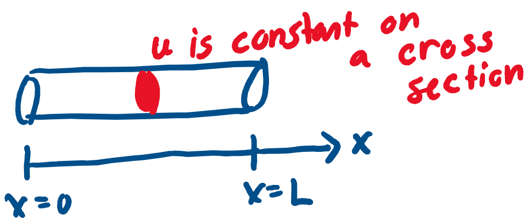
Section 9.5: Heat Conduction and Separation of Variables (Pt 1)

Wednesday, September 17, 2025 10:26 AM

Goal: Use Fourier Series to solve a PDE.



Rod of length L

$u(x,t)$ - temperature of rod

t - time

Assume that the temperature is constant through a cross section.

We can model the temperature of the rod using the 1D Heat Eqn.

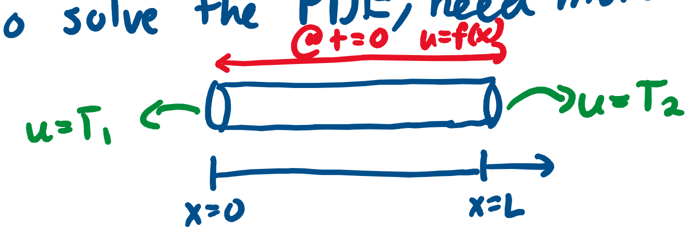
$$u_t = k u_{xx} \quad \text{on} \quad 0 \le x \le L,\ t > 0$$

Here k is called the thermal diffusivity ($k>0$) depends on the material of the rod.

Common Thermal Diffusivities

Material	k (cm^2/s)
Silver	1.70
Copper	1.15
Aluminum	0.85
Iron	0.15
Concrete	0.005

To solve the PDE, need more information



Assume at $x=0$, held at constant temperature T_1

at $x=L$, held at constant temperature T_2.

At $t=0$, initially the rod has initial temperature $f(x)$.

At $t=0$, initially the rod has initial temperature $T(x)$.

Boundary Value Problem (BVP)

$$\begin{cases} u_t = k u_{xx} & \text{on } 0 \le x \le L,\ t>0 \\ u(0,t) = T_1 \\ u(L,t) = T_2 \\ u(x,0) = f(x) \end{cases}$$

Definition: If $u(0,t) = u(L,t) = 0$, we say that the PDE has homogeneous boundary conditions.

NOTE: The heat eqn $u_t = k u_{xx}$ is linear, so the principle of superposition still holds.

To solve this BVP we use the method of separation of variables.

Assume that $u(x,t)$ can be separated into 2 parts.

$$u(x,t) = \underbrace{X(x)}_{\text{function of one variable } x} \cdot \underbrace{T(t)}_{\text{function of one variable } t.}$$

Plug into PDE: $u_t = k u_{xx}$

$$\frac{\partial}{\partial t}\left(X(x)T(t)\right) = k \frac{\partial^2}{\partial x^2}\left(X(x)T(t)\right)$$

$$X(x)\frac{\partial}{\partial t}(T(t)) = k\,T(t)\frac{\partial^2}{\partial x^2}(X(x))$$

$$X(x)T'(t) = k\,T(t)X''(x)$$

Get all the X terms on one side and T terms on other.

$$\frac{T'}{kT} = \frac{X''}{X}$$

Key observation: These can only be equal if both sides are equal to the

Called this constant $-\lambda$

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equal if both sides are equal to the same constant.

Note: λ is called the separation constant

Assume $\lambda > 0$

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda$$

$$\frac{X''}{X} = -\lambda$$
$$X'' = -\lambda X$$
$$X'' + \lambda X = 0$$

Assume $X = e^{rx}$

Characteristic Eqn:

$$r^2 + \lambda = 0$$
$$r = \pm i\sqrt{\lambda}$$
$$X = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$$

$$\frac{T'}{kT} = -\lambda$$
$$T' = -\lambda kT$$
$$T' + \lambda kT = 0$$

Assume $T = e^{rt}$

Characteristic Eqn:

$$r + k\lambda = 0$$
$$r = -\lambda k$$
$$T = c_3 e^{-k\lambda t}$$

Now boundary conditions

$$u(0,t) = u(L,t) = 0$$

What does this mean for X and T?

$$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0$$
$$u(L,t) = X(L)\,T(t) = 0 \Rightarrow X(L) = 0$$

So $\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 \quad X(L) = 0 \end{cases}$

So $X(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$

$$X(0) = c_1\cos(0) + \cancel{c_2\sin(0)}^{\,0} = 0 \rightarrow c_1 = 0$$

$$X(L) = c_2\sin(\sqrt{\lambda}\cdot L) = 0 \qquad (\text{since } c_1 = 0)$$

$\mathbb{X}(L) = c_2 \sin(\sqrt{\lambda} \cdot L) = 0$ (since $c_1 = 0$)

Assume $c_2 \neq 0$ (b/c if not we have the trivial solution)

$\sin(\sqrt{\lambda} \cdot L) = 0$

$\sqrt{\lambda}\, L = n\pi$ for $n = 1, 2, \ldots$

$\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, \ldots$

Family of solutions $\mathbb{X}_n$ is

$$\mathbb{X}_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, 3 \ldots$$

Plug $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ into the equation for $T(t)$

$$T_n' + k\lambda_n T_n = 0$$

Family of solutions T_n is

$$T_n(t) = e^{-k\lambda_n t} = e^{-n^2\pi^2 k t/L^2} \qquad n = 1, 2, 3, \ldots$$

Recall we assumed that

$$u(x,t) = \mathbb{X}(x)\, T(t)$$

So the family of solution is

$$u_n(x,t) = e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right) \text{ for } n = 1, 2, \ldots$$

each solution satisfies $\begin{cases} u_t = k u_{xx} \\ u(0,t) = u(L,t) = 0 \end{cases}$

By the Principle of Superposition, the general solution is a linear combination of these

$$u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t)$$

$$= \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1} b_n e \quad \left(\frac{\quad}{L} \right)$$

Now let's use the last condition: $u(x,0) = f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} b_n \cancel{e^{-kn^2\pi^2 \cdot 0/L^2}}\, 1 \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

$$\underbrace{\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x)}$$

This is a Fourier sine series of $f(x)$

$\Rightarrow$ b_n are the Fourier coefficients,

with $b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

So the solution of the BVP with homogeneous boundary condition is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

with $b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

Ex: Solve the BVP on $0 \le x \le \pi$, $t > 0$

$$\begin{cases} u_t = 3u_{xx} \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = 4\sin(2x) \end{cases}$$

So $f(x) = 4\sin(2x)$ $\quad L = \pi \quad k = 3$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-3n^2\pi^2 t/\pi^2} \sin\left(\frac{n\pi x}{\pi}\right)$$

$$= \sum_{n=1}^{\infty} b_n e^{-3n^2 t} \sin(nx)$$

where $b_n = \frac{2}{\pi}\underbrace{\int_0^{\pi} 4\sin(2x)\sin(nx)\,dx}$

Use the orthogonality of $\sin(nx)$

π if $n = m$

Use the orthogonality of $\sin(nx)$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} \pi & \text{if } n=m \\ 0 & \text{if } n \neq m \end{cases}$$

$$2\int_{0}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} \pi & \text{if } n=m \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_{0}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} \pi/2 & \text{if } n=m \\ 0 & \text{if } n \neq m \end{cases}$$

$$= \frac{2}{\pi}\cdot 4 \begin{cases} \pi/2 & \text{if } n=2 \\ 0 & \text{if } n \neq 2 \end{cases}$$

$$= \begin{cases} 4 & \text{if } n=2 \\ 0 & \text{if } n \neq 2 \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-3\cdot n^2 t}\sin(nx) = 4e^{-3(2)^2 t}\sin(2x) = 4e^{-12t}\sin(2x)$$