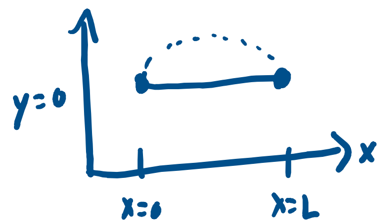
Section 9.6: Vibrating Strings and the One-Dimensional Wave Equation (Part 1)

Sunday, October 26, 2025 9:11 PM

Goal: Solve the 1D Wave Eqn

I. Vibrating Strings:



String length L
fixed at endpoints
@ t=0, pluck the string and measure the displacement $y(x,t)$

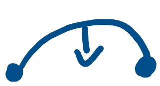
Model the displacement with the 1D-Wave Eqn:

$$y_{tt} = a^2 y_{xx} \quad \text{where } a^2 = \frac{\text{tension}}{\text{density}} > 0$$

Physical intuition

y_{tt} is acceleration of string

y_{xx} is curvature of string (in space)



$y_{xx} < 0$

So $y_{tt} = a^2 y_{xx} < 0$

string accelerates downward

$y_{xx} > 0$

$y_{tt} = a^2 y_{xx} > 0$

String accelerates upward

Intuition for both cases is that the string wants to restore to equilibrium

Boundary Value Problem (for Wave Eqn)

$$\begin{cases} y_{tt} = a^2 y_{xx} & 0 \le x \le L,\ t > 0 \\ y(0,t) = y(L,t) = 0 & \text{(fixed endpoints)} \end{cases}$$

$$\begin{cases} y(0,t) = y(L,t) = 0 & \text{(fixed endpoints)} \\ y(x,0) = f(x) & \text{(initial displacement)} \\ y_t(x,0) = g(x) & \text{(initial velocity)} \end{cases}$$

To solve this BVP, it is easier to split into 2 BVPs and then add their solutions.

Problem A

$$\begin{cases} y_{tt} = a^2 y_{xx} \\ y(0,t) = y(L,t) = 0 \\ y(x,0) = f(x) \\ y_t(x,0) = 0 \end{cases}$$

Problem B

$$\begin{cases} y_{tt} = a^2 y_{xx} \\ y(0,t) = y(L,t) = 0 \\ y(x,0) = 0 \\ y_t(x,0) = g(x) \end{cases}$$

Let y_A solve Problem A and y_B solve Problem B.

Thus the eventual solution of the wave eqn will be

$$y(x,t) = y_A(x,t) + y_B(x,t)$$

Let's start by solve Problem A with Separation of Variables.

Assume that $y_A(x,t) = X(x)T(t)$ solves Problem A.

Plug $y_A(x,t)$ into PDE:

$$y_{tt} = a^2 y_{xx}$$

$$\frac{\partial^2}{\partial t^2}(XT) = a^2 \frac{\partial^2}{\partial x^2}(XT)$$

$$X T'' = a^2 X'' T$$

... X terms and T terms.

Separate X terms and T terms.

$$\frac{T''}{a^2T} = \frac{X''}{X} = -\lambda \quad \text{(separation constant)}$$

So we get

$$\frac{T''}{a^2T} = -\lambda$$

$$T'' = -\lambda a^2 T$$

$$T'' + \lambda a^2 T = 0$$

Characteristic Eqn:

$$r^2 + \lambda a^2 = 0$$

$$r = \pm i a\sqrt{\lambda}$$

$$\Rightarrow T(t) = c_1 \sin(a\sqrt{\lambda}\, t) + c_2 \cos(a\sqrt{\lambda}\, t)$$

and

$$\frac{X''}{X} = -\lambda$$

$$X'' = -\lambda X$$

$$X'' + \lambda X = 0$$

Characteristic eqn:

$$r^2 + \lambda = 0$$

$$r = \pm i\sqrt{\lambda}$$

$$\Rightarrow X(x) = c_3 \sin(\sqrt{\lambda}\, x) + c_4 \cos(\sqrt{\lambda}\, x)$$

Now let's consider the boundary conditions

$$y(0,t) = 0 = X(0)T(t) \rightarrow X(0) = 0$$

$$y(L,t) = 0 = X(L)T(t) \rightarrow X(L) = 0$$

Note $T(t) \neq 0$ b/c then we have the trivial solution.

So for X we have the ODE

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases} \longrightarrow X(x) = c_3 \sin(\sqrt{\lambda}\, x) + c_4 \cos(\sqrt{\lambda}\, x)$$

When $X(0) = c_3 \sin(0) + c_4 \cos(0) = 0 \Rightarrow c_4 = 0$

So $X(x) = c_3 \sin(\sqrt{\lambda}\, x)$

So $X(x) = c_3 \sin(\sqrt{\lambda}\, x)$

When $X(L) = c_3 \sin(\sqrt{\lambda}\, L) = 0$

Again $c_3 \neq 0$ b/c if not we get the trivial solution.

$$\sin(\sqrt{\lambda}\, L) = 0$$
$$\sqrt{\lambda}\, L = \pi n \quad n = 1, 2, 3, \ldots$$
$$\lambda = \left(\frac{\pi n}{L}\right)^2 \quad n = 1, 2, 3, \ldots$$

So for T we have

$$T'' + a^2 \lambda T = 0 \Rightarrow T(t) = c_1 \sin(a\sqrt{\lambda}\, t) + c_2 \cos(a\sqrt{\lambda}\, t)$$

With $\lambda = \left(\frac{\pi n}{L}\right)^2$, $\quad T(t) = c_1 \sin\left(\frac{\pi n a}{L} t\right) + c_2 \cos\left(\frac{\pi n a}{L} t\right)$

Now the remaining boundary conditions.

$$y(x,0) = f(x) = X(x)T(0)$$

Let's pin this for later

$$y_t(x,0) = 0 = X(x)T'(0) \rightarrow T'(0) = 0$$

Again $X(x) \neq 0$ because it will give us the trivial solution.

So $T'(t) = c_1 \frac{\pi n a}{L} \cos\left(\frac{\pi n a}{L} t\right) - c_2 \frac{\pi n a}{L} \sin\left(\frac{\pi n a}{L} t\right)$

$$T'(0) = c_1 \frac{\pi n a}{L} \cos(0) - c_2 \frac{\pi n a}{L} \sin(0) = 0$$
$$c_1 \frac{\pi n a}{L} = 0 \Rightarrow c_1 = 0$$

So $T(t) = c_2 \cos\left(\frac{\pi n a}{L} t\right)$.

Alright so $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ and $T_n = \cos\left(\frac{\pi n a}{L} t\right)$

So for each n,

So for each n,

$$y_n(x,t) = X_n(x)\,T_n(t) = \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{\pi n a}{L}t\right)$$

By the principle of superposition

$$y_A(x,t) = \sum_{n=1}^{\infty} A_n\, y_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{\pi n a}{L}t\right)$$

Let's go back to the eqn that we pinned.

$$y(x,0) = f(x)$$

$$y_A(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi n x}{L}\right)\cos(0) = \underbrace{\sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi n x}{L}\right) = f(x)}_{\text{Fourier Sine Series}}$$

So $A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

So $y_A(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{\pi n a}{L}t\right)$

where $A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

Now let's solve Problem B, which was

$$\begin{cases} y_{tt} = a^2 y_{xx} \\ y(0,t) = y(L,t) = 0 \\ y(x,0) = 0 \\ y_t(x,0) = g(x) \end{cases}$$

Note: the first two eqns are the same as Problem A. So

① We will get the same ODE for X.

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases}$$

which gave us $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and

which gave us $\lambda_n = \left(\frac{n\pi}{L}\right)$ and
$X_n = \sin\left(\frac{n\pi x}{L}\right)$ for $n = 1, 2, 3, \ldots$

② We also get $T'' + a\lambda T = 0 \Rightarrow T(t) = c_1 \sin\left(\frac{an\pi}{L}t\right) + c_2 \cos\left(\frac{an\pi}{L}t\right)$.

Let's impose the boundary condition: $y(x,0) = 0$

$y(x,0) = X(x)T(0) = 0 \Rightarrow T(0) = 0$

Again $X(x) \neq 0$ because we get the trivial solution.

$T(0) = c_1 \sin(0) + c_2 \cos(0) = 0 \Rightarrow c_2 = 0$

So $T(t) = c_1 \sin\left(\frac{n\pi a}{L}t\right) \Rightarrow T_n = \sin\left(\frac{n\pi a}{L}t\right)$

So $y_n(x,t) = \sin\left(\frac{n\pi at}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$

By superposition,

$$y_B(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi at}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

Now for the last boundary condition: $y_t(x,0) = g(x)$

$$g(x) = y_t(x,0) = \frac{\partial}{\partial t}[y_B]\Big|_{t=0} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\frac{\partial}{\partial t}\left[\sin\left(\frac{n\pi at}{L}\right)\right]\Big|_{t=0}$$

$$= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\left(\cos\left(\frac{n\pi at}{L}\right)\cdot\frac{n\pi a}{L}\right)\Big|_{t=0}$$

$$g(x) = \sum_{n=1}^{\infty} \underbrace{B_n \cdot \frac{n\pi a}{L}}_{\text{Let this be } b_n} \sin\left(\frac{n\pi x}{L}\right)$$

$$\underbrace{g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)}_{\text{Fourier sine series}}$$

Fourier sine series

where $b_n = \frac{2}{L}\int_0^L g(x)\,dx$

But I need B_n not b_n. So $B_n \cdot \frac{n\pi a}{L} = \frac{2}{L}\int_0^L g(x)\,dx$

$$B_n = \frac{L}{n\pi a}\cdot\frac{2}{L}\int_0^L g(x)\,dx$$

$$= \frac{2}{\pi n a}\int_0^L g(x)\,dx$$

Full solution: $y(x,t) = y_A(x,t) + y_B(x,t)$

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

where $A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$$B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$