Section 9.4: Application of Fourier Series (Pt 2)

Now let's consider where there is friction for the mass on a spring system

So last time, we consider Newton's Second Law

$$F = ma = mx''$$

Hooke's Law

$$F_s = -kx \quad (F_s \text{ for the spring force})$$

Now let's add the friction force:

$$F_f = -cx' \quad (\text{proportional to velocity})$$

So $\quad F = F_s + F_f \quad$ (b/c Newton Law is the sum of all forces)

$$mx'' = -kx - cx'$$

$$mx'' + cx' + kx = 0$$

Let's guess $x = e^{rt}$

Characteristic eqn: $mr^2 + cr + k = 0$

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

Let's look at the discriminant

① $c^2 - 4mk < 0 \Rightarrow c$ is very small

$\Rightarrow$ underdamping

② $c^2 - 4mk > 0 \Rightarrow c$ is very larger

$\Rightarrow$ overdamping

③ $c^2 - 4mk = 0 \Rightarrow c = \pm\sqrt{4mk}$

$\Rightarrow$ critically damped

$\Rightarrow$ All returning to their equilibrium possible as soon as possible.

Again the solution of this kind of system will be $x(t) = x_c(t) + x_{sp}(t)$

$x_c(t)$: solves homog eqn
- underdamped
- criticclly damped
- overdamped

$x_{sp}(t)$: solves the non-homogeneous part and oscillates for all t

We can find $x_{sp}(t)$ for $mx'' + cx' + kx = 0$ like we did for $mx'' + kx = 0$ where assumed a forcing term (sine or cosine).

Good news for real-world cases $c > 0$

So for $mx'' + cx' + kx = 0$

$$x_{sp}(t) = \sum_{n=1}^{\infty} \frac{b_n \sin\left(\left(\frac{n\pi}{L}\right)t - \alpha_n\right)}{\sqrt{\left(k - m\left(\frac{n\pi}{L}\right)^2\right)^2 + \left(c\left(\frac{n\pi}{L}\right)\right)^2}}, \quad \tan(\alpha_n) = \frac{c\left(\frac{n\pi}{L}\right)}{k - n\left(\frac{n\pi}{L}\right)^2}$$

where $0 < \alpha_n < \pi$

Example 1: Let $m = 9$, $k = 144$, $F(t)$ is odd, period 2π with $F(t) = 3$ on $(0, \pi)$ and $F(t) = -3$ on $(-\pi, 0)$. Determine if pure resonance occurs.

Step 1: What's the natural frequency?

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{144}{9}} = 4$$

Step 2: Fourier series

B/c $F(t)$ is odd

$$F(t) = \sum_{n=1}^{\infty} b_n \sin(nt) \quad w/ \quad b_n = \frac{2}{\pi}\int_0^{\pi} 3\sin(nt)\,dt$$

$$= \frac{-6}{\pi n}\cos(nt)\Big]_0^{\pi}$$

b/c n is integer $\Rightarrow \cos(n\pi) = (-1)^n$

$$b_n = -\frac{6}{\pi}\left[\frac{\cos(\pi n) - 1}{n}\right] = -\frac{6((-1)^n - 1)}{\pi n}$$

Note: If n is even, $b_n = -\frac{6(1-1)}{\pi} = 0$

If n is odd, $b_n = -\frac{6(-1-1)}{\pi} = \frac{12}{\pi}$

Thus $F(t)$ only contains odd harmonics $n = 1, 3, \ldots$

Step 3: Comparing forcing and natural frequencies

Each sine term drives the system at $\omega_n = n$.

Resonance requires $\omega_n = \omega_0 \Rightarrow n = 4$

But $b_4 = 0 \Rightarrow$ No resonance

Physical explanation: Even though the system's natural frequency is 4 rad/s, the external force never "pushes" at that rate - only at 1, 3, 5, ... rad/s

Since none of these match 4, the system never accumulates energy at its natural rhythm. The response stays finite, oscillatory, and periodic.

Ex 2: Let $m=2$, $c=0.6$, $k=162$, $F(t)$ is odd, periodic 2π where $F(t)=3$ on $(0,\pi)$, $F(t)=-3$ on $(-\pi, 0)$

Find $x_{sp}(t)$.

Note this is the same function from Ex 1.

So $b_n = \frac{2}{\pi}\int_0^{\pi} 3\sin(nt)\,dt = \begin{cases} 12/\pi n & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$

$$x_{sp}(t) = \sum_{n=1}^{\infty} \frac{b_n \sin(n\pi/L\, t - \alpha_n)}{\sqrt{(k - m(n\pi/L)^2)^2 + (c(n\pi/L))^2}}$$

$$= \sum_{n=1}^{\infty} \frac{12}{\pi n} \frac{\sin(n\pi/\pi - \alpha_n)}{\sqrt{(162 - 2(n\pi/\pi)^2)^2 + (0.6(n\pi/\pi))^2}}$$

$$= \sum_{n=1}^{\infty} \frac{12}{\pi n} \frac{\sin(n - \alpha_n)}{\sqrt{(162 - 2n^2)^2 + (0.6n)^2}}$$

where α_n is

$$\tan(\alpha_n) = \frac{0.6(n\pi/\pi)}{k - n(n\pi/\pi)^2} = \frac{0.6n}{k - n^3}$$

$$\alpha_n = \tan^{-1}\left(\frac{0.6n}{k - n^3}\right)$$