \to \infty$

So $\cos(x)$ doesn't matter.

$\lim\limits_{x\to\infty} \frac{\cos(3x)}{\ln(x)} =$ 0.

xii. $\lim\limits_{x\to 0} \frac{\sqrt{x+1}-1}{\tan(x/2)}$

$= \lim\limits_{x\to 0} \frac{\sqrt{x+1}-1}{\frac{\sin(x/2)}{\cos(x/2)}}$

$= \lim\limits_{x\to 0} \frac{(\sqrt{x+1}-1)\cos(x/2)}{\sin(x/2)}$

$= \lim\limits_{x\to 0} \frac{(\sqrt{x+1}-1)\cos(x/2)}{\sin(x/2)} \cdot \frac{(\sqrt{x+1}+1)}{(\sqrt{x+1}+1)}$

$= \lim\limits_{x\to 0} \frac{(x+1-1)\cos(x/2)}{\sin(x/2)(\sqrt{x+1}+1)}$

$= \lim\limits_{x\to 0} \frac{x}{\sin(x/2)} \cdot \lim\limits_{x\to 0} \frac{\cos(x/2)}{\sqrt{x+1}+1}$

$= \frac{1/2}{1/2} \cdot \lim\limits_{x\to 0} \frac{x}{\sin(x/2)} \cdot \lim\limits_{x\to 0} \frac{\cos(x/2)}{\sqrt{x+1}+1}$

$= \frac{1}{1/2} \cdot \lim\limits_{x\to 0} \frac{x/2}{\sin(x/2)} \cdot \lim\limits_{x\to 0} \frac{\cos(x/2)}{\sqrt{x+1}+1}$

$= 2 \cdot 1 \cdot \frac{1}{2}$

$\lim\limits_{x\to 0} \frac{\sqrt{x+1}-1}{\tan(x/2)} =$ $\boxed{1}$

xiii. $\lim\limits_{x\to 0} \frac{\sin(1/x)}{1/x} = \lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right)$

Squeezing Thm:

$-1 \le \sin\left(\frac{1}{x}\right) \le 1$

$-x \le x\sin\left(\frac{1}{x}\right) \le x$

Take the limit of $-x$ and x as $x \to 0$

$0 \le \lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) \le 0$

$\lim\limits_{x\to 0} \frac{\sin(1/x)}{1/x} =$ 0

$\Rightarrow \lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

xiv. $\lim_{x\to 8} \frac{\sqrt[3]{x}-2}{x-8} = \lim_{x\to 8} \frac{\sqrt[3]{x}-2}{(\sqrt[3]{x^3}-2^3)} = \lim_{x\to 8} \frac{\cancel{\sqrt[3]{x}-2}}{\cancel{(\sqrt[3]{x}-2)}((\sqrt[3]{x})^2+2\sqrt[3]{x}+2^2)}$

$$= \lim_{x\to 8} \frac{1}{x^{2/3}+2x^{1/3}+4}$$

$$= \frac{1}{4+2\cdot 4+4} = \frac{1}{16}$$

$\lim_{x\to 8} \frac{\sqrt[3]{x}-2}{x-8} =$ 1/16

xv. $\lim_{x\to\infty} \frac{x^3+3x^2}{2x^3+\sqrt{9x^6+4x^2}}$

$$\approx \lim_{x\to\infty} \frac{x^3}{2x^3+\sqrt{9x^6}}$$

$$= \lim_{x\to\infty} \frac{x^3}{2x^3+3x^3}$$

$$= \lim_{x\to\infty} \frac{x^3}{5x^3}$$

$$= \lim_{x\to\infty} \frac{1}{5} = \frac{1}{5}$$

$\lim_{x\to\infty} \frac{x^3+3x^2}{2x^3+\sqrt{9x^6+4x^2}} =$ 1/5

xvi. $\lim_{x\to-\infty} \frac{x^3+3x^2}{2x^3+\sqrt{9x^6+4x^4}}$

$$\approx \lim_{x\to-\infty} \frac{x^3}{2x^3+\sqrt{9x^6}}$$

$$= \lim_{x\to-\infty} \frac{x^3}{2x^3-3x^3}$$

$$= \lim_{x\to-\infty} \frac{x^3}{-x^3}$$

$$= \lim_{x\to-\infty} -1$$

$$= -1$$

$\lim_{x\to-\infty} \frac{x^3+3x^2}{2x^3+\sqrt{9x^6+4x^4}} =$ -1

xvii. $\lim_{x\to 0^-} \frac{x^3+3x^2}{2x^3+\sqrt{9x^6+4x^4}}$

$$= \lim_{x\to 0^-} \frac{x^3+3x^2}{2x^3+\sqrt{x^4(9x^2+4)}}$$

$$= \lim_{x\to 0^-} \frac{x^2(x+3)}{2x^3-x^2\sqrt{9x^2+4}}$$

$$= \lim_{x\to 0^-} \frac{x^2(x+3)}{x^2(2x-\sqrt{9x^2+4})}$$

$$= \lim_{x\to 0^-} \frac{x+3}{2x-\sqrt{9x^2+4}} = \frac{3}{-\sqrt{4}} = -3/2$$

$\lim_{x\to 0^-} \frac{x^3+3x^2}{2x^3+\sqrt{9x^6+4x^4}}$ = $-3/2$

xviii. $\lim_{x\to 0^-} \frac{2x+5x^3}{3x+\sqrt{25x^6+x^2}}$

$$= \lim_{x\to 0^-} \frac{x(2+5x^2)}{3x+\sqrt{x^2(25x^4+1)}}$$

$$= \lim_{x\to 0^-} \frac{x(2+5x^2)}{3x-x\sqrt{25x^4+1}}$$

$$= \lim_{x\to 0^-} \frac{\cancel{x}(2+5x^2)}{\cancel{x}(3-\sqrt{25x^4+1})}$$

$$= \frac{2}{3-\sqrt{1}} = \frac{2}{2} = 1$$

$\lim_{x\to 0^-} \frac{2x+5x^3}{3x+\sqrt{25x^6+x^2}}$ = 1

xix. $\lim_{x\to 0^+} \frac{2x+5x^3}{3x+\sqrt{25x^6+x^2}}$

$$= \lim_{x\to 0^+} \frac{x(2+5x^2)}{3x+\sqrt{x^2(25x^4+1)}}$$

$$= \lim_{x\to 0^+} \frac{x(2+5x^2)}{3x+x\sqrt{25x^4+1}}$$

$$= \lim_{x\to 0^+} \frac{\cancel{x}(2+5x^2)}{\cancel{x}(3+\sqrt{25x^4+1})} = \frac{2}{3+\sqrt{1}} = \frac{2}{4} = \frac{1}{2}$$

$\lim_{x\to 0^+} \frac{2x+5x^3}{3x+\sqrt{25x^6+x^2}}$ = $1/2$

xx. $\lim_{x\to\infty} (\sqrt{x+1} - \sqrt{x}) \cdot \sqrt{x}$

$$= \lim_{x\to\infty} \frac{(\sqrt{x+1} - \sqrt{x})\sqrt{x}}{1} \cdot \frac{(\sqrt{x+1} + \sqrt{x})}{(\sqrt{x+1} + \sqrt{x})}$$

$$= \lim_{x\to\infty} \frac{(x+1-x)\sqrt{x}}{(\sqrt{x+1} + \sqrt{x})}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+1} + \sqrt{x}}$$

$$\approx \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x}}{2\sqrt{x}} = \frac{1}{2}$$

$\lim_{x\to\infty} (\sqrt{x+1} - \sqrt{x}) \cdot \sqrt{x} =$ 1/2

7. You are supposed to understand the meaning of the Intermediate Value Theorem (IVT), and to be able to use it to show that a certain equation has a root in a specified interval.

Example Problems

(a) Using the IVT determine on which of the following intervals the equation $x^3 - 3x = 5$ has a root?

(A) (0,1)
(B) (1,2)
(C) (2,3)
(D) (3,4)
(E) (4,5)

$f(x) = x^3 - 3x - 5 = 0$

Plug in $x = 0, 1, 2, 3, 4, 5$ and determine the sign of their value. (i.e. positive, negative)

Stop when you have + value then − value (or vice versa)

$f(0) = -5$
$f(1) = 1-3-5 = -7$
$f(2) = 8-6-5 = -11$
$f(3) = 27-9-5 = +13$

Answer: C (2,3)

(b) From the IVT, which of the following interval, can one conclude, contains a solution of the equation $2x^3 + 3x^2 = 5x - 3$?

(A) (-3,-2)
(B) (-2,-1)
(C) (-1,0)
(D) (0,1)
(E) (1,2)

$f(x) = 2x^3 + 3x^2 - 5x + 3 = 0$

Plug in $x = -3, -2, -1, 0, 1, 2$ and determine the sign of their value. (i.e. positive, negative)

Stop when you have + value then − value (or vice versa)

$f(-3) = 2(-27) + 27 + 15 + 3 = -27 + 18 = -$

$f(-2) = 2(-8) + 3(4) + 10 + 3 = -16 + 12 + 13 = +$

Answer: A $(-3, 2)$

(c) From the IVT, which of the following interval, can one conclude, contains a solution of the equation $x^3 - 3x^2 = -2x + 1$?

(A) (-1,0)
(B) (0,1)
(C) (1,2)
(D) (2,3)
(E) (3,4)

$f(x) = x^3 - 3x^2 + 2x - 1 = 0$

Plug in $x = -1, 0, 1, 2, 3, 4$ and determine the sign of their value. (i.e. positive, negative)

Stop when you have + value then − value (or vice versa)

$f(-1) = -1 - 3 - 2 + 1 = -$

$f(0) = -1 = -$

$f(1) = 1 - 3 + 2 - 1 = -$

$f(2) = 8 - 3(4) + 2 - 1 = -$

$f(3) = 27 - 3(9) + 6 - 1 = +$

Answer: D $(2, 3)$

8. You are supposed to understand the meaning of the defining formula of the derivative, and being able to determine the values of the related limits.

8. You are supposed to understand the meaning of the defining formula of the derivative, and being able to determine the values of the related limits.

Example Problems

(a) Suppose we have a function $f(x)$ with $f'(2) = 5$. Determine the following values:

i. $g'(1)$ where $g(x) = f(2x)$

$$g'(x) = f'(2x) \cdot 2$$
$$g'(1) = f'(2) \cdot 2 = 5 \cdot 2 = 10$$

$g'(1) = $ 10

ii. $\lim_{h\to 0} \frac{f(2+4h)-f(2)}{3h} \cdot \frac{4}{4} = \lim_{h\to 0} \frac{f(2+4h)-f(2)}{4h} \cdot \frac{4}{3}$

$$= \frac{4}{3} \lim_{h\to 0} \frac{f(2+4h)-f(2)}{4h}$$

since $h \to 0$

Then $\underset{u}{\overset{\shortparallel}{4h}} \to 0$

So $u \to 0$

Let $u = 4h$

$$= \frac{4}{3} \lim_{u\to 0} \frac{f(2+u)-f(2)}{u} = \frac{4}{3} f'(2)$$
$$= \frac{4}{3} \cdot 5$$

$\lim_{h\to 0} \frac{f(2+4h)-f(2)}{3h} = \boxed{\frac{20}{3}}$

iii. $\lim_{h\to 0} \frac{f(2+4h)-f(2-5h)}{7h}$

$$= \lim_{h\to 0} \frac{f(2+4h)-f(2)+f(2)-f(2-5h)}{7h}$$
$$= \lim_{h\to 0} \frac{f(2+4h)-f(2)}{7h} + \lim_{h\to 0} \frac{f(2)-f(2-5h)}{7h}$$
$$= \frac{4}{4} \lim_{h\to 0} \frac{f(4h+2)-f(2)}{7h} - \frac{5}{5} \lim_{h\to 0} \frac{f(-5h+2)-f(2)}{7h}$$
$$= \frac{4}{7} \lim_{h\to 0} \frac{f(4h+2)-f(2)}{4h} + \frac{5}{7} \lim_{h\to 0} \frac{f(-5h+2)-f(2)}{-5h}$$
$$= \frac{4}{7} f'(2) + \frac{5}{7} f'(2) = \frac{9}{7} f'(2) = \frac{9}{7}(5) = \boxed{\frac{45}{7}}$$

(b) Suppose we have a function $f(x)$ whose derivative at $x = 5$ is equal to 3. That is to say, we have

$$f'(5) = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} = 3$$

then evaluate the following limit

$$\lim_{h\to 0} \frac{f(5+2h) - f(5-4h)}{h}$$

$$= \lim_{h\to 0} \frac{f(5+2h) - f(5) + f(5) - f(5-4h)}{h}$$

$$= \lim_{h\to 0} \frac{f(5+2h) - f(5)}{h} - \lim_{h\to 0} \frac{f(5-4h) - f(5)}{h}$$

$$= \frac{2}{2} \lim_{h\to 0} \frac{f(2h+5) - f(5)}{h} - \frac{4}{4} \lim_{h\to 0} \frac{f(-4h+5) - f(5)}{h}$$

$$= 2 \lim_{h\to 0} \frac{f(2h+5) - f(5)}{2h} + 4 \lim_{h\to 0} \frac{f(-4h+5) - f(5)}{-4h}$$

$$= 2f'(5) + 4f'(5)$$

$$= 2(3) + 4(3) = 18$$

$$\lim_{h\to 0} \frac{f(5+2h) - f(5-4h)}{h} = \underline{\quad 18 \quad}$$

(c) Let $f(x)$ be the function defined as follows:

$$f(x) = \begin{cases} -(x-1)^2 + 5 & \text{if } x < 0 \\ 2x + 4 & \text{if } x \geq 0 \end{cases}$$

Does $f'(0)$ exists? If it exists, compute the value of $f'(0)$.

✓ $\lim_{x\to 0^-} -(x-1)^2 + 5 = -(-1)^2 + 5 = -1 + 5 = 4$

$\lim_{x\to 0^+} 2x + 4 = 4$

$f'(x)$ @ $x = 0$ we use $f(x) = 2x + 4$

$f'(x) = 2$

$f'(0) = 2$

Answer: $f'(0) = 2$

(d) We have a function $f(x)$ such that

$$\begin{cases} f(4) = 3 \\ f'(4) = 2 \end{cases}$$

Compute the following limit:

$$\lim_{h \to 0} \frac{f(4+h)\sqrt{4+h} - 6}{h}$$

HINT: What is the relation between the derivative of the product function $g(x) = f(x)\sqrt{x}$ at $x = 4$ and the given limit above?

$$g'(4) = \lim_{h \to 0} \frac{g(4+h) - g(4)}{h}$$

$$= \lim_{h \to 0} \frac{f(4+h)\sqrt{4+h} - f(4)\sqrt{4}}{h}$$

$$= \lim_{h \to 0} \frac{f(4+h)\sqrt{4+h} - (3)2}{h}$$

$$= \lim_{h \to 0} \frac{f(4+h)\sqrt{4+h} - 6}{h} \checkmark$$

$$g'(x) = f'(x)\sqrt{x} + f(x)\frac{1}{2\sqrt{x}}$$

$$g'(4) = f'(4)\sqrt{4} + f(4)\frac{1}{2\sqrt{4}}$$

$$= 2 \cdot 2 + \frac{3}{4}$$

$$= \frac{16}{4} + \frac{3}{4}$$

$$\lim_{h \to 0} \frac{f(4+h)\sqrt{4+h} - 6}{h} = \underline{\quad 19/4 \quad}$$

(e) Compute the following limit

$$\lim_{h \to 0} \frac{\ln(1+3h)}{7h}$$

Remember $\ln(1) = 0$

$$\lim_{h \to 0} \frac{\ln(1+3h) - \ln(1)}{7h}$$

$$= \frac{3}{3} \lim_{h \to 0} \frac{\ln(1+3h) - \ln(1)}{7h}$$

$$= \frac{3}{7} \lim_{h \to 0} \frac{\ln(1+3h) - \ln(1)}{3h}$$

Let $g(x) = \ln(x)$

So $\underbrace{\frac{3}{7} \lim_{h \to 0} \frac{\ln(1+3h) - \ln(1)}{3h}}_{g'(1)}$

$$g(x) = \ln(x)$$

$$g'(x) = \frac{1}{x}$$

$$\frac{3}{7} g'(1) = \frac{3}{7} \cdot \frac{1}{1} = \frac{3}{7}$$

$$\lim_{h \to 0} \frac{\ln(1+3h)}{7h} = \underline{\quad 3/7 \quad}$$

9. When a function is defined piecewise nd depending on some variables, you are supposed to know how to determine those variables so that the function becomes continuous entirely over the specified interval (e.g., everywhere).

Example Problems

(a) Find the values of a and b so that the function

$$f(x) = \begin{cases} x^2 - a & \text{if } x \le 1 \\ \dfrac{3x^2+12x-b}{x^2+2x-3} & \text{if } x > 1 \end{cases}$$

is continuous on $(-\infty, \infty)$.

$\to \dfrac{3x^2+12x-b}{(x-1)(x+3)}$

Note that
$x^2+2x-3 \neq 0$
$(x-1)(x+3) \neq 0$
$x \neq 1,\ x \neq -3$
So $x > 1$ the function is continuous

Since $x = +1$ is a VA.
$\lim_{x \to 1^+} (x^2+2x-3) = 0$
So let's find the b for
$\lim_{x \to 1^+} \underbrace{(3x^2+12x-b)}_{\text{numerator}} = 0$

$3+12-b=0$
$15=b$

Next find
$\lim_{x \to 1^+} \dfrac{3x^2+12x-15}{x^2+2x-3}$
$= \lim_{x \to 1} \dfrac{3\cancel{(x-1)}(x+5)}{(x+3)\cancel{(x-1)}}$
$= \lim_{x \to 1} \dfrac{3(x+5)}{(x+3)} = \dfrac{3 \cdot 6}{4} = \dfrac{9}{2}$

$\lim_{x \to 1} x^2 - a = \dfrac{9}{2}$
$1 - a = 9/2$
$1 - 9/2 = a$
$a = -7/2$

a = $-7/2$

b = 15

(b) Find the values of a and b so that the function

$$f(x) = \begin{cases} x - b + 7 & \text{if } x \le 3 \\ \dfrac{ax+b}{x-3} & \text{if } x > 3 \end{cases}$$

is continuous on $(-\infty, \infty)$.

$\to$ Note $x - 3 \neq 0$
$x \neq 3$
So $x > 3$ function is continuous

Since $x = 3$ is a VA.
$\lim_{x \to 3^+} (x-3) = 0$
So let's work w/ numerator
$\lim_{x \to 3^+} (ax+b) = 0$
$a(3)+b=0$
$3a+b=0$
$b=-3a$

Next clean up
$\lim_{x \to 3^+} \dfrac{ax+b}{x-3}$
$= \lim_{x \to 3^+} \dfrac{ax-3a}{x-3}$
$= \lim_{x \to 3^+} \dfrac{a\cancel{(x-3)}}{\cancel{x-3}}$
$= a$

$\lim_{x \to 3^-} (x-b+7) = a$
$3-b+7=a$
Remember $b=-3a$
$3+3a+7=a$
$10=-2a$
$a=-5$
And $b=-3(-5)=15$

a = -5

b = 15

10. You are supposed to be able to find the horizontal/vertical asymptote(s) of a given function.

Example Problems

(a) Find the horizontal/veritcal asymptote(s) of the following functions

i. $y = f(x) = \dfrac{x^3 + 4x^2 + x - 6}{x(x^2 - 1)}$

HA: $\lim\limits_{x \to \pm\infty} \dfrac{x^3+4x^2+x-6}{x(x^2-1)}$

$= \lim\limits_{x \to \pm\infty} \dfrac{x^3+4x^2+x-6}{x^3-x}$

$\approx \lim\limits_{x \to \pm\infty} \dfrac{x^3}{x^3}$

$= \lim\limits_{x \to \pm\infty} 1 = 1$

VA: $x(x^2-1)=0$

$x(x-1)(x+1)=0$

$x = 0, 1, -1$

Possible VA

Check w/ limits

$\lim\limits_{x \to 0^-} f(x) = \dfrac{-6}{0^- \cdot (-1)} = -\infty$

$\lim\limits_{x \to 0^+} f(x) = \dfrac{-6}{0^-(-1)} = \infty$

$\checkmark$ $x=0$

$\lim\limits_{x \to 1^-} f(x) = \dfrac{0}{1(0)}$ ✗ → Hole

$\lim\limits_{x \to -1^-} f(x) = \dfrac{-1+4-1-6}{-1\,(0^+)} = \dfrac{4}{0^+} = \infty$

$\lim\limits_{x \to -1^+} f(x) = \dfrac{4}{0^-} = -\infty$

$\checkmark$ $x=-1$

HA: $y=1$

VA: $x = 0, -1$

ii. $y = f(x) = \dfrac{7 + e^x}{4 - e^x}$

HA: $\lim\limits_{x \to \infty} \dfrac{7+e^x}{4-e^x}$

$= \lim\limits_{x \to \infty} \dfrac{(7+e^x)/e^x}{(4-e^x)/e^x}$

$= \lim\limits_{x \to \infty} \dfrac{\frac{7}{e^x}+1}{\frac{4}{e^x}-1} = -1$

$\lim\limits_{x \to -\infty} \dfrac{7+e^x}{4-e^x} = \dfrac{7}{4}$

VA: $4-e^x=0$

$e^x = 4$

$x = \ln(4)$

Check w/ limits

$\lim\limits_{x \to \ln(4)^-} f(x) = \dfrac{7+e^{\ln(4)}}{0^-} = -\infty$

$\lim\limits_{x \to \ln(4)^+} f(x) = \dfrac{7+e^{\ln(4)}}{0^+} = \infty$

HA: $y = -1, 7/4$

VA: $x = \ln(4)$

Not factorable

$x^2(\underbrace{25x^4-9x^2+1}) = 0$

$x = 0$

iii. $y = f(x) = \dfrac{x^3+x^2}{3x^2+\sqrt{25x^6+x^2}}$

HA: $\lim\limits_{x\to\pm\infty} \dfrac{x^3+x^2}{3x^2+\sqrt{25x^6+x^2}}$

$\approx \lim\limits_{x\to\pm\infty} \dfrac{x^3}{3x^2+\sqrt{25x^6}}$

$= \lim\limits_{x\to\pm\infty} \dfrac{x^3}{3x^2\pm 5x^3}$

$\approx \lim\limits_{x\to\pm\infty} \dfrac{x^3}{\pm 5x^3}$

$= \lim\limits_{x\to\pm\infty} \dfrac{1}{\pm 5} = \pm\dfrac{1}{5}$

VA: $3x^2+\sqrt{25x^6+x^2} = 0$

$\sqrt{25x^6+x^2} = -3x^2$

$25x^6+x^2 = (-3x^2)^2$

$25x^6+x^2 = 9x^4$

$25x^6-9x^4+x^2 = 0$

Check w/ limits

BUT note we can factor x from the numerator and denominator and get

$f(x) = \dfrac{x^2+x}{3x+\sqrt{25x^4+1}}$

$\lim\limits_{x\to 0^-} f(x) = \dfrac{0^-}{1} = 0 \neq \infty$

So not VA @ $x=1$

HA: $y = -1/5, 1/5$

VA: NONE

iv. $y = f(x) = \dfrac{4e^{2x}-16}{e^{2x}-3e^x-10}$

HA: $\lim\limits_{x\to-\infty} \dfrac{4e^{2x}-16}{e^{2x}-3e^x-10} = \dfrac{-16}{-10} = \dfrac{8}{5}$

$\lim\limits_{x\to\infty} \dfrac{4e^{2x}-16}{e^{2x}-3e^x-10}$

$= \lim\limits_{x\to\infty} \dfrac{(4e^{2x}-16)/e^{2x}}{(e^{2x}-3e^x-10)/e^{2x}}$

$= \lim\limits_{x\to\infty} \dfrac{4-\frac{16}{e^{2x}}}{1-\frac{3}{e^x}-\frac{10}{e^{2x}}} = 4$

b/c $\lim\limits_{x\to\infty} \dfrac{\#}{e^x} = 0$

VA: $e^{2x}-3e^x-10 = 0$

$u = e^x$

$u^2-3u-10 = 0$

$(u-5)(u+2) = 0$

$u = 5, 2$

$e^x = 5$ and $e^x = 2$

$x = \ln 5$ $\quad$ $x = \ln 2$

Check w/ limits

$\left.\begin{array}{l}\lim\limits_{x\to\ln(5)^-} f(x) = \dfrac{+}{0^-} = -\infty \\ \lim\limits_{x\to\ln(5)^+} f(x) = \dfrac{+}{0^+} = \infty\end{array}\right\} \checkmark$

$\lim\limits_{x\to\ln(2)^-} f(x) = \dfrac{4e^{2\ln(2)}-16}{0^-}$

$= \dfrac{4e^{\ln(4)}-16}{0^-}$

$= \dfrac{4\cdot 4-16}{0^-} = \dfrac{0}{0^-}$

$\Rightarrow$ Hole at $x=\ln(2)$

HA: $y = 8/5, 4$

VA: $x = \ln(5)$

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v. $y = f(x) = \dfrac{x-7}{\sqrt{2x^2+7x}}$

HA: $\lim_{x\to\infty} \dfrac{x-7}{\sqrt{2x^2+7x}}$

$\approx \lim_{x\to\infty} \dfrac{x}{\sqrt{2x^2}}$

$= \lim_{x\to\infty} \dfrac{x}{\sqrt{2}\cdot x} = \dfrac{1}{\sqrt{2}}$

$\lim_{x\to-\infty} \dfrac{x-7}{\sqrt{2x^2+7x}}$

$\approx \lim_{x\to-\infty} \dfrac{x}{-\sqrt{2}x} = \dfrac{-1}{\sqrt{2}}$

VA: $\sqrt{2x^2+7x} = 0$

$2x^2+7x=0$

$x(2x+7)=0$

$x = 0, -7/2$

Check w/ limits

$\lim_{x\to 0^-} f(x) = \dfrac{-7}{0^-} = +\infty$ ✓

$\lim_{x\to 0^+} f(x) = \dfrac{-7}{0^+} = -\infty$

$\lim_{x\to -\frac{7}{2}^-} f(x) = \dfrac{-}{0^-} = \infty$ ✓

$\lim_{x\to -\frac{7}{2}^+} f(x) = \dfrac{-}{0^+} = -\infty$

HA: $y = 1/\sqrt{2}, -1/\sqrt{2}$

VA: $x = 0, -7/2$

vi. $y = f(x) = \dfrac{e^{2x}-1}{(e^x-1)(e^x-2)}$

$= \dfrac{\cancel{(e^x-1)}(e^x+1)}{\cancel{(e^x-1)}(e^x-2)}$

$= \dfrac{e^x+1}{e^x-2}$

HA: $\lim_{x\to-\infty} \dfrac{e^x+1}{e^x-2} = \dfrac{-1}{2}$

$\lim_{x\to\infty} \dfrac{e^x+1}{e^x-2}$

$= \lim_{x\to\infty} \dfrac{(e^x+1)/e^x}{(e^x-2)/e^x}$

$= \lim_{x\to\infty} \dfrac{1+\frac{1}{e^x}}{1-\frac{2}{e^x}} = 1$

VA: $e^x-2=0$

$e^x=2$

$x = \ln 2$

Check w/ limits

$\lim_{x\to\ln(2)^-} \dfrac{e^x+1}{e^x-2} = \dfrac{+}{0^-} = -\infty$ ✓

$\lim_{x\to\ln(2)^+} \dfrac{e^x+1}{e^x-2} = \dfrac{+}{0^+} = \infty$

HA: $y = -1/2, 1$

VA: $x = \ln(2)$

11. You are supposed to be able to compute the derivative of a polynomial (square root/rational) function, and understand that its value represents the slope of the tangent line to the graph of the function.

Example Problems

(a) Find the equation of the line that is tangent to the curve $y = \sin(x)$ over $(0, \pi)$ and that is parallel to the line $y = -\frac{1}{2}x + 5$.

$f'(x) = \cos(x)$

We want parallel to $y = -\frac{1}{2}x + 5$

$f'(x) = -\frac{1}{2}$

$\cos(x) = -\frac{1}{2}$

$x = \frac{2\pi}{3}$ b/c $(0, \pi)$

Find y when $x = \frac{2\pi}{3}$

$y = \sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

Pt: $\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$

$y - \frac{\sqrt{3}}{2} = -\frac{1}{2}\left(x - \frac{2\pi}{3}\right)$

$y - \frac{\sqrt{3}}{2} = \frac{-1}{2}x + \frac{\pi}{3}$

$y = \frac{-1}{2}x + \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

Answer: $y = -\frac{1}{2}x + \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

(b) Find an equation for the line tangent to the graph of $f(x) = \sin(x)$ over the interval $(0, \pi)$, which is perpendicular to the line $y = 2x + 1$.

$f'(x) = \cos(x)$

We want perpendicular to $y = 2x + 1$

$f'(x) = -\frac{1}{2}$

$\cos(x) = -\frac{1}{2}$

$x = \frac{2\pi}{3}$ b/c $(0, \pi)$

Find y when $x = \frac{2\pi}{3}$

$y = \sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)$

$= \frac{\sqrt{3}}{2}$

Pt: $\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$

$y - \frac{\sqrt{3}}{2} = \frac{-1}{2}\left(x - \frac{2\pi}{3}\right)$

$y - \frac{\sqrt{3}}{2} = \frac{-1}{2}x + \frac{\pi}{3}$

$y = \frac{-1}{2}x + \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

Answer: $y = -\frac{1}{2}x + \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

(c) Find an equation for the line tangent to the graph of $f(x) = \tan(x)$ over the interval $(-\pi/2, 0)$ which is parallel to the line $y = 2x + 1$.

$f'(x) = \sec^2(x)$

We want parallel to $y = 2x+1$

So $f'(x) = 2$

$\sec^2(x) = 2$

$\sec(x) = \pm\sqrt{2}$

$\frac{1}{\cos(x)} = \pm\sqrt{2}$

$\cos(x) = \frac{1}{\pm\sqrt{2}}$

$\cos(x) = \pm\frac{\sqrt{2}}{2}$

$x = \frac{\pi}{4}$ but $(-\pi/2, 0)$

$\Downarrow$

$x = -\pi/4$

Find y when $x = -\pi/4$

$y = \tan(-\pi/4) = -1$

Pt: $(-\pi/4, -1)$

$y - (-1) = 2\left(x - \frac{\pi}{4}\right)$

$y + 1 = 2x - \frac{\pi}{2}$

$y = 2x - \frac{\pi}{2} - 1$

Answer: $y = 2x - \frac{\pi}{2} - 1$

(d) Find the equation of the line(s) which is tangent to the parabola $y = x^2$ and passes through the point $(1, -3)$.

$f(x) = x^2$

$f'(x) = 2x$

$f'(1) = 2$

$y - y_1 = f'(x_1)(x - x_1)$

$y - (-3) = f'(1)(x - 1)$

$y + 3 = 2(x - 1)$

$y + \cancel{3} = 2x - 2$

$\quad\ \cancel{-3} \qquad -3$

$y = 2x - 5$

Answer: $y = 2x - 5$

12. You are supposed to understand the meaning of the continuity and differentiability, and their difference.

Example Problems

(a) Determine when the following function is continuous/differentiable.

i. $f(x) = \begin{cases} \dfrac{x-2}{|x-2|} & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases}$

$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

$\lim_{x \to 2^-} \frac{x-2}{|x-2|}$

$= \lim_{x \to 2^-} \frac{\cancel{x-2}}{-(\cancel{x-2})} = -1$

$\lim_{x \to 2^+} \frac{x-2}{|x-2|}$

$= \lim_{x \to 2^+} \frac{x-2}{x-2} = 1$

Continuous? Nope

Differentiable? Nope

ii. $g(x) = x|x-2|$

$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^+} g(x) = g(2)$

$\lim_{x \to 2^-} x|x-2|$

$= \lim_{x \to 2^-} x(-(x-2)) = 0$

$\lim_{x \to 2^+} x|x-2|$

$= \lim_{x \to 2^+} x(x-2) = 0$

Continuous? Yes

Differentiable? Nope b/c $|x-2|$ sharper corner

iii. $h(x) = x(x-2)|x-2|$

$\lim_{x \to 2^-} x(x-2)|x-2| = \lim_{x \to 2^-} x(x-2)[-(x-2)] = 0$

$\lim_{x \to 2^+} x(x-2)|x-2| = \lim_{x \to 2^+} x(x-2)(x-2) = 0$

$= h(2)$

Continuous? Yes

Differentiable? Yes b/c $(x-2)$ extra terms smooth the graph

Also $h'(2) = 0$

(b) Consider the following piece-wise defined function:

$$h(x) = \begin{cases} -4x & \text{if } x \leq 0 \\ x(x-2)|x-2| & \text{if } x > 0 \end{cases}$$

Judge contunuity/ differentiability of the function (i) at $x = 0$ and (ii) at $x = 2$.

$\lim_{x\to 0^-} -4x = 0$

$\lim_{x\to 0^+} x(x-2)|x-2| = 0$

$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$ continuous

$f'(0) = 0 \Rightarrow$ differentiable @ $x = 0$

By (iii) on pg. 31, we know

Continuous at $x = 0$? Yes

Differentiable at $x = 0$? Yes

Continuous at $x = 2$? Yes

Differentiable at $x = 2$? Yes

What happens if we change the first part of the description of $h(x)$ to "$3x$ if $x \leq 0$"?

The statements for $x = 2$ don't change.

The continuity statement for $x = 0$ doesn't change

since $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$0 = \lim_{x\to 0^-} (3x)$

BUT f' doesn't exist

cause the positive

slope creates a sharp turn

(c) Judge continuity/differentiability of the following function at $x = 0$.

(c) Judge continuity/differentiability of the following function at $x = 0$.

i. $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$

$$-1 \leq \sin\left(\tfrac{1}{x}\right) \leq 1$$

$$-x \leq x\sin\left(\tfrac{1}{x}\right) \leq x$$

$$\lim_{x\to 0}(-x) \leq \lim_{x\to 0} x\sin\left(\tfrac{1}{x}\right) \leq \lim_{x\to 0}(x)$$

$$0 \leq \lim_{x\to 0} x\sin\left(\tfrac{1}{x}\right) \leq 0$$

$$\lim_{x\to 0} x\sin\left(\tfrac{1}{x}\right) = 0 = f(0)$$

Continuous at $x = 0$? Yes

Differentiable at $x = 0$? No b/c f' DNE

ii. $f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$

$$-1 \leq \sin\left(\tfrac{1}{x}\right) \leq 1$$

$$-x^2 \leq x^2\sin\left(\tfrac{1}{x}\right) \leq x^2$$

$$\lim_{x\to 0}(-x^2) \leq \lim_{x\to 0} x^2\sin\left(\tfrac{1}{x}\right) \leq \lim_{x\to 0}(x^2)$$

$$0 \leq \lim_{x\to 0} x^2\sin\left(\tfrac{1}{x}\right) \leq 0$$

$$\lim_{x\to 0} x^2\sin\left(\tfrac{1}{x}\right) = 0 = f(0)$$

Continuous at $x = 0$? Yes

Differentiable at $x = 0$? Yes b/c f' exists

13. You are supposed to know how to compute the derivative of the sum, product, and quotient of two functions. You are suppose to know the following relation between the derivative of the original function and its inverse.

Fact: Suppose that we have the function f and its inverse f^{-1} and that $f(a) = b$ (i.e., $a = f^{-1}(b)$.) Then

$$\{f^{-1}\}'(b) = \frac{1}{f'(\underset{\downarrow}{a})}$$

$$f^{-1}(b)$$

Example Problems

$\cdot \quad f(-3)=0$

(a) Suppose that f, g and h are functions that satisfy the following conditions: $\begin{cases} f(0) = 2, f'(0) = 3, f'(-3) = 2. \quad -3) \\ g(0) = -1, g'(0) = 5, \\ h(0) = 1, h'(0) = -2 \end{cases}$

Evaluate the derivative of the following function at $x = 0$.

$$F(x) = \frac{f^{-1}(x)}{g(x) - h(x)}$$

$$F'(x) = \frac{[f^{-1}(x)]'(g(x)-h(x)) - [f^{-1}(x)](g'(x)-h'(x))}{(g(x)-h(x))^2}$$

$$F'(x) = \frac{\frac{1}{f'(f^{-1}(x))}(g(x)-h(x)) - [f^{-1}(x)](g'(x)-h'(x))}{(g(x)-h(x))^2}$$

$$F'(0) = \frac{\frac{g(0)-h(0)}{f'(f^{-1}(0))} - [f^{-1}(0)](g'(0)-h'(0))}{(g(0)-h(0))^2}$$

$$f(-3)=0 \iff f^{-1}(0)=-3$$

$$= \frac{\frac{-1-1}{f'(-3)} - (-3)(5-(-2))}{(-1-1)^2} = \frac{\frac{-2}{2}+3(7)}{(-2)^2} = \frac{-1+21}{4} = 5$$

F'(0) = 5

i. Suppose that f is a function that satisfy the following conditions:

Formula:

$$f(2) = 3 \text{ and } f'(2) = 5$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Evaluate the derivative of the following function at $x = 3$

$f(2) = 3 \iff f^{-1}(3) = 2$

$$F(x) = \frac{1}{f^{-1}(x)} = [f^{-1}(x)]^{-1}$$

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(2)} = \frac{1}{5}$$

$$F'(x) = -1[f^{-1}(x)]^{-2}(f^{-1}(x))'$$

$$F'(3) = -1[f^{-1}(3)]^{-2}(f^{-1}(3))'$$

$$F'(3) = -1[2]^{-2}\left(\frac{1}{5}\right)$$

$$= -\frac{1}{2^2} \cdot \frac{1}{5}$$

$$= -\frac{1}{20}$$

F'(3) = $-1/20$

ii. Suppose that f is a function that satisfy the following conditions:

Formula:

$$f(5) = 2 \text{ and } f'(5) = 3$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Evaluate the derivative of the following function at $x = 2$

$f(5) = 2 \iff f^{-1}(2) = 5$

$$F(x) = \frac{1}{f^{-1}(x)} = [f^{-1}(x)]^{-1}$$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(5)} = \frac{1}{3}$$

$$F'(x) = -1[f^{-1}(x)]^{-2}(f^{-1}(x))'$$

$$F'(2) = -[f^{-1}(2)]^{-2}(f^{-1}(2))'$$

$$F'(2) = -1[5]^{-2}\left(\frac{1}{3}\right)$$

$$= \frac{-1}{5^2} \cdot \frac{1}{3}$$

$$= -1/75$$

F'(3) = $-1/75$