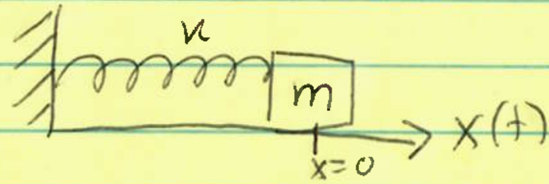
Section 9.4: Application of Fourier Series

Consider the undamped mass-on-a spring system



m - mass
k - spring constant
x(t) - displacement

Recall that Newton's Second Law

$$F = ma = mx''$$

Hooke's Law (the action of the spring the mass)

$$F = -kx$$

So

$$mx'' = -kx \iff mx'' + kx = 0$$

What's the natural frequency?

Well we can solve this ODE w/ characteristic eqn.

$$mr^2 + k = 0$$

$$r^2 = \frac{-k}{m} \iff r = \pm i\sqrt{\frac{k}{m}}$$

In most books and sources, they relabel $\sqrt{k/m} = \omega_0$ which is the natural frequency.

If $r = \pm i\omega_0$ then the general solution is

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

$\omega_0 > 0$ means the curves oscillates more
$\omega_0 < 0$ " " " " less

So great we have the solution for the homogeneous case, how about the particular say

$$mx'' + kx = F(t)$$

well if $F(t)$ is $2L$-periodic then we can approx w/ Fourier series

$$F(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

If we pick one term of the forcing say $F_n = a_n \cos\left(\frac{n\pi t}{L}\right)$

Try a particular solution of the same form

i.e. $x_{p,n}(t) = A_n \cos\left(\frac{n\pi}{L} t\right)$,

By finding $x''_{p,n}(t)$ and plugging into

$$m x'' + kx = F(t)$$

we get $A_n = \dfrac{a_n}{k - m(n\pi/L)^2}$

Similar, if you had a sine forcing term $b_n = \sin\left(\frac{n\pi t}{L}\right)$

we get $B_n = \dfrac{b_n}{k - m(n\pi/L)^2}$

By superposition, the particular solution $(x_{sp}(t))$

$$= \frac{a_0}{2k} + \sum_{n=1}^{\infty} \frac{a_n \cos(n\pi/L\, t) + b_n \sin(n\pi t/L)}{k - m(n\pi/L)^2}$$

Ex 1: Find the steady periodic solution of $x'' + 5x = F(t)$ where $F(t) = t^2$ for $0 < t < \pi$, extended evenly with period 2π. $\hookrightarrow L = \pi$

Step 1: Compute coefficients.

Note: $F(t) = t^2$ is even $\Rightarrow$ only cosine terms will appear

$\Rightarrow b_n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{2}{\pi}\int_0^{\pi} t^2\,dt = \frac{2}{\pi}\cdot\frac{t^3}{3}\Big]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\cos(nt)\,dt = \frac{2}{\pi}\int_0^{\pi} t^2\cos(nt)\,dt$$

u		dv
t^2	$+$	$\cos(nt)$
$2t$	$-$	$\sin(nt)/n$
2	$+$	$-\cos(nt)/n^2$
0		$-\sin(nt)/n^3$

Tabular Integration

$$= \frac{2}{\pi}\left[\underbrace{\frac{t^2\sin(nt)}{n}}_{0} + \frac{2t\cos(nt)}{n^2} - \underbrace{\frac{2\sin(nt)}{n^3}}_{0}\right]_0^{\pi}$$

$$= \frac{2}{\cancel{\pi}}\cdot\frac{2\cancel{\pi}\cos(n\pi)}{n^2} = \frac{4(-1)^n}{n^2}$$

Step 2: Plug into $x_{sp}(t)$.

$$x_{sp}(t) = \frac{a_0}{2k} + \sum_{n=1}^{\infty}\frac{a_n\cos(n\pi t/L)}{k-(n\pi/L)^2}$$

$$= \frac{2\pi^2/3}{2(5)} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\,\frac{\cos(nt)}{5-n^2}$$

Ex 2: Find $x_{sp}(t)$ if $x'' + 3x = F(t)$ where $F(t) = t$ on $(0, \pi)$ and extended odd w/ period 2π.

Step 1: Compute coefficients.

Note odd extension $\Rightarrow$ only sine terms $\Rightarrow a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} t\sin(nt)\,dt$$ b/c t is odd and $\sin(nt)$ is odd $\Rightarrow$ int even

$u = t$ $\quad$ $dv = \sin(nt)\,dt$

$du = dt$ $\quad$ $v = -\cos(nt)/n$

$$= \frac{2}{\pi}\left(-t\frac{\cos(nt)}{n}\Big]_0^{\pi} + \int_0^{\pi} \frac{\cos(nt)}{n}\,dt \right)$$

$$= \frac{2}{\pi}\left(-\pi\frac{\cos(n\pi)}{n} - (0) + \cancel{\frac{\sin(nt)}{n^2}\Big]_0^{\pi}} \right)$$

$$= \frac{2}{\cancel{\pi}} \cdot \frac{-\cancel{\pi}\cos(n\pi)}{n} = \frac{-2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}$$

Step 2: Plug into $x_{sp}(t) = \sum_{n=1}^{\infty} \frac{b_n \sin(nt)}{k - n^2} = \sum_{n=1}^{\infty} \frac{2\sin(nt)}{n(3-n^2)}$

~~Ex 3: A mass-spring system has $m = 9$, $k = 144$. $F(t)$ is odd, period 2π, with $F(t) = 3$ for $0 < t < \pi$. Does pure resonance occur? is when $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{144}{9}} = \frac{12}{3} = 4$~~