Section 9.5: Heat Conduction and Separation of Variables (Pt 2)

Thursday, October 23, 2025 3:40 PM

Last time, we solved the BVP for the Heat Eqn

$$\begin{cases} u_t = k u_{xx} & \text{on } 0 \le x \le L, t > 0 \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

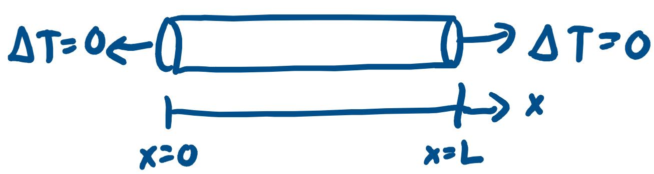
We found the solution using Separation of Variables

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

where $b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

Today let's look at a different BVP that has different boundary conditions.

<u>Insulated Endpoint Conditions</u>



length L

Endpoints are insulated (i.e. heat doesn't leave out the end)

$\Delta T = 0$ in normal direction.

So BVP of this case is

$$\begin{cases} u_t = k u_{xx} & 0 < x < L, t > 0 \\ u_x(0,t) = u_x(L,t) = 0 & \rightarrow \text{insulated endpts} \\ u(x,0) = f(x) \end{cases}$$

Again we solve with separation of variables.

Assume $u(x,t) = X(x)\, T(t)$

Plug into PDE $u_t = k u_{xx}$

Assume $u(x,t) = X(x)T(t)$

Plug into PDE $u_t = k u_{xx}$

$$\frac{\partial}{\partial t}\left(X(x)T(t)\right) = k\frac{\partial^2}{\partial x^2}\left(X(x)T(t)\right)$$

$$X(x)T'(t) = kX''(x)T(t)$$

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$\frac{X''}{X} = -\lambda$	$\frac{T'}{kT} = -\lambda$
$X'' + \lambda X = 0$	$T' + k\lambda T = 0$

Note this is the same as the BVP we did last time.

Now, let's apply the insulated endpoints conditions

$$u_x(0,t) = X'(0)T(t) = 0 \quad \Rightarrow X'(0) = 0$$

$$u_x(L,t) = X'(L)T(t) = 0 \quad \Rightarrow X'(L) = 0$$

So now, the ODE for X is

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(L) = 0 \end{cases}$$

<u>Characteristic Eqn:</u> $r^2 + \lambda = 0$

$$r = \pm i\sqrt{\lambda}$$

<u>General Solution:</u> $X = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$

Now conditions, $X' = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

$$X'(0) = \cancel{-c_1\sqrt{\lambda}\sin(0)}^{\,0} + c_2\sqrt{\lambda}\cos(0) = 0$$

$$c_2\sqrt{\lambda} = 0$$

$$c_2 = 0 \quad \text{since } \lambda > 0$$

So $X = c_1\cos(\sqrt{\lambda}x)$ and $X' = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x)$

$$X'(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$$

$$\bar{X}'(L) = \underbrace{-c_1}_{\neq 0}\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$$

$\neq 0$ b/c then trivial solution

$$\sin(\sqrt{\lambda}L) = 0$$

$$\sqrt{\lambda}L = \pi n \quad n = 0, 1, 2, \ldots$$

$$\lambda_n = \left(\frac{\pi n}{L}\right)^2 \quad n = 0, 1, 2, \ldots$$

So $\bar{X}_n = \cos\left(\frac{\pi n x}{L}\right)$

Note we keep $n=0$ in this case b/c if $n=0$ $\lambda_0 = 0$

$$\sin(\sqrt{\lambda_n}L) = \sin(0) = 0$$

then $\bar{X}_0 = \cos(0) = 1$

Solve the T equation with $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

$$T_n' + k\lambda_n T = 0$$

Characteristic eqn: $r + k\lambda_n = 0 \Rightarrow r = -k\lambda_n$

$$= -k\left(\frac{n\pi}{L}\right)^2$$

So $T_n = e^{-kn^2\pi^2 t/L} \quad n = 0, 1, 2, \ldots$

Family of solutions for u is

$$u_n(x,t) = \bar{X}_n T_n = e^{-kn^2\pi^2 t/L^2}\cos\left(\frac{n\pi x}{L}\right) \quad n = 0, 1, 2, \ldots$$

So by the Principle of Superposition

$$u(x,t) = \sum_{n=0}^{\infty} a_n u_n(x,t)$$

$$= \sum_{n=0}^{\infty} a_n e^{-kn^2\pi^2 t/L^2}\cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-kn^2\pi^2 t/L^2}\cos\left(\frac{n\pi x}{L}\right)$$

Let's plug in the initial condition:

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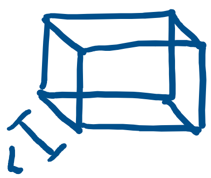
$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cancel{e^{-kn^2\pi^2(0)/L^2}}^{\,1} \cos\left(\frac{n\pi x}{L}\right) = f(x)$$

$$\underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)}_{\text{Fourier cosine series of } f(x)} = f(x)$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

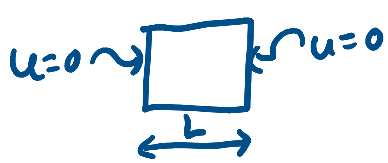
Heating a Slab



Assume material is homogeneous
- inside heats uniformly
- outside temperature $T=0$ for all time
- thickness L

Dynamics are dominated by what happens in the z-direction.

Look at a cross-section. No heat gradient in ↕ direction



Only heat gradient in ⟷ direction

So this reduces to a 1D equation.

Ex: A copper slab is 4cm thick with both faces are kept at 0°C. Initially the temp is 100°C. What is the temperature at center 3s later?

Copper slab $\Rightarrow k = 1.15\ \text{cm}^2/\text{s}$

$L = 4\text{cm}$

$0°C \Rightarrow u(0,t) = u(L,t) = 0$

$100°C \Rightarrow u(x,0) = 100°C$

So
$$\begin{cases} u_t = 1.15\, u_{xx} & 0 \le x \le 4,\ t>0 \\ u(0,t) = u(4,t) = 0 \\ u(x,0) = 100 \end{cases}$$

Ex: Insulated Endpoints

$13u_t = u_{xx} \qquad 0 \le x \le 2,\ t > 0$

Ex. Insulated Endpoints

$$\begin{cases} 3u_t = u_{xx} & 0 \le x \le 2,\ t > 0 \\ u_x(0,t) = u_x(2,t) = 0 \\ u(x,0) = \cos^2(2\pi x) \end{cases}$$

Here $L = 2$, $k = 1/3$, $f(x) = \cos^2(2\pi x)$

Solution has the form:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{1}{3}n^2\pi^2 t/2} \cos\left(\frac{n\pi x}{2}\right)$$

where $a_0 = \frac{2}{2}\int_0^2 \cos^2(2\pi x)\,dx$

$$a_n = \frac{2}{2}\int_0^2 \cos^2(2\pi x)\cos\left(\frac{n\pi x}{2}\right)dx$$

To find these integrals we need to remember

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\cos^2\theta = \frac{1}{2}\left[1 + \cos(2\theta)\right]$$

So $a_0 = \int_0^2 \cos^2(2\pi x)\,dx = \frac{1}{2}\int_0^2 (1 + \cos(4\pi x))\,dx$

$$= \frac{1}{2}\left[x + \frac{\sin(4\pi x)}{4\pi}\right]_0^2$$

$$= \frac{1}{2}\left[2 + \frac{\sin(8\pi)}{4\pi}\right] = 1 + 0 = 1$$

$$a_n = \int_0^2 \cos^2(2\pi x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_0^2 \left[1 + \cos(4\pi x)\right]\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\int_0^2 \cos\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_0^2 \cos(4\pi x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \cancel{\frac{1}{2}\left[\frac{\sin(n\pi x/2)}{n\pi/2}\right]_0^2} \to 0 + \underbrace{\frac{1}{2}\int_0^2 \cos(4\pi x)\cos\left(\frac{n\pi x}{2}\right)dx}_{\text{Want to use orthogonality}}$$

First let $u = \frac{\pi x}{2}$

$du = \frac{\pi}{2}dx \iff dx = \frac{2}{\pi}du$

$$= \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi} \cos(8u)\cos(nu)\,du$$

$$= \frac{\quad}{\pi} \cdot \frac{\quad}{2} \int_0 \cos(8u)\cos(nu)\,du$$

Orthogonality condition

$$\int_{-\pi}^{\pi} \cos(mu)\cos(nu)\,du = \begin{cases} \pi & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases}$$

$$2\int_{0}^{\pi} \cos(mu)\cos(nu)\,du = \begin{cases} \pi & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases}$$

$$\int_{0}^{\pi} \cos(mu)\cos(nu)\,du = \begin{cases} \pi/2 & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases}$$

$$= \frac{1}{\pi} \cdot \begin{cases} \pi/2 & \text{if } n=8 \\ 0 & \text{if } n\neq 8 \end{cases}$$

$$= \begin{cases} 1/2 & \text{if } n=8 \\ 0 & \text{if } n\neq 8 \end{cases}$$

So $u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\frac{1}{3}n^2\pi^2 t/2} \cos\left(\frac{n\pi x}{2}\right)$

$$= \frac{1}{2} + \frac{1}{2} e^{-16\pi^2 t/3} \cos(4\pi x)$$