Section 9.6: Vibrating Strings and the One-Dimensional Wave Equation (Part 2)

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Part 2 of String (Wave)

Last class: We wrote down the BVP for a vibrating string

$$\begin{cases} y_{tt} = a^2 y_{xx} \quad 0 \le x \le L,\ t > 0 & \text{(wave eqn)} \\ y(0,t) = y(L,t) = 0 & \text{(fixed endpts)} \\ y(x,0) = f(x) & \text{(initial displacement)} \\ y_t(x,0) = g(x) & \text{(initial velocity)} \end{cases}$$

With Separation of Variables we found

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

with $A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

$$B_n = \frac{2}{n\pi a}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Ex 1: Solve the BVP $\underbrace{a^2}_{} \to a = 10$

$$\begin{cases} y_{tt} = \underbrace{100}_{=a^2} y_{xx} \quad 0 < x < \underbrace{1}_{=L},\ t > 0 \\ y(0,t) = y(1,t) = 0 \\ y(x,0) = 0 = f(x) \\ y_t(x,0) = x = g(x) \end{cases}$$

This has the form of BVP of Heat Eqn.

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} A_n \cos\left(n\pi \cdot 10\ t\right) \sin(n\pi x) + \sum_{n=1}^{\infty} B_n \sin\left(n\pi \cdot 10\ t\right) \sin(n\pi x)$$

where $A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

$$= \frac{2}{1}\int_0^1 0 \sin\left(\frac{n\pi x}{1}\right) dx$$

$$= \frac{2}{1}\int_0^1 0 \sin\left(\frac{n\pi x}{1}\right)dx$$

$$= 2\int_0^1 0\, dx = 0$$

Now $y(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi \cdot 10 \cdot t)\sin(n\pi x)$

$$B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{n\pi\cdot 10}\int_0^1 x\sin(n\pi x)\,dx$$

u	dv
x $+$	$\sin(n\pi x)$
1 $-$	$-\cos(n\pi x)/n\pi$
0	$-\sin(n\pi x)/n^2\pi^2$

$$= \frac{1}{5n\pi}\left[-\frac{x\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{n^2\pi^2}\right]_0^1$$

$$= \frac{1}{5n\pi}\left[-\frac{\cos(n\pi)}{n\pi} + \cancel{\frac{\sin(n\pi)}{n^2\pi^2}}^{\,0} - (0+0)\right]$$

Remember n is integer

Remember $\cos(n\pi) = (-1)^n$, $n = 1, 2, 3, \ldots$

$$= \frac{-1}{5n\pi}\cdot\frac{\cos(n\pi)}{n\pi} = \frac{(-1)(-1)^n}{5n^2\pi^2} = \frac{(-1)^{n+1}}{5n^2\pi^2}$$

$$y(x,t) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{5\pi^2 n^2}\sin(10\pi n t)\sin(\pi n x)$$

<u>Ex 2:</u> Solve

$$\begin{cases} y_{tt} = 1\, y_{xx} \quad \rightarrow a^2 = 1,\ a = 1 & 0 \le x \le 1,\ t > 0 \\ y(0,t) = y(1,t) = 0 \\ y(x,0) = -3 \\ y_t(x,0) = 5\sin(3\pi x) \end{cases}$$

See $a = 1$ $L = 1$ $f(x) = -3$ $g(x) = 5\sin(3\pi x)$

So $y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi(1)t}{1}\right)\sin\left(\frac{n\pi x}{1}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi(1)t}{1}\right)\sin\left(\frac{n\pi x}{1}\right)$

$$\text{So } y(x,t) = \sum_{n=1} A_n \cos\left(\frac{\quad}{1}\right)\sin\left(\frac{\quad}{1}\right) + \sum_{n=1} B_n \sin\left(\frac{\quad}{1}\right)\sin\left(\frac{\quad}{1}\right)$$

$$= \sum_{n=1}^{\infty} A_n \cos(n\pi t)\sin(n\pi x) + \sum_{n=1}^{\infty} B_n \sin(n\pi t)\sin(n\pi x)$$

where $A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$$= \frac{2}{1}\int_0^1 -3\sin(n\pi x)\,dx$$

$$= 6\,\frac{\cos(\pi n x)}{\pi n}\Bigg]_0^1$$

$$= 6\left[\frac{\cos(\pi n)}{\pi n} - \frac{\cos(0)}{\pi n}\right]$$

Remember $\cos(\pi n) = (-1)^n$

$$= \frac{6}{\pi n}\left[\cos(\pi n) - 1\right]$$

If n is odd, -1

If n is even, $+1$.

$$= \frac{6}{\pi n}\cdot\begin{cases} -1-1 & \text{if } n \text{ is odd} \\ 1-1 & \text{if } n \text{ is even}\end{cases}$$

$$= \begin{cases} \frac{6}{\pi n}\cdot(-2) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even}\end{cases}$$

Now $B_n = \frac{2}{\pi n a}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$$= \frac{2}{\pi n\cdot 1}\int_0^1 5\sin(3\pi x)\sin(n\pi x)\,dx$$

$u = \pi x$

$du = \pi\,dx \Leftrightarrow dx = \frac{du}{\pi}$

Orthogonality

$$\int_0^\pi \sin(mu)\sin(nu)\,du = \begin{cases} \pi/2 & \text{if } m=n \\ 0 & \text{if } m\neq n\end{cases}$$

$$= \frac{2}{\pi n}\int_0^\pi 5\sin(3u)\sin(nu)\,\frac{du}{\pi}$$

$$= \frac{2}{\pi n}\cdot\frac{5}{\pi}\begin{cases} \pi/2 & \text{if } 3=n \\ 0 & \text{if } 3\neq n\end{cases}$$

$$= \begin{cases} \frac{5}{\pi n} & \text{if } n=3 \end{cases}$$

$$= \begin{cases} \dfrac{5}{\pi n} & \text{if } n = 3 \\ 0 & \text{if } n \neq 3 \end{cases}$$

So $$y(x,t) = \sum_{\substack{n=1 \\ n \text{ is odd}}}^{\infty} \left(-\frac{12}{n\pi}\right) \cos(n\pi t)\sin(n\pi x) + \frac{5}{\pi n}\sin(3\pi t)\sin(3\pi x)$$