FA25_MA165_Study_Guide_Exam_2

Wednesday, October 15, 2025 9:56 AM

FA25_MA1
65_Study...

Please show **all** your work! Answers without supporting work will not be given credit.
Write answers in spaces provided.

Name: Solutions

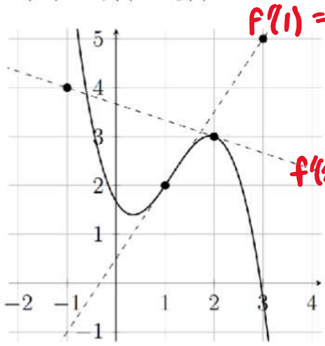
1. You are supposed to be able to use the chain rule properly and precisely, even when the function is obtained as the composition of several functions. You are supposed to understand the relation of the derivative of a one-to-one function and that of its inverse. You are supposed to be able to compute the derivatives of the inverse trigonometric functions.

Example Problems

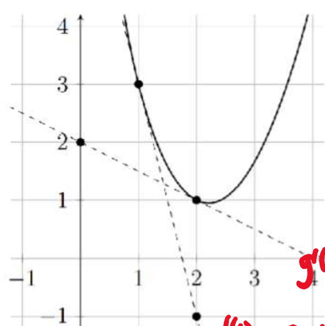
(1.1) Let $f(x)$ and $g(x)$ be two differentiable functions with the following graphs

$f'(1) = \frac{5-2}{3-1} = \frac{3}{2}$

$f'(2) = \frac{4-3}{-1-2} = \frac{-1}{-3}$



Graph of $f(x)$



Graph of $g(x)$

$g'(2) = \frac{2-1}{0-2} = -\frac{1}{2}$

$g'(1) = \frac{3-1}{1-2} = -2$

The dashed lines in the figures are tangent to the graphs at the indicated points.

(1.1.1) What is the derivative of $g(f(x))$ at $x = 1$?

$g'(f(1))\,f'(1)$

$= g'(2)\left(\frac{3}{2}\right) = \left(-\frac{1}{2}\right)\left(\frac{3}{2}\right)$

$[g \circ f]'(1)) =$ $-3/4$

(1.1.2) What is the derivative of $g(f(3-x))$ at $x = 2$?

$g'(f(3-x))\,f'(3-x)(-1)$

$g'(f(1))\,f'(1)(-1) = -(-3/4)$

$[g \circ f]'(3-2) =$ $3/4$

(1.1.3) What is the derivative of $[g(f(3-x))]^2$ at $x = 2$?

$([g \circ f](3-2))^{2\prime} =$ $3/2$

$2[g(f(3-x))]\,g'(f(3-x))\,f'(3-x)(-1)$

$2[g(f(1))]\,g'(f(1))\,f'(1)(-1) = 2[g(2)]\left(\frac{3}{4}\right) = 2(1)\left(\frac{3}{4}\right)$

(1.2) Let $h(x) = f(g(x^4))$. Assume

$$\begin{cases} f(5) = 13 & f'(5) = -3 \\ g(16) = 5 & g'(16) = 7 \end{cases}$$

Compute $h'(2)$.

$$h'(x) = f'(g(x^4))\, g'(x^4) \cdot \frac{d}{dx}(x^4)$$
$$= f'(g(x^4))\, g'(x^4) \cdot 4x^3$$
$$h'(2) = f'(g(16))\, g'(16) \cdot 4(2)^3$$
$$= f'(5)\, g'(16) \cdot 4(2)^3$$
$$= (-3)(7) \cdot 4(8)$$

$$\begin{array}{r} 32 \\ \times 21 \\ \hline 32 \\ +\ 64x \\ \hline 672 \end{array}$$

$h'(2) =$ -672

(1.3) Suppose that $F(x) = \dfrac{f(x)}{g(f(x))}$ and that the functions f and g satisfy the following conditions:

$$\begin{cases} f(2) = 4 & f'(2) = 3 \\ g(2) = -1 & g'(2) = 4 \\ g(4) = 3 & g'(4) = 2 \end{cases}$$

Find $F'(2)$.

$$F'(x) = \frac{f'(x)\, g(f(x)) - f(x) \frac{d}{dx}[g(f(x))]}{[g(f(x))]^2}$$
$$= \frac{f'(x)\, g(f(x)) - f(x)\, g'(f(x))\, f'(x)}{[g(f(x))]^2}$$
$$F'(2) = \frac{f'(2)\, g(f(2)) - f(2)\, g'(f(2))\, f'(2)}{[g(f(2))]^2}$$
$$= \frac{3\, g(4) - 4\, g'(4) \cdot 3}{[g(4)]^2}$$

$$F'(2) = \frac{3 \cdot 3 - 4 \cdot 2 \cdot 3}{3^2}$$
$$= \frac{9 - 24}{9}$$
$$= -\frac{15}{9} = -\frac{5}{3}$$

$F'(2) =$ $-5/3$

(1.4) Compute the derivative of the following function:

(1.4.1) $y = \sin(\sin(\sin(x)))$

$$y' = \cos(\sin(\sin(x)))\frac{d}{dx}(\sin(\sin(x)))$$

$$= \cos(\sin(\sin(x)))\cos(\sin(x))\frac{d}{dx}(\sin(x))$$

$$= \underbrace{\cos(\sin(\sin(x)))\cos(\sin(x))\cos(x)}$$

$y' =$ ______________________

(1.4.2) $y = \cos(2\pi \cdot 3^x)$

$$y' = -\sin(2\pi \cdot 3^x)\frac{d}{dx}(2\pi \cdot 3^x)$$

$$= [-\sin(2\pi \cdot 3^x)] \cdot 2\pi \frac{d}{dx}(3^x)$$

$$= \underbrace{-2\pi \sin(2\pi \cdot 3^x)[3^x \ln(3)]}$$

$y' =$ ______________________

(1.4.3) $y = \left(\dfrac{t-2}{2t+1}\right)^9$

$$y' = 9\left(\frac{t-2}{2t+1}\right)^8 \frac{d}{dt}\left(\frac{t-2}{2t+1}\right)$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8\left[\frac{(t-2)'(2t+1)-(t-2)(2t+1)'}{(2t+1)^2}\right]$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8\left[\frac{1\cdot(2t+1)-(t-2)\cdot 2}{(2t+1)^2}\right]$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8\left[\frac{2t+1-2t+4}{(2t+1)^2}\right]$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{5}{(2t+1)^2} = \frac{45(t-2)^8}{(2t+1)^{10}}$$

$y' =$ ____________________

(1.4.4) $y = \sin^{-1}(\sqrt{s})$

$$y' = \frac{1}{\sqrt{1-(\sqrt{s})^2}} \cdot \frac{d}{ds}(\sqrt{s})$$

$$= \frac{1}{\sqrt{1-s}} \cdot \frac{d}{ds}(s^{1/2})$$

$$= \frac{1}{\sqrt{1-s}}\left(\frac{1}{2}\right)s^{-1/2} = \frac{1}{2\sqrt{s}\sqrt{1-s}}$$

$y' =$ ____________________

(1.4.5) $y = \tan^{-1}(\sqrt{x})$

$$y' = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{1+x} \cdot \frac{d}{dx}(x^{1/2})$$

$$= \frac{1}{1+x} \cdot \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2\sqrt{x}(1+x)}$$

$y' =$

(1.4.6) $y = \ln|\sec(\theta) + \tan(\theta)|$

$$y' = \frac{1}{\sec\theta + \tan\theta} \frac{d}{dx}(\sec\theta + \tan\theta)$$

$$= \frac{1}{\sec\theta + \tan\theta}(\sec\theta\tan\theta + \sec^2\theta)$$

$$= \frac{\sec\theta\cancel{(\tan\theta + \sec\theta)}}{\cancel{\sec\theta + \tan\theta}} = \sec\theta$$

$y' =$ $\sec\theta$

(1.4.7) $y = \ln(e^{sin(x)} + e^{-\sin(x)})$

$$y' = \frac{1}{e^{\sin(x)}+e^{-\sin(x)}} \frac{d}{dx}\left(e^{\sin(x)} + e^{-\sin(x)}\right)$$

$$= \frac{1}{e^{\sin(x)}+e^{-\sin(x)}} \left(e^{\sin(x)} \frac{d}{dx}(\sin(x)) + e^{-\sin(x)} \frac{d}{dx}(-\sin(x))\right)$$

$$= \frac{1}{e^{\sin(x)}+e^{-\sin(x)}} \left(e^{\sin(x)} \cos(x) - e^{-\sin(x)} \cos(x)\right)$$

$$= \frac{\cos(x)\left[e^{\sin(x)} - e^{-\sin(x)}\right]}{e^{\sin(x)}+e^{-\sin(x)}}$$

$y' =$ ____________________

(1.4.8) $y = \ln\left(\frac{e^x+1}{e^x-1}\right) = \ln(e^x+1) - \ln(e^x-1)$

$$y' = \frac{1}{e^x+1} \frac{d}{dx}(e^x+1) - \frac{1}{e^x-1} \frac{d}{dx}(e^x-1)$$

$$= \frac{1}{e^x+1} \cdot e^x - \frac{1}{e^x-1} \cdot e^x$$

$$= e^x\left[\frac{e^x-1}{(e^x+1)(e^x-1)} - \frac{e^x+1}{(e^x+1)(e^x-1)}\right]$$

$$= e^x\left[\frac{e^x-1-e^x-1}{e^{2x}-1}\right] = \frac{-2e^x}{e^{2x}-1}$$

$y' =$ ____________________

(1.4.9) $y = \ln[2x \cdot (\ln x)^3]$

$$y' = \frac{1}{2x(\ln(x))^3} \frac{d}{dx}\left[2x(\ln(x))^3\right]$$

$$= \frac{1}{2x(\ln(x))^3}\left[2(\ln(x))^3 + 2x(3)(\ln(x))^2 \frac{d}{dx}(\ln(x))\right]$$

$$= \frac{1}{2x(\ln(x))^3}\left[2(\ln(x))^3 + 6x(\ln(x))^2 \cdot \frac{1}{x}\right]$$

$$= \frac{2(\ln(x))^3 + 6(\ln(x))^2}{2x(\ln(x))^3}$$

$$= \frac{2(\ln(x))^2\left[\ln(x) + 3\right]}{2x(\ln(x))^3} = \frac{\ln(x)+3}{x\ln(x)}$$

$y' =$

(1.4.10) $f(x) = \ln(\ln(\ln(x)))$

$$y' = \frac{1}{\ln(\ln(x))} \frac{d}{dx}\left[\ln(\ln(x))\right]$$

$$= \frac{1}{\ln(\ln(x))} \cdot \frac{1}{\ln(x)} \frac{d}{dx}\left[\ln(x)\right]$$

$$= \frac{1}{\ln(\ln(x))} \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}$$

$f'(x) = \dfrac{1}{x\ln(x)\ln(\ln(x))}$

(1.5) Suppose that $F(x) = \dfrac{1}{g(f^{-1}(x))}$ and that the functions f (which is one-to-one, and hence has its inverse) and g satisfy the following conditions. Find $F'(5)$.

$$\begin{cases} f(3) = 5 & f'(3) = 7 \\ g(3) = 2 & g'(3) = 11 \\ g(5) = 4 & g'(5) = 9 \end{cases}$$

Note $f^{-1}(5) = x$
$f(x) = 5$
$x = 3$

$$F(x) = \left[g(f^{-1}(x))\right]^{-1}$$

$$F'(x) = (-1)\left[g(f^{-1}(x))\right]^{-2} \frac{d}{dx}\left[g(f^{-1}(x))\right]$$
$$= (-1)\left[g(f^{-1}(x))\right]^{-2} g'(f^{-1}(x)) \frac{d}{dx}(f^{-1}(x))$$
$$= (-1)\left[g(f^{-1}(x))\right]^{-2} g'(f^{-1}(x)) \frac{1}{f'(f^{-1}(x))}$$
$$F'(5) = (-1)\left[g(f^{-1}(5))\right]^{-2} g'(f^{-1}(5)) \frac{1}{f'(f^{-1}(5))}$$
$$= (-1)\left[g(3)\right]^{-2} g'(3) \cdot \frac{1}{f'(3)}$$
$$= (-1)\, 2^{-2} \cdot 11 \cdot \frac{1}{7} = -\frac{11}{28}$$

$F'(5) =$ $-11/28$

(1.6) Consider the function $y = f(x) = x^3 + 6x + \sin(\pi x)$. It is one-to-one, and hence has its inverse function f^{-1}. Observe that the point $(1, 7)$ is on the graph of $y = f(x)$. Find the equation of the tangent line to the graph of $y = f^{-1}(x)$ at the point $(7, 1)$.

$$x = y^3 + 6y + \sin(\pi y)$$
$$1 = \left[3y^2 + 6 + \cos(\pi y) \cdot \pi\right] \frac{dy}{dx}$$
@ (7,1)
$$1 = \left[3 + 6 + \pi(-1)\right] \frac{dy}{dx}$$
$$\frac{1}{9-\pi} = \frac{dy}{dx}$$

$$y - 1 = \frac{1}{9-\pi}(x - 7)$$
$$y - 1 = \frac{x}{9-\pi} - \frac{7}{9-\pi}$$
$$y = \frac{x}{9-\pi} - \frac{7}{9-\pi} + 1$$
$$= \frac{x}{9-\pi} - \frac{7}{9-\pi} + \frac{9-\pi}{9-\pi}$$
$$y = \frac{x}{9-\pi} + \frac{2-\pi}{9-\pi}$$

Tangent line:

2. You are supposed to know how to compute the derivative of a function of the form $y = f(x)^{g(x)}$.

Example Problems

(2.1) Find the derivative of the following function.

(2.1.1) $y = x^{\cos(x)}$

$$\ln(y) = \ln(x^{\cos(x)})$$

$$\ln(y) = \cos(x)\ln(x)$$

$$\frac{1}{y}\frac{dy}{dx} = -\sin(x)\ln(x) + \cos(x)\cdot\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = y\left[-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right]$$

$$= x^{\cos(x)}\left[-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right]$$

$y' =$ ____________________

(2.1.2) $y = (\ln(x))^{\tan(3x)}$

$$\ln(y) = \ln\left[(\ln(x))^{\tan(3x)}\right]$$

$$\ln(y) = \tan(3x)\ln(\ln(x))$$

$$\frac{1}{y}\frac{dy}{dx} = \sec^2(3x)\frac{d}{dx}(3x)\ln(\ln(x)) + \tan(3x)\frac{1}{\ln(x)}\frac{d}{dx}(\ln(x))$$

$$\frac{1}{y}\frac{dy}{dx} = (\sec^2(3x))(3)\ln(\ln(x)) + \tan(3x)\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = y\left[3\sec^2(3x)\ln(\ln(x)) + \frac{\tan(3x)}{x\ln(x)}\right]$$

$$\frac{dy}{dx} = (\ln(x))^{\tan(3x)}\left[3\sec^2(3x)\ln(\ln(x)) + \frac{\tan(3x)}{x\ln(x)}\right]$$

$y' =$ ____________________

(2.1.3) $y = (\ln(x))^{\ln(x)}$

$$\ln(y) = \ln\left[(\ln(x))^{\ln(x)}\right]$$
$$\ln(y) = \ln(x)\ln(\ln(x))$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln(\ln(x)) + \cancel{\ln(x)}\cdot\frac{1}{\cancel{\ln(x)}}\cdot\frac{d}{dx}(\ln(x))$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln(\ln(x)) + \frac{1}{x}$$
$$\frac{dy}{dx} = y\left[\frac{\ln(\ln(x))+1}{x}\right] = \underbrace{(\ln(x))^{\ln(x)}\left[\frac{\ln(\ln(x))+1}{x}\right]}$$

$y' =$ ______________________

(2.1.4) $y = (\ln(x))^{\sin(x)}$

$$\ln(y) = \ln\left[(\ln(x))^{\sin(x)}\right]$$
$$\ln(y) = \sin(x)\ln(\ln(x))$$
$$\frac{1}{y}\frac{dy}{dx} = \cos(x)\ln(\ln(x)) + \sin(x)\frac{1}{\ln(x)}\frac{d}{dx}(\ln(x))$$
$$\frac{1}{y}\frac{dy}{dx} = \cos(x)\ln(\ln(x)) + \sin(x)\left(\frac{1}{\ln(x)}\right)\left(\frac{1}{x}\right)$$
$$\frac{dy}{dx} = y\left[\cos(x)\ln(\ln(x)) + \frac{\sin(x)}{x\ln(x)}\right]$$
$$= (\ln(x))^{\sin(x)}\left[\cos(x)\ln(\ln(x)) + \frac{\sin(x)}{x\ln(x)}\right]$$

$y' =$ ______________________

(2.1.5) $y = (x+5)^{\sin(x)}$

$$\ln(y) = \ln\left[(x+5)^{\sin(x)}\right]$$
$$\ln(y) = \sin(x)\ln(x+5)$$
$$\frac{1}{y}\frac{dy}{dx} = \cos(x)\ln(x+5) + \sin(x)\frac{1}{x+5}$$
$$\frac{dy}{dx} = y\left[\cos(x)\ln(x+5) + \frac{\sin(x)}{x+5}\right]$$
$$= (x+5)^{\sin(x)}\left[\cos(x)\ln(x+5) + \frac{\sin(x)}{x+5}\right]$$

$y' =$ ____________________

(2.1.6) $y = \left(1+\frac{3}{x}\right)^{2x}$

$$\ln(y) = \ln\left[\left(1+\frac{3}{x}\right)^{2x}\right]$$
$$\ln(y) = 2x\ln(1+3x^{-1})$$
$$\frac{1}{y}\frac{dy}{dx} = 2\ln(1+3x^{-1}) + 2x\frac{1}{1+3x^{-1}}\frac{d}{dx}(1+3x^{-1})$$
$$\frac{1}{y}\frac{dy}{dx} = 2\ln\left(1+\frac{3}{x}\right) + \frac{2x}{1+\frac{3}{x}}(-3x^{-2})$$
$$\frac{1}{y}\frac{dy}{dx} = 2\ln\left(1+\frac{3}{x}\right) + \frac{2\cancel{x}}{1+\frac{3}{x}}\cdot\frac{-3}{x^{\cancel{2}}}$$
$$\frac{dy}{dx} = y\left[2\ln\left(1+\frac{3}{x}\right) - \frac{6}{x+3}\right]$$

$y' =$ ____________________

$$= \left(1+\frac{3}{x}\right)^{2x}\left[2\ln\left(1+\frac{3}{x}\right) - \frac{6}{x+3}\right]$$

$y = f(1) = (1+\frac{3}{1})^1 = 4$

(2.2) Set $y = f(x) = \left(1 + \frac{3}{x^2}\right)^x$. Find $f'(1)$.

$\ln(y) = x \ln(1+3x^{-2})$

$\frac{1}{y}\frac{dy}{dx} = \ln(1+3x^{-2}) + x\left(\frac{1}{1+3x^{-2}}\right) \cdot \frac{d}{dx}(1+3x^{-2})$

$\frac{1}{y}\frac{dy}{dx} = \ln(1+3x^{-2}) + x\left(\frac{1}{1+3x^{-2}}\right)(-6x^{-3})$

$\frac{1}{4}\frac{dy}{dx} = \ln(1+3) + 1\left(\frac{1}{4}\right)(-6)$

$\frac{dy}{dx} = 4\left[\ln(4) - \frac{6}{4}\right]$

$f'(1) =$ $4\ln(4) - 6$ or $8\ln(2) -$

3. You are supposed to understand the method of implicit differentiation to compute the derivative. For example, you should be able to determine the equation of the tangent line to the graph of a function implicitly defined, computing the derivative using the implicit differentiation.

Example Problems

(3.1) Suppose that f is a differentiable function defined on $(-\infty, \infty)$ satisfying the following equation

$$f(x) + x^2 f(x)^3 = 10$$

and that $f(1) = 2$. Find $f'(1)$.

$f'(x) + 2x[f(x)]^3 + x^2 \cdot 3[f(x)]^2 \cdot f'(x) = 0$

$f'(1) + 2[f(1)]^3 + 1 \cdot 3[f(1)]^2 \cdot f'(1) = 0$

$f'(1) + 2[2]^3 + 3 \cdot 2^2 \cdot f'(1) = 0$

$f'(1) + 12 f'(1) = -16$

$13 f'(1) = -16$

$f'(1) =$ $-16/13$

(3.2) Find the slope of the tangent to the curve given by the equation

$$(x-y)^2 + x = 2$$

at point $(x, y) = (1, 2)$.

$\frac{d}{dx}\left((x-y)^2 + x\right) = \frac{d}{dx}(2)$

$2(x-y)\frac{d}{dx}(x-y) + 1 = 0$

@ $(1,2)$ $\quad 2(x-y)\left[1 - \frac{dy}{dx}\right] + 1 = 0$

$2(1-2)\left[1 - \frac{dy}{dx}\right] + 1 = 0$

$-2 + 2\frac{dy}{dx} + 1 = 0$

$-1 + 2\frac{dy}{dx} = 0$

$2\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{2}$

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \underline{\quad 1/2 \quad}$$

(3.3) Find the slope of the tangent line to the curve defined by

$$\sin(\pi(2x+y)) = xy$$

at the point $(0, 1)$.

$\frac{d}{dx}\left(\sin(\pi(2x+y))\right) = \frac{d}{dx}(xy)$

$\cos(\pi(2x+y))\frac{d}{dx}(\pi(2x+y)) = y + x\frac{dy}{dx}$

$\cos(\pi(2x+y))\,\pi\frac{d}{dx}(2x+y) = y + x\frac{dy}{dx}$

$\cos(\pi(2x+y))\cdot\pi\left[2 + \frac{dy}{dx}\right] = y + x\frac{dy}{dx}$

$\cos(\pi(1))\cdot\pi\left[2 + \frac{dy}{dx}\right] = 1$

$-\pi\left[2 + \frac{dy}{dx}\right] = 1$

$-2\pi - \pi\frac{dy}{dx} = 1$

$-\pi\frac{dy}{dx} = 1 + 2\pi$

$\frac{dy}{dx} = \frac{1+2\pi}{-\pi}$

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{-(1+2\pi)}{\pi}$$

(3.4) Find the equation of the tangent line to the curve defined by

$$3x^2y + \pi\cos(xy) = 2\pi$$

at the point $(1, \pi)$.

$$\frac{d}{dx}(3x^2y + \pi\cos(xy)) = \frac{d}{dx}(2\pi)$$

$$6xy + 3x^2\frac{dy}{dx} - \pi\sin(xy)\frac{d}{dx}(xy) = 0$$

$$6xy + 3x^2\frac{dy}{dx} - \pi\sin(xy)\left[1 + \frac{dy}{dx}\right] = 0$$

$$6\pi + 3\frac{dy}{dx} - \pi\,\cancel{\sin(\pi)}^{\,0}\left[1 + \frac{dy}{dx}\right] = 0$$

$$6\pi + 3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-6\pi}{3}$$

$$= -2\pi$$

$\left.\frac{dy}{dx}\right|_{(1,\pi)} =$ -2π

(3.5) Find the equation of the tangent line to the curve defined by

$$2(x+y)^{1/3} = y$$

at the point (4,4).

$$\frac{d}{dx}\left(2(x+y)^{1/3}\right) = \frac{d}{dx}(y)$$

$$\frac{2}{3}(x+y)^{-2/3}\frac{d}{dx}(x+y) = \frac{dy}{dx}$$

$$\frac{2}{3}(x+y)^{-2/3}\left[1 + \frac{dy}{dx}\right] = \frac{dy}{dx}$$

$$\frac{2}{3}(8)^{-2/3}\left[1 + \frac{dy}{dx}\right] = \frac{dy}{dx}$$

$$\frac{2}{3}\cdot\frac{1}{2^2}\left[1 + \frac{dy}{dx}\right] = \frac{dy}{dx}$$

$$\frac{1}{6} + \frac{1}{6}\frac{dy}{dx} = \frac{dy}{dx}$$

$$\frac{1}{6} = \frac{5}{6}\frac{dy}{dx}$$

$$1/5 = \frac{dy}{dx}$$

$\left.\frac{dy}{dx}\right|_{(4,4)} =$ $1/5$

(3.6) Find $\frac{dy}{dx}$ given $e^{x/y} = 7x - y$.

$$\frac{d}{dx}\left(e^{x/y}\right) = \frac{d}{dx}(7x - y)$$

$$e^{x/y}\,\frac{d}{dx}\left(\frac{x}{y}\right) = 7 - \frac{dy}{dx}$$

$$e^{x/y}\left[\frac{1\cdot y - x\frac{dy}{dx}}{y^2}\right] = 7 - \frac{dy}{dx}$$

$$\frac{y e^{x/y}}{y^2} - \frac{x e^{x/y}}{y^2}\frac{dy}{dx} = 7 - \frac{dy}{dx}$$

$$\frac{e^{x/y}}{y} - 7 = \frac{x e^{x/y}}{y^2}\frac{dy}{dx} - \frac{dy}{dx}$$

$$\frac{e^{x/y}}{y} - 7 = \left(\frac{x e^{x/y}}{y^2} - 1\right)\frac{dy}{dx}$$

$$\frac{e^{x/y} - 7y}{\cancel{y}} = \left(\frac{x e^{x/y} - y^2}{y^{\cancel{2}}}\right)\frac{dy}{dx}$$

$\frac{dy}{dx}\Big|$ = $\dfrac{\left(e^{x/y} - 7y\right)y}{x e^{x/y} - y^2}$

4. You are supposed to be understand the meaning of the derivative as the rate of change. For example, if the function represents the position, then its 1st derivative is its velocity, the 2nd is its acceleration, etc.

Example Problems:

(4.1) A particle moves along a coordinate axis in such a way that its position is described by

$$s(t) = 2\cos(t) + \sqrt{3}\cdot t$$

for $0 < t < 2\pi$. At what times(s) t in between 0 and 2π is the particle's acceleration equal to $\sqrt{3}$? $s''(t)$

$$s'(t) = -2\sin(t) + \sqrt{3}$$

$$s''(t) = -2\cos(t)$$

$$s''(t) = -2\cos(t) = \sqrt{3}$$

$$\cos(t) = \frac{-\sqrt{3}}{2}$$

* S	A
* T	C

$t = \frac{5\pi}{6}, \frac{7\pi}{6}$

$t =$ ______

(4.2) A particle moves along a coordinate axis in such a way that its position is described by

$$s(t) = 2\sin(t) + \sqrt{2}\cdot t$$

for $0 < t < 2\pi$. At what times(s) t in between 0 and 2π does the particle's movement change from going forward to going backwards? That is to say, when does the particle's velocity change from positive to negative?

$s'(t) = 2\cos(t) + \sqrt{2} = 0$

$\cos(t) = -\frac{\sqrt{2}}{2}$

$t = \frac{3\pi}{4}, \frac{5\pi}{4}$

* S | A

* T | C

$t =$ $\frac{3\pi}{4}, \frac{5\pi}{4}$

(4.3) The following function describes the motion of a dolphin over the time period $t \in [0, 2\pi]$

$$P(t) = e^{-t}\sin\left(t - \frac{\pi}{2}\right)$$

where the level of the ocean surface is set to be 0. Accordingly when the value of p is negative (resp. positive), the dolphin is under the water (resp. above the ater). The dolphin is under the water at the beginning, comes out of the water, and then goes back in. Find the speed of the dolphin with which it goes back into the water.

NOTE: The speed is the absolute value of the velocity.

$P(t) = 0$

$e^{-t}\sin\left(t - \frac{\pi}{2}\right) = 0$

$\sin\left(t - \frac{\pi}{2}\right) = 0$

b/c $e^{-t} \neq 0$

$t - \frac{\pi}{2} = \pi = \frac{2\pi}{2}$

$t = \frac{3\pi}{2}$

Next step find $v(t) = P'(t)$

$P'(t) = -e^{-t}\sin\left(t - \frac{\pi}{2}\right) + e^{-t}\cos\left(t - \frac{\pi}{2}\right)$

$P'\left(\frac{3\pi}{2}\right) = \cancel{-e^{-3\pi/2}\sin\left(\frac{3\pi}{2} - \frac{\pi}{2}\right)} \to 0$

$+ e^{-3\pi/2}\cos\left(\frac{3\pi}{2} - \frac{\pi}{2}\right)$

$P'\left(\frac{3\pi}{2}\right) = -e^{-3\pi/2}$

Final Answer: $e^{-3\pi/2}$

5. You are supposed to be able to compute the derivative of a function involving the logarithmic functions, first simplifying the formula using the laws of the logarithms. You are also supposed to know the technique of logarithmic differentiation.

Example Problems:

(5.1) Compute the derivatives of the following functions.

(5.1.1) $y = \ln(x\sqrt{x^2 - 10})$

$$y' = \frac{1}{x\sqrt{x^2-10}} \frac{d}{dx}\left(x\sqrt{x^2-10}\right)$$

$$= \frac{1}{x\sqrt{x^2-10}}\left[\sqrt{x^2-10} + x\frac{d}{dx}\left((x^2-10)^{1/2}\right)\right]$$

$$= \frac{1}{x\sqrt{x^2-10}}\left[\sqrt{x^2-10} + x\left(\frac{1}{\cancel{2}}\right)(x^2-10)^{-1/2}(\cancel{2}x)\right]$$

$$= \frac{1}{x\sqrt{x^2-10}}\left[\sqrt{x^2-10} + \frac{x^2}{\sqrt{x^2-10}}\right] = \frac{1}{x} + \frac{x}{x^2-10}$$

$y' =$ ____________________

(5.1.2) $y = \ln(e^x + xe^x)$

$y = \ln(e^x(1+x))$

$y = \ln(e^x) + \ln(1+x)$

$y = x + \ln(1+x)$

$y' = 1 + \frac{1}{1+x}$

$y' =$ $1 + \frac{1}{1+x}$

(5.1.3) $y = \dfrac{(x^3-1)^4 e^x}{(x^2+4)^3}$

$$\ln(y) = \ln\left[\frac{(x^3-1)^4 e^x}{(x^2+4)^3}\right]$$

$$\ln(y) = 4\ln(x^3-1) + \overbrace{\ln(e^x)}^{x} - 3\ln(x^2+4)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{4}{x^3-1}(3x^2) + 1 - \frac{3}{x^2+4}(2x)$$

$$\frac{dy}{dx} = \frac{(x^3-1)^4 e^x}{(x^2+4)}\left[\frac{12x^2}{x^3-1} - \frac{6x}{x^2+4} + 1\right]$$

$y' =$ ______________________________

(5.2) Compute $\frac{1}{y}\cdot\frac{dy}{dx}$ when the function y is given by the following formula.

$$y = \frac{(x^2+1)^3 \cdot \sqrt[5]{7x+1}}{\ln(1-x^2)}$$

$$\ln(y) = \ln\left[\frac{(x^2+1)^3(7x+1)^{1/5}}{\ln(1-x^2)}\right]$$

$$\ln(y) = 3\ln(x^2+1) + \frac{1}{5}\ln(7x+1) - \ln\left[\ln(1-x^2)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x^2+1}(2x) + \frac{1}{5}\cdot\frac{7}{7x+1} - \frac{1}{\ln(1-x^2)}\cdot\frac{1}{1-x^2}(-2x)$$

$$= \frac{6x}{x^2+1} + \frac{7}{5(7x+1)} + \frac{2x}{(1-x^2)\ln(1-x^2)}$$

(5.3) Consider

$$y = \frac{(x^2+1)^4 \cdot \sqrt[3]{7x+1}}{\ln(1+2x^2)}$$

What is the value of $\frac{1}{y} \cdot \frac{dy}{dx}$ when $x = 0$?

$$\ln(y) = \ln\left[\frac{(x^2+1)^4 (7x+1)^{1/3}}{\ln(1+2x^2)}\right]$$

$$\ln(y) = 4\ln(x^2+1) + \frac{1}{3}\ln(7x+1) - \ln[\ln(1+2x^2)]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{4}{x^2+1}(2x) + \frac{1}{3}\cdot\frac{7}{7x+1} - \frac{1}{\ln(1+2x^2)}\cdot\frac{1}{1+2x^2}(4x)$$

$$@x=0 \quad \frac{1}{y}\frac{dy}{dx} = 0 + \frac{1}{3}\cdot\frac{7}{1} - 0$$

$$\left(\frac{1}{y}\cdot\frac{dy}{dx}\right)\Big|_{x=0} = 7/3$$

(5.4) Let $f(x) = \ln[2x \cdot (3\ln x)^4]$. Compute $f'(e)$.

$$f'(x) = \frac{1}{2x(3\ln x)^4}\left[2(3\ln x)^4 + 2x(4)(3\ln x)^3\cdot\frac{1}{x}\right]$$

$$f'(e) = \frac{1}{2e(3\ln e)^4}\left[2(3\ln e)^4 + 2\cancel{e}(4)(3\ln e)^3\cdot\frac{1}{\cancel{e}}\right]$$

$$= \frac{1}{2e\cdot 3^4}\left[2\cdot 3^4 + 8\cdot 3^3\right]$$

$$= \frac{\cancel{2}\cdot\cancel{3^3}(3+4)}{\cancel{2}e(3)^{\cancel{4}}}$$

$$f'(e) = \frac{7}{3e}$$

6. You are supposed to be able to compute the limit of some indeterminate form, by relating its computation to the definition of a derivative.

Example Problems:

(6.1) Compute the following limits.

(6.1.1) $\lim_{h\to 0} \frac{\left[\sin\left(\frac{\pi}{2}+h\right)\right]^{\frac{\pi}{2}+h}-1}{h} = \lim_{h\to 0} \frac{F(h)-F(0)}{h}$ **Note:** $F(0)=1$

$F(h) = \left[\sin\left(\frac{\pi}{2}+h\right)\right]^{\pi/2+h}$

$\ln[F(h)] = \left(\frac{\pi}{2}+h\right)\ln\left[\sin\left(\frac{\pi}{2}+h\right)\right]$

$\frac{F'(h)}{F(h)} = 1\cdot \ln\left[\sin\left(\frac{\pi}{2}+h\right)\right] + \left(\frac{\pi}{2}+h\right)\frac{1}{\sin\left(\frac{\pi}{2}+h\right)}\cos\left(\frac{\pi}{2}+h\right)$

$\frac{F'(0)}{F(0)} = 1\cdot \ln\left[\sin\left(\frac{\pi}{2}\right)\right] + \cancel{\frac{\pi}{2}\cdot\frac{1}{\sin(\pi/2)}\cdot\cos(\pi/2)} \rightarrow 0$

$F'(0) = 1\cdot \ln(1) = 0$

$\lim_{h\to 0} \frac{\left[\sin\left(\frac{\pi}{2}+h\right)\right]^{\frac{\pi}{2}+h}-1}{h} =$ $\underline{\quad 0 \quad}$

(6.1.2) $\lim_{h\to 0} \frac{\left[\sin\left(\frac{\pi}{2}+h\right)\right]^{7h}-1}{h} = \lim_{h\to 0} \frac{F(h)-F(0)}{h}$ $F(0)=1$

$F(h) = \left[\sin\left(\frac{\pi}{2}+h\right)\right]^{7h}$

$\ln[F(h)] = 7h\ln\left[\sin\left(\frac{\pi}{2}+h\right)\right]$

$\frac{F'(h)}{F(h)} = 7\ln\left[\sin\left(\frac{\pi}{2}+h\right)\right] + 7h\cdot\frac{1}{\sin\left(\frac{\pi}{2}+h\right)}\cdot\cos\left(\frac{\pi}{2}+h\right)$

$\frac{F'(0)}{F(0)} = 7\ln\left[\sin\left(\frac{\pi}{2}\right)\right] + \cancel{7(0)\frac{\cos(\pi/2)}{\sin(\pi/2)}} \rightarrow 0$

$\frac{F'(0)}{1} = 7\ln[1] = 0$

$\lim_{h\to 0} \frac{\left[\sin\left(\frac{\pi}{2}+h\right)\right]^{7h}}{h} =$ $\underline{\quad 0 \quad}$

(6.1.3) $\lim_{h\to 0} \frac{\left[\sin\left(\frac{\pi}{6}+h\right)\right]^{1+3h} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{F(h)-F(0)}{h}$ $\quad F(0)=\frac{1}{2}$

$F(h) = \left[\sin\left(\frac{\pi}{6}+h\right)\right]^{1+3h}$

$\ln[F(h)] = (1+3h)\ln\left[\sin\left(\frac{\pi}{6}+h\right)\right]$

$\frac{F'(h)}{F(h)} = 3\ln\left[\sin\left(\frac{\pi}{6}+h\right)\right] + (1+3h)\cdot\frac{1}{\sin\left(\frac{\pi}{6}+h\right)}\cos\left(\frac{\pi}{6}+h\right)$

$\frac{F'(0)}{F(0)} = 3\ln\left[\sin(\pi/6)\right] + 1\cdot\frac{\cos(\pi/6)}{\sin(\pi/6)}$

$\frac{F'(0)}{1/2} = 3\ln\left[1/2\right] + \frac{\sqrt{3}/2}{1/2}$

$F'(0) = \frac{1}{2}\left[3\ln(2^{-1}) + \sqrt{3}\right]$

$\lim_{h\to 0} \frac{\left[\sin\left(\frac{\pi}{6}+h\right)\right]^{1+3h} - \frac{1}{2}}{h} = \underline{-\frac{3}{2}\ln(2) + \frac{\sqrt{3}}{2}}$

(6.1.4) $\lim_{h\to 0} \frac{(e+2h)^{4+3h} - e^4}{h} = \lim_{h\to 0} \frac{F(h)-F(0)}{h}$ $\quad F(0)=e^4$

$F(h) = (e+2h)^{4+3h}$

$\ln[F(h)] = (4+3h)\ln(e+2h)$

$\frac{F'(h)}{F(h)} = 3\ln(e+2h) + (4+3h)\cdot\frac{1}{e+2h}\cdot 2$

$\frac{F'(0)}{F(0)} = 3\ln(e) + 4\cdot\frac{1}{e}\cdot 2$

$\frac{F'(0)}{e^4} = 3 + \frac{8}{e}$

$F'(0) = e^4\left(3+\frac{8}{e}\right)$

$\lim_{h\to 0} \frac{(e+2h)^{4+3h} - e^4}{h} = \underline{8e^3 + 3e^4}$

(6.2) $\lim_{h\to 0} \frac{(3+2h)^{5+3h}-3^5}{h} = \lim_{h\to 0} \frac{F(h)-F(0)}{h}$ $\quad F(0)=3^5$

$F(h)=(3+2h)^{5+3h}$

$\ln[F(h)]=(5+3h)\ln[3+2h]$

$\frac{F'(h)}{F(h)}=3\ln[3+2h]+(5+3h)\cdot\frac{1}{3+2h}\cdot 2$

$\frac{F'(0)}{F(0)}=3\ln(3)+5\cdot\frac{1}{3}\cdot 2$

$\frac{F'(0)}{3^5}=3\ln(3)+\frac{10}{3}$

$F'(0)=3^5[3\ln(3)+\frac{10}{3}]$

$\lim_{h\to 0} \frac{(3+2h)^{5+3h}-3^5}{h} = \underline{3^6\ln(3)+3^4(10)}$

(6.3) $\lim_{h\to 0} \frac{(3+h)^{2-h}-9}{h} = \lim_{h\to 0} \frac{F(h)-F(0)}{h}$ $\quad F(0)=9$

$F(h)=(3+h)^{2-h}$

$\ln[F(h)]=(2-h)\ln[3+h]$

$\frac{F'(h)}{F(h)}=-\ln[3+h]+(2-h)\cdot\frac{1}{3+h}$

$\frac{F'(0)}{F(0)}=-\ln(3)+2\cdot\frac{1}{3}$

$\frac{F'(0)}{9}=-\ln(3)+\frac{2}{3}$

$F'(0)=9[-\ln(3)+\frac{2}{3}]$

$\lim_{h\to 0} \frac{(3+h)^{2-h}-9}{h} = \underline{-9\ln(3)+6}$

7. You are supposed to be able to compute the derivatives of the inverse trigonometric functions.

Examples:

(7.1) Derive the formula for the derivative of each inverse trigonometric function using implicit differentiation.

(7.1.1) $y = \sin^{-1}(x)$

$\sin(y) = x/1$

$\cos(y)\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cos(y)}$

$= \frac{1}{\sqrt{1-x^2}}$

$y' =$ $1/\sqrt{1-x^2}$

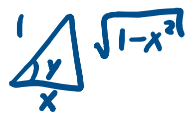
(7.1.2) $y = \cos^{-1}(x)$

$\cos(y) = x$

$-\sin(y)\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{-1}{\sin(y)}$

$= -1/\sqrt{1-x^2}$



$y' =$ $-1/\sqrt{1-x^2}$

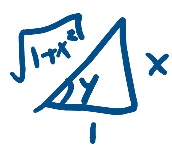
(7.1.3) $y = \tan^{-1}(x)$

$\tan(y) = x/1$

$\sec^2(y)\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sec^2(y)}$

$\frac{dy}{dx} = \cos^2(y)$



$y' =$ $\frac{1}{(\sqrt{1+x^2})^2} = \frac{1}{1+x^2}$

(7.2) Find the formula for the derivative for the following functions.

(7.2.1) $f(x) = \sin(\cos^{-1}(x))$

$$f'(x) = \cos(\cos^{-1}(x)) \cdot \frac{d}{dx}(\cos^{-1}(x))$$

$$= \cos(\cos^{-1}(x)) \cdot \frac{-1}{\sqrt{1-x^2}}$$

$$= \frac{-x}{\sqrt{1-x^2}}$$

$f'(x) =$ $-x/\sqrt{1-x^2}$

(7.2.2) $f(x) = \tan(\sin^{-1}(x))$

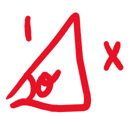
$$f'(x) = \sec^2(\sin^{-1}(x)) \frac{d}{dx}(\sin^{-1}(x))$$

$$= \sec^2(\underbrace{\sin^{-1}(x)}_{\theta}) \cdot \frac{1}{\sqrt{1-x^2}}$$

$$f'(x) = \frac{1}{(1-x^2)\sqrt{1-x^2}}$$

$\sin(\theta) = \frac{x}{1}$

$\sec\theta = \frac{1}{\sqrt{1-x^2}}$



$y = \sqrt{1-x^2}$

$\sec^2\theta = \frac{1}{1-x^2}$

$x^2 + y^2 = 1$

$y = \sqrt{1-x^2}$

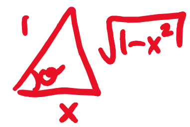
$f'(x) =$ $\frac{1}{(1-x^2)\sqrt{1-x^2}}$

(7.2.3) $f(x) = \tan(\cos^{-1}(x))$

$$f'(x) = \sec^2(\cos^{-1}(x))\frac{d}{dx}(\cos^{-1}(x))$$

$$= \Big[\sec(\underbrace{\cos^{-1}(x)}_{\theta})\Big]^2 \frac{-1}{\sqrt{1-x^2}}$$

$\cos\theta = x$



$$= \left(\frac{1}{x}\right)^2\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$f'(x) =$ $\dfrac{-1}{x^2\sqrt{1-x^2}}$

(7.2.4) $f(x) = \cos(\sin^{-1}(2x))$

$$f'(x) = -\sin(\sin^{-1}(2x))\frac{d}{dx}(\sin^{-1}(2x))$$

$$= -\sin(\sin^{-1}(2x))\frac{1}{\sqrt{1-(2x)^2}}\frac{d}{dx}(2x)$$

$$= -\sin(\sin^{-1}(2x))\frac{1}{\sqrt{1-4x^2}}\cdot 2$$

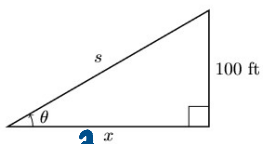
$$= -\frac{(2x)(2)}{\sqrt{1-4x^2}}$$

$f'(x) =$ $-4x/\sqrt{1-4x^2}$

(7.3) THREE "Related Rates" problems will be on the exam.

Example Problems:

(7.1) A kite 100 ft above the ground flies horizontally away from the person holding the kite's string. Assume that the string is stretched into a straight line. At a particular moment in time, the elevation angle θ of the kite (the angle between the strong and the ground) is $\pi/3$, and the elevation angle is decreasing at a rate of 0.01 rad/s. At this moment, what is the speed of the kite?



KNOW: $\frac{d\theta}{dt} = -0.01$

WANT: $\left.\frac{dx}{dt}\right|_{\theta=\pi/3}$

Find x^2:

$\tan\left(\frac{\pi}{3}\right) = \frac{100}{x}$

$\frac{\sqrt{3}}{1} = \frac{100}{x}$

$\sqrt{3}x = 100$

$x = \frac{100}{\sqrt{3}}$

$x^2 = \frac{100^2}{3}$

$\tan\theta = \frac{100}{x}$

$\tan\theta = 100x^{-1}$

$\sec^2\theta \frac{d\theta}{dt} = -100x^{-2}\frac{dx}{dt}$

$\sec^2\left(\frac{\pi}{3}\right)(-0.01) = -\frac{100}{100^2/3}\frac{dx}{dt}$

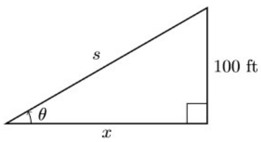
$(2)^2 \cdot \frac{-1}{\cancel{100}} = -\frac{\cancel{100}}{\cancel{100^2}/3}\frac{dx}{dt}$

$4 = 3\frac{dx}{dt}$

$\frac{4}{3} = \frac{dx}{dt}$

Answer: 4/3

(7.2) A kite 100 ft above the ground flies horizontally away from the person holding the kite's string. Assume that the string is stretched into a straight line. At a particular moment in time, the elevation angle θ of the kite (the angle between the strong and the ground) is $\pi/3$, and the elevation angle is decreasing at a rate of 0.01 rad/s. At this moment, what is the rate at which the string is unwinding from the spool?



$\sin\theta = \frac{100}{s} = 100s^{-1}$ KNOW: $\frac{d\theta}{dt} = -0.01$

$\cos\theta\frac{d\theta}{dt} = -100\,s^{-2}\frac{ds}{dt}$ WANT: $\left.\frac{ds}{dt}\right|_{\theta=\frac{\pi}{3}}$

Find s^2:

$\sin\left(\frac{\pi}{3}\right) = \frac{100}{s}$

$\frac{\sqrt{3}}{2} = \frac{100}{s}$

$\sqrt{3}s = 200$

$s = \frac{200}{\sqrt{3}}$

$s^2 = \frac{(200)^2}{3} = \frac{4\cdot 100^2}{3}$

$\cos\left(\frac{\pi}{3}\right)(-0.01) = -\frac{100}{(400/3)}\frac{ds}{dt}$

$\frac{1}{2}\cdot\frac{+1}{\cancel{100}} = \frac{+\cancel{100}}{4\cdot\cancel{100^2}/3}\frac{ds}{dt}$

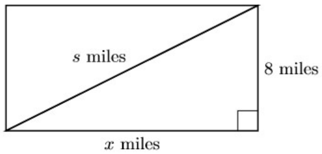
$\frac{1}{2} = \frac{3}{4}\frac{ds}{dt}$

$\frac{4}{3}\cdot\frac{1}{2} = \frac{ds}{dt}$

$\frac{2}{3} = \frac{ds}{dt}$

Answer: 2/3

(7.3) A plane is flying directly away from a bicyclist at 850 mph at an altitude of 8 miles. What is the rate of change of the distance between the plane and the bicyclist at the moment when the distance is 17 miles?



$$\frac{dx}{dt} = 850 \text{ mph}$$

$$\left.\frac{ds}{dt}\right|_{s=17}$$

$$8^2 + x^2 = 17^2$$
$$64 + x^2 = 289$$
$$x^2 = 225$$
$$x = 15$$

$$x^2 + 8^2 = s^2$$

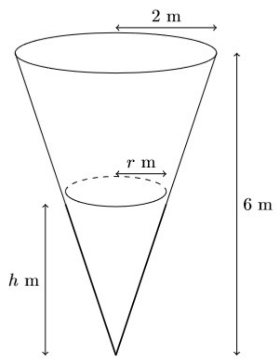
$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$

$$\frac{\cancel{2}(15)(850)}{\cancel{2}(17)} = \frac{ds}{dt}$$

$$\frac{ds}{dt} = \frac{(15)(850)}{17}$$

Answer: 12750/17

(7.4) A water tank has the shape of an inverted circular cone with base radius 2 m and height 6 m. If water is being pumped into the tank at a rate of 1 m^3 /min, find the rate at which the water level is rising when the water is 3 m deep.
HINT: The volume V of a reversed circular cone with radius r for the top circle and height h is given by $V = \frac{1}{3}\pi r^2 h$.



$\frac{r}{h} = \frac{2}{6} = \frac{1}{3} \iff r = \frac{h}{3}$

KNOW: $\frac{dV}{dt} = 1$

Want: $\left.\frac{dh}{dt}\right|_{h=3}$

$V = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h$

$= \frac{1}{27}\pi h^3$

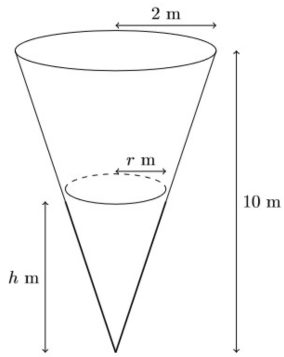
$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$

$1 = \frac{1}{\cancel{9}}\pi(\cancel{3})^2 \frac{dh}{dt}$

$\frac{1}{\pi} = \frac{dh}{dt}$

Answer: $1/\pi$

(7.5) Suppose that a water tank has the shape of an inverted circular cone with radius 2 m and height 10 m. When the water is 4 m deep in the tank, the depth of the water is increasing at a rate of 1 m/sec. What is the rate at which the volume of the water in the tank is increasing at the same time?



$$\frac{r}{h} = \frac{2}{10} = \frac{1}{5} \iff r = \frac{h}{5}$$

$$V = \frac{1}{3}\pi\left(\frac{h}{5}\right)^2 h$$

$$= \frac{1}{3}\pi\frac{h^2}{25}\cdot h$$

$$= \frac{\pi}{75}h^3$$

<u>KNOW</u>: $\frac{dh}{dt} = 1$

<u>WANT</u>: $\left.\frac{dV}{dt}\right|_{h=4}$

$$\frac{dV}{dt} = \frac{3\pi}{75}h^2\frac{dh}{dt}$$

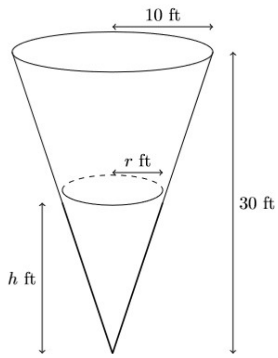
$$= \frac{\pi}{25}(4)^2\cdot 1 = \frac{16}{25}\pi$$

Answer: $\frac{16}{25}\pi$

(7.6) Water is being drained out of a conical tank with radius of 10 ft and a height of 30 ft. If the water is being drained at a constant rate of 100 ft^3/min, how fast is the depth of the water in the tank decreasing at the instant when the depth is 5 ft deep.

HINT: The volume V of a reversed circular cone with radius r for the top circle and height h is given by

$$V = \frac{1}{3}\pi r^2 h.$$



$$\frac{r}{h} = \frac{10}{30} = \frac{1}{3} \iff r = \frac{h}{3}$$

KNOW: $\frac{dV}{dt} = 100$

WANT: $\left.\frac{dh}{dt}\right|_{h=5}$

$$V = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h$$

$$= \frac{\pi}{27}h^3$$

$$\frac{dV}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}$$

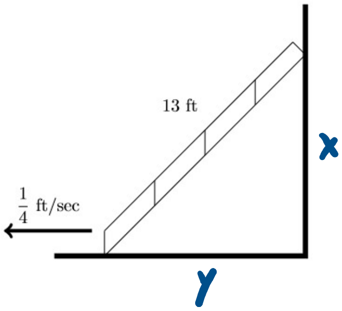
$$100 = \frac{\pi}{9}(5)^2\frac{dh}{dt}$$

$$100 = \frac{25\pi}{9}\frac{dh}{dt}$$

$$\frac{36}{\pi} = \frac{dh}{dt}$$

Answer: $36/\pi$

(7.7) A ladder 13 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of $\frac{1}{4}$ ft/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 12 ft from the wall?



$$x^2+y^2=13^2$$
$$x^2+y^2=169$$

KNOW: $\frac{dy}{dt}=\frac{1}{4}$

WANT: $\left.\frac{dx}{dt}\right|_{y=12}$

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$$2(5)\frac{dx}{dt}+2(12)\left(\frac{1}{4}\right)=0$$

$$10\frac{dx}{dt}+6=0$$

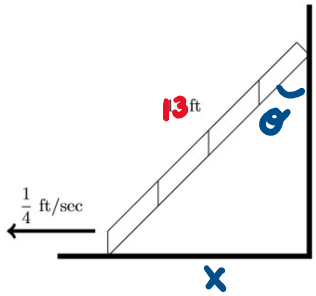
$$\frac{dx}{dt}=\frac{-6}{10}=-\frac{3}{5}$$

Find x.

$$x^2+12^2=13^2$$
$$x^2+144=169$$
$$x^2=25$$
$$x=5$$

Answer: $-3/5$

(7.8) A ladder 13 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of $\frac{1}{4}$ ft/sec, how fast is the angle θ between the top of the ladder and the wall increasing when the bottom of the ladder is 12 ft from the wall?



$\sin\theta = \frac{x}{13}$

$\cos\theta \frac{d\theta}{dt} = \frac{1}{13}\frac{dx}{dt}$

$\frac{5}{\cancel{13}}\frac{d\theta}{dt} = \frac{1}{\cancel{13}}\cdot\frac{1}{4}$

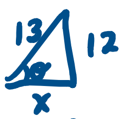
$\frac{d\theta}{dt} = \frac{1}{20}$

KNOW: $\frac{dx}{dt} = \frac{1}{4}$

WANT: $\left.\frac{d\theta}{dt}\right|_{x=12}$

Find $\cos\theta$.

$\sin\theta = \frac{12}{13}$



$x^2 + 12^2 = 13^2$

$x^2 + 144 = 169$

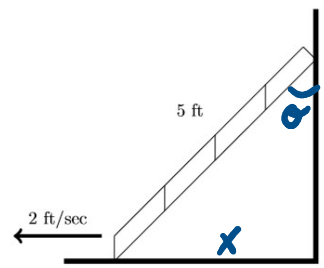
$x^2 = 25$

$x = 5$

$\cos\theta = \frac{5}{13}$

Answer: 1/20

(7.9) A 5 ft long ladder is leaned against a wall. The base of the ladder is being pulled away from the wall at a rate of 2 ft/sec. If θ is the angle formed by the ladder with the wall, how fast is θ changing (measured in rad/sec, i.e., radians per second) when the base of the ladder is 3 ft from the wall.



KNOW: $\frac{dx}{dt} = 2$

WANT: $\left.\frac{d\theta}{dt}\right|_{x=3}$

$\sin\theta = \frac{x}{5}$

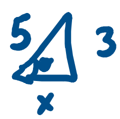
$\cos\theta \frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt}$

$\frac{4}{\cancel{5}}\frac{d\theta}{dt} = \frac{1}{\cancel{5}} \cdot 2$

$\frac{d\theta}{dt} = \frac{2}{4} = \frac{1}{2}$

Find $\cos\theta$.

$\sin\theta = \frac{3}{5}$



$x^2 + 3^2 = 5^2$

$x^2 + 9 = 25$

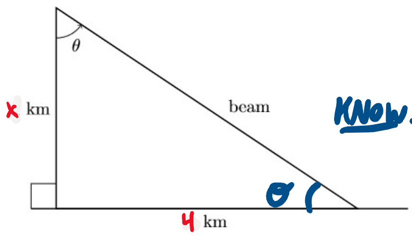
$x^2 = 16$

$x = 4$

$\cos\theta = \frac{4}{5}$

Answer: 1/2

(7.10) A lighthouse is located on an island 4 km away from the nearest point p on a straight shoreline and its light makes 8 revolutions per minute. How fast is the beam of light moving along the shoreline when it is 3 km from P?



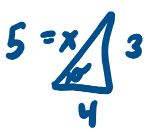
$1 \text{ rev} = 2\pi \text{ rad}$

KNOW: $\frac{d\theta}{dt} = \left(8 \frac{\cancel{rev}}{min}\right)\left(\frac{2\pi}{1} \cdot \frac{rad}{\cancel{rev}}\right)$

$= 16\pi \frac{rad}{min}$

Find $\sec\theta$.

$\tan\theta = \frac{3}{4}$



$3^2 + 4^2 = x^2$

$25 = x^2$

$x = 5$

$\sec\theta = \frac{5}{4}$

WANT: $\left.\frac{dx}{dt}\right|_{x=3}$

$\tan\theta = \frac{x}{4}$

$4\tan\theta = x$

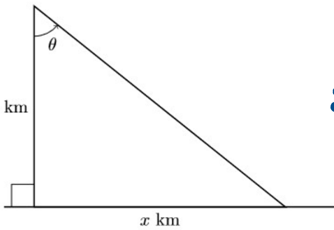
$4\sec^2\theta \frac{d\theta}{dt} = \frac{dx}{dt}$

$4\left(\frac{5}{4}\right)^2(16\pi) = \frac{dx}{dt}$

$4 \cdot \frac{25}{\cancel{16}} \cdot \cancel{16}\pi = \frac{dx}{dt}$

Answer: 100π

(7.11) A lighthouse is located on an island 4 km away from the nearest point P on a straight shoreline. The beam of light is moving along the shoreline at the speed of 25π km/min when it is 3 km away from the point P. How many revolutions per minute is the beam making at the lighthouse?



$\tan\theta = \frac{x}{4}$

KNOW: $\frac{dx}{dt} = 25\pi$

$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{4}\frac{dx}{dt}$

WANT: $\left.\frac{d\theta}{dt}\right|_{x=3}$

$\left(\frac{5}{4}\right)^2 \frac{d\theta}{dt} = \frac{1}{4}(25\pi)$

$\frac{\cancel{25}}{4^{\cancel{2}}}\frac{d\theta}{dt} = \frac{\cancel{25}\pi}{\cancel{4}}$

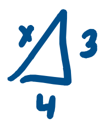
$\frac{d\theta}{dt} = 4\pi \ \frac{\text{rad}}{\text{min}}$

$= \left(4\pi \ \frac{\text{rad}}{\text{min}}\right)\left(\frac{1}{2\pi} \ \frac{\text{rev}}{\text{rad}}\right)$

$= 2 \ \frac{\text{rev}}{\text{min}}$

Find $\sec\theta$.

$\tan\theta = \frac{3}{4}$

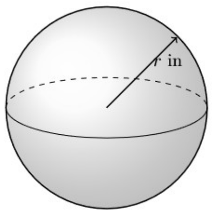


$3^2 + 4^2 = x^2$

$25 = x^2$

Answer: 2

(7.12) Suppose the surface area of a sphere is increasing at a constant rate of 10 in^2/sec. What will the radius of the sphere be at the instant when the radius of the sphere is increasing at 5 in/sec?



$S = 4\pi r^2$

KNOW: $\frac{dS}{dt} = 10$

$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

WANT: r when $\frac{dr}{dt} = 5$

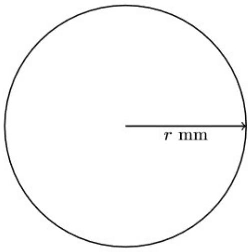
$10 = 8\pi \cdot r \cdot 5$

$\frac{10}{40\pi} = r$

$\frac{1}{4\pi} = r$

Answer: $r = 1/4\pi$

(7.13) Water is falling on a surface, wetting a circular area that is expanding at a rate of 9 mm^2/sec. How fast is the radius of the wet area expanding when the radius is 135 mm?



$A = \pi r^2$

$\underline{\text{KNOW}}$: $\frac{dA}{dt} = 9$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$\underline{\text{WANT}}$: $\left.\frac{dr}{dt}\right|_{r=135}$

$9 = 2\pi(135)\frac{dr}{dt}$

$\frac{9}{270\pi} = \frac{dr}{dt}$

Answer: $9/(270\pi)$

(7.14) A particle is moving along the curve $xy = 16$. As it reaches the point (8,2), the y-coordinate is decreasing at a rate of 4 cm/sec. What is the rate of change of the x-coordinate of the particle at that instant?

$$\frac{d}{dt}(xy) = \frac{d}{dt}(16)$$

$$y\frac{dx}{dt} + x\frac{dy}{dt} = 0$$

$$2\left(\frac{dx}{dt}\right) + 8(-4) = 0$$

$$2\left(\frac{dx}{dt}\right) - 32 = 0$$

Answer: $dx/dt = 16$

(7.15) A particle moves along the curve $y = x^2$. As it passes through the point (2,4), its x-coordinate increases at a rate of $\sqrt{5}$ cm/sec. How fast is the distance from the particle to the origin changing?

$$d = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$$= \sqrt{x^2+y^2}$$

$$D = x^2+y^2$$

$$D = x^2 + (x^2)^2$$

$$= x^2 + x^4$$

$$\frac{dD}{dt} = (2x + 4x^3)\frac{dx}{dt}$$

$$= [2(2) + 4(2)^3]\sqrt{5}$$

$$= (4 + 32)\sqrt{5}$$

Answer: $36\sqrt{5}$